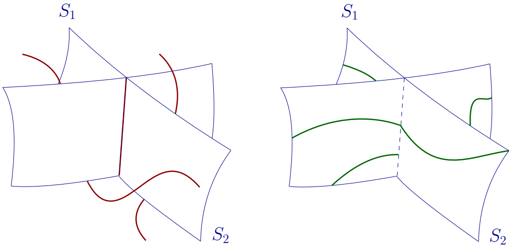
Calabi-Yau four wall-crossing

Arkadij Bojko
Mirror Symmetry Workshop 2025



Contents

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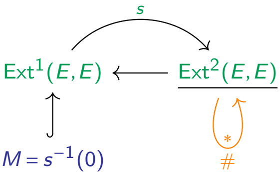
2. Locally around E, M looks like:



where $\#$ - the Serre isomorphism, $B : \mathrm{Ext}^2(E,E)^{\otimes 2} \to \mathbb{C}$ the induced pairing, and $B(s,s) = 0$.

Orientations

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Remark

The argument for projective X consists of two parts - 1) prove orientability for $U(n)$-connections, 2) transport it to sheaves and complexes on X. 1) was corrected by Joyce-Upmeier under extra conditions, 2) is done by CGJ. For quasi-projective X, I reduced orientations for compactly supported sheaves to orientations in 1) on a compact manifold.

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Example

All X obtained as complete intersections in toric varieties satisfy this.

Compatibility of orientations

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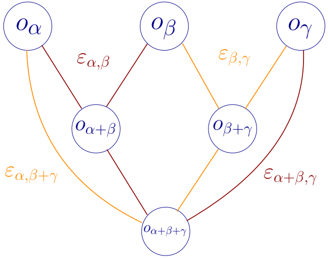
3. These signs satisfy associativity and more



Virtual integrals

1. When M is orientable, Borisov-Joyce(15') and later Oh-Thomas(20') construct

$$[M]^{\mathrm{vir}} \in H_{2-\chi(E,E)}(M,\mathbb{Z}),$$
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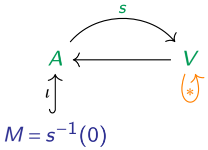
Example

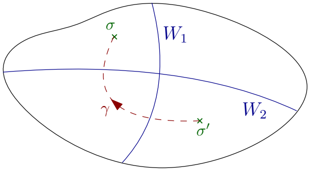
Assume the local model holds globally for a smooth A and an $SO(2n,\mathbb{C})$ vector bundle V:



where $B(s,s)=0$ for the induced pairing B. If $V=\Lambda\oplus\Lambda^*$, then $\iota_*[M]^{\mathrm{vir}}=e(\Lambda)\cap[A]$.

Gross-Joyce-Tanaka conjecture

1. Consider two stability conditions in a stability manifold:



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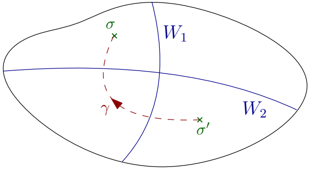

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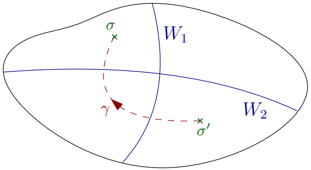

2. Let ϕ, ϕ' be the associated phases of objects
3. For sheaves E, E' that are σ- resp. σ'-semistable, there can be short exact sequences

$$0 \longrightarrow E_1 \longrightarrow E' \longrightarrow E_2 \longrightarrow 0 \quad \text{s.t.} \quad \phi(E_1) > \phi(E') > \phi(E_2),$$

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Lie algebras in wall-crossing

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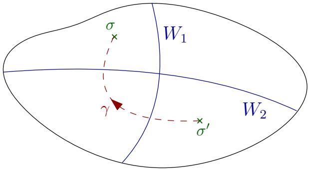

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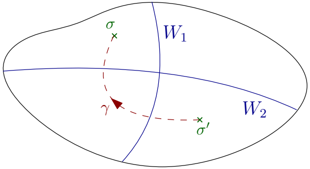

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5. Vertex algebras are refinements of Lie algebras.

