Intrinsic mirror symmetry and theta functions via root stacks.

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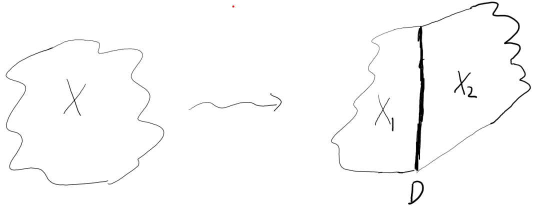
A big picture

$$
\begin{tikzcd}
\text{A-model} \arrow[r, shift left, "\text{Mirror Symmetry}"] & \text{B-model} \arrow[l, shift left]
\end{tikzcd}
$$

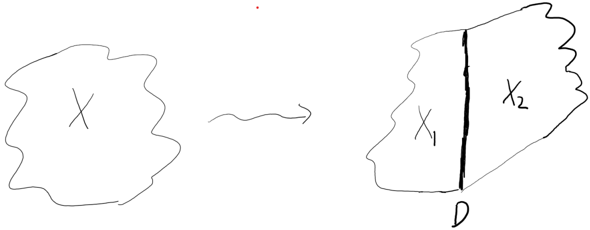
$$
\begin{tikzcd}
\text{Symplectic Geometry of } X \arrow[r, shift left, "\text{Mirror Symmetry}"] \arrow[d, leftrightarrow, "\text{Degeneration}"'] & \text{Complex geometry of } X^\vee \arrow[l, shift left] \arrow[d, leftrightarrow, "\text{Fibration}"] \\
\begin{array}{c}\text{Symplectic geometry of} \\ (X_1,D)(X_2,D)\end