Borcherds bicharacter construction

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3. An action of $H_* := H_*(B\mathbb{G}_m)$ (a Hopf algebra) on A_* induced by $\rho : B\mathbb{G}_m \times \mathscr{M}_X \to \mathscr{M}_X$ scaling automorphisms. Using $(H_*)^* \cong \Bbbk[\![z]\!]$, it induces a map

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Theorem (Borcherds(99'))

Let A_* be a bialgebra with a compatible action of the Hopf algebra H_*. For a given

$$- \cap B_z \in A_* \otimes A_* \longrightarrow A_* \otimes A_*((z))$$

satisfying the axioms of a bicharacter, there is a vertex algebra given by

$$Y_z(v, w) = m\Big[e^{zT} \otimes \mathrm{id}\Big(v \otimes w \cap B_z\Big)\Big]\,.$$

Joyce's bicharacter

1. Continue using A_*, H_* introduced as homologies.

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Theorem (Joyce (17'))

Let

$$\mathscr{E}\mathrm{xt} = \mathrm{R}\mathscr{H}\mathrm{om}_{\mathscr{M}_X \times \mathscr{M}_X}\big(\mathscr{E}_1, \mathscr{E}_2\big)$$

and $\varepsilon : K^0(X) \times K^0(X) \to \{\pm 1\}$ *be the signs* $\varepsilon_{\alpha,\beta}$ *determined by comparing orientations. The cap product*

$$\cap B_z = \cap \varepsilon\, z^{\mathrm{rk}} c_{z^{-1}}(\mathscr{E}\mathrm{xt})$$

is a bicharacter and therefore gives rise to a vertex algebra.

Theorem (Borcherds' foundational paper)

Let A_* be a vertex algebra. Then $L_* = A_{*+2}/T(A_*)$ is a Lie algebra with

$$[v, w] = \mathrm{Res}_{z=0}\Big\{Y_z(v, w)\Big\}.$$

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1. In our case, there is a comparison (Joyce 17'):

$$L_* \sim H_{*+\mathrm{vdim}_{\mathbb{R}}}(\mathcal{M}_X^{\mathrm{rig}}),$$

so L_* contains all $[M_\alpha^\sigma]^{\mathrm{vir}}$ and their generalizations $\langle \mathcal{M}_\alpha^\sigma \rangle$.

Conjecture (GJT(20'))

In L_0, have

$$\langle \mathcal{M}_\alpha^\sigma \rangle = \sum_{\underline{\alpha} \vdash_{\mathscr{A}} \alpha} \widetilde{U}(\underline{\alpha};\sigma',\sigma)\Big[\Big[\cdots\Big[\langle \mathcal{M}_{\alpha_1}^{\sigma'}\rangle,\langle \mathcal{M}_{\alpha_2}^{\sigma'}\rangle\Big],\dots\Big],\langle \mathcal{M}_{\alpha_n}^{\sigma'}\rangle\Big].$$

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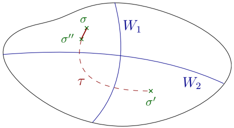
2. **Application (Virasoro constraints)**: Sheaf-theoretic Virasoro constraints for $\langle \mathcal{M}_\alpha^\sigma \rangle$ cut out a Lie subalgebra $\Longrightarrow$ preserved by wall-crossing

1. The proof can be separated into two problems:



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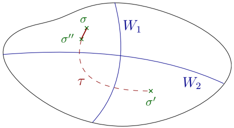

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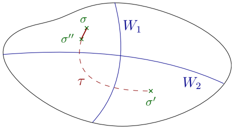

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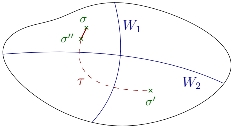

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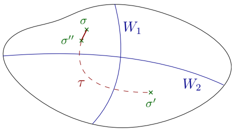

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(II) Prove that the invariants $\langle \mathscr{M}_\alpha^\tau \rangle$ are defined independent of choices made.

2. New tools and conceptual framework for proving these two steps developed in B. (25').
3. Contains the first established proofs of wall-crossing in the CY4 setting.

Theorem (B. (25'))

Both (I) and (II) hold in the case of representations of CY4 quivers and sheaves or pairs on local Calabi-Yau fourfolds.

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and $[P^\tau_{L,\alpha}]^{\mathrm{vir}}$ for L a sufficiently negative line bundle.

Theorem (B. (25'))

For a very ample D and $L' = L \otimes \mathscr{O}_X(-D)$, the subspace $\iota : P^\tau_{L,\alpha} \hookrightarrow P^\tau_{L',\alpha}$ cut out by a section of $\mathbb{V}$. A CY4 version of a quantum-Lefschetz-type argument gives

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3. The theorem from the previous slide is extended using the Jouanolou trick in

Theorem (B.-Kuhn-Liu-Thimm (in progress))

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First applications

Historically first application

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2. Starting point: For a line bundle L on X, construct

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using the projections $X \xleftarrow{\pi_X} X \times \mathrm{Hilb}^n(X) \xrightarrow{\pi_H} \mathrm{Hilb}^n(X)$ and $\mathscr{F}$ the universal 0-dimensional sheaf.
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3. The wall-crossing formula for $\mathrm{Hilb}^n(X)$ becomes

$$\sum_{n\geq 0}[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}q^n = \exp\Big\{\sum_{n>0}\Big[\langle \mathscr{M}_{np}\rangle, -\Big]q^n\Big\}e^{(1,0)}.$$

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3. In fact, need invariants only for $L = \mathscr{O}_X(D)$ when D is a smooth divisor - formula conjectured by Cao–Kool (17') and proved by Park (21').

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Example

Use

$$\mathrm{Hilb}^n(\mathbb{C}^4)=\mathrm{Rep}^{\sigma}_{(1,n)}(C_4)$$

where

$C_4 =$

and equivariant wall-crossing for CY4 quivers from B. (25').

1. One limit of $K_n(L,y)$ is

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2. The generating series was conjectured to be:

$$I(q) := \sum_{n\geq 0} I_n q^n = \exp\left[q\frac{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)}{\lambda_1\lambda_2\lambda_3\lambda_4}\right]$$

where $e^{\lambda_i} = t_i$.

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$$I(q) := \sum_{n\geq 0} I_n q^n = \exp\left[q\frac{(\lambda_1+\lambda_2)(\lambda_2+\lambda_3)(\lambda_1+\lambda_3)}{\lambda_1\lambda_2\lambda_3\lambda_4}\right]$$
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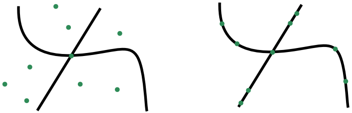

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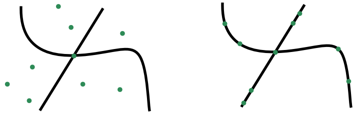

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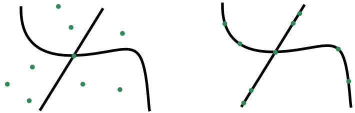

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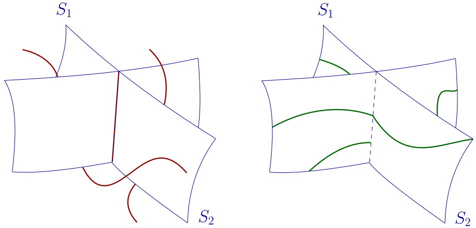

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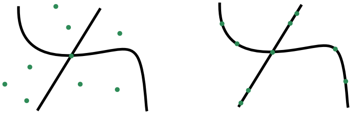

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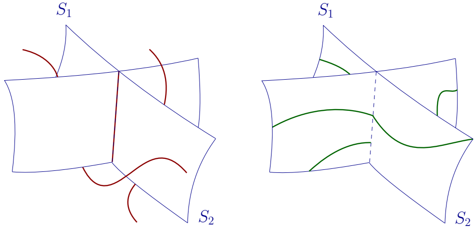

4. I will denote the moduli spaces with $\mathrm{ch}(F) = (\gamma, \delta)$ by $\mathrm{PT}^{(i)}_{\gamma,\delta}$ even if $\gamma = 0$.

Refinements and tautological stable pair correspondences

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$$\left\{\begin{array}{c}\text{Integrals}\\ \text{over DT moduli spaces}\end{array}\right\} = \left\{\begin{array}{c}\text{Integrals}\\ \text{over } \mathrm{PT}^{(0)} \text{ moduli spaces}\end{array}\right\}\cdot\left\{\begin{array}{c}\text{Integrals}\\ \text{over } \mathrm{Hilb}^n(X)\end{array}\right\}.$$

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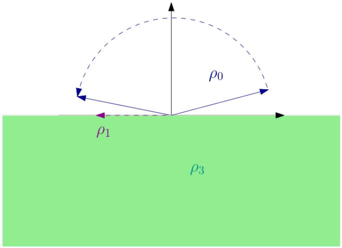
   follows by a similar calculation as in B. (21') relying on BKLT.
4. Would like to formalize it and extend it to $\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing.

$\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing

1. Without insertions get

$$\sum_{\delta}\left[\mathrm{PT}^{(0)}_{\gamma,\delta}\right]^{\mathrm{vir}} q^{\delta} = \exp\left\{\sum_{\Delta}\left[\langle \mathscr{M}_{\Delta}\rangle, -\right] q^{\Delta}\right\}\sum_{\delta_0}\left[\mathrm{PT}^{(1)}_{\gamma,\delta_0}\right]^{\mathrm{vir}} q^{\delta_0}\,.$$

from the variation of stabilities



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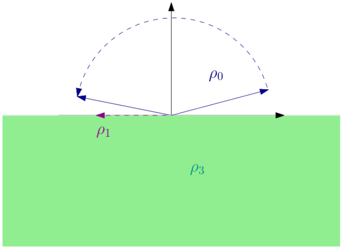

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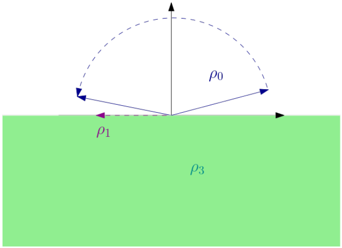

2. The operation $\cap c_{\mathrm{rk}}(L_{\gamma,\delta})$ no longer behaves well.
3. Instead, construct "vertex algebras" twisted $u^{\mathrm{rk}}c_{u^{-1}}(L_{\gamma,\delta})$

Additive deformations of vertex algebras

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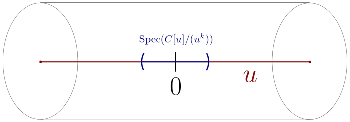

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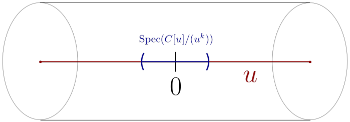
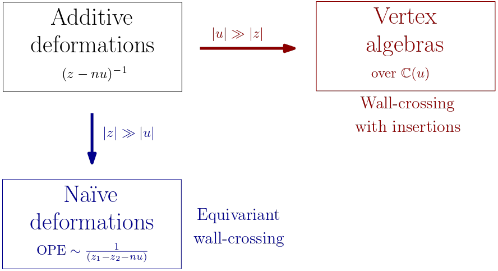


3. To get Lie algebras, need residues, so choose an expansion:



1. Taking residues (after expanding) defines a u-dependent Lie bracket

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3. Taking the appropriate coefficients of u on both sides, often get relations between invariants.

Applications: $\mathrm{PT}/\{\mathscr{O}_X\}$

Example

$\mathrm{PT}/\{\mathscr{O}_X\}$ wall-crossing: Set $\gamma = 0$ and do $\mathrm{PT}^{(0)}/\mathrm{PT}^{(1)}$ wall-crossing with $\pi_*(\delta) = 0$ for the geometry

$$\begin{array}{ccc} X & \longleftarrow & L := \pi^* L_B \\ \downarrow \pi & & \downarrow \\ B & \longleftarrow & L_B \end{array} .$$

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Conjecture (Cao-Toda (21'))

$$\sum_{\pi_*(\delta)=0} \langle \mathrm{PT}_\delta \rangle^L q^\delta = \prod_{\substack{\beta:\ \pi_*(\beta)=0 \\ k>0}} \left(1 - (-1)^k q^{(\beta,k)}\right)^{k \cdot n_{0,\beta}(L)} \prod_{\beta:\ \pi_*\beta=0} M(q^{(\beta,0)})^{n_{1,\beta}(X)}.$$

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For now have

Theorem (B.(upcoming))

$$\langle \mathrm{PT}_\delta \rangle^L = \mathrm{Pol}\Big(\langle \mathscr{M}_{\beta,m} \rangle^L, \langle \mathscr{M}_{\beta,m} \rangle^{\mathscr{O}_X} : \pi_*(\beta) = 0, m > 0\Big), \quad \langle M^{ss}_\delta \rangle^L = \int_{\langle \mathscr{M}_\delta \rangle} c_1\big(L^{[\delta]}\big).$$

Example

Consider the situation $\pi : X \to B$ for an elliptic fibration and $L = \mathscr{O}_X$.

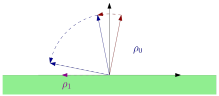
Conjecture (BKP (upcoming))

If $(\gamma,\delta) = \pi^*(\beta,n) + mp$ *for* $(\beta,n) \in H^{\geq 4}(B)$ *then*

$$\sum_d \langle \mathrm{PT}^{(0)}_{\gamma,\delta+dE}\rangle^{\mathscr{O}_X} q^d = \sum_{d_0} \langle \mathrm{PT}^{(1)}_{\gamma,\delta+d_0E}\rangle^{\mathscr{O}_X} q^{d_0} \sum_{\Delta\geq 0} \langle \mathrm{PT}_{\Delta E}\rangle q^{\Delta}$$

Claim (B. (upcoming))

The wall-crossing formula holds for the total path of



Will come back to prove the vanishing of the last segment of the arc.

Claim (BKP (upcoming))

The previous conjecture is equivalent to

$$\sum_{d} \langle \mathrm{DT}_{\beta,d} \rangle^{\mathcal{O}_B} q^d = \sum_{d_0} \langle \mathrm{PT}_{\beta,d_0} \rangle^{\mathcal{O}_B} q^{d_0} M(-q)^{\chi(B)}.$$

on the base if some extra geometric assumptions are satisfied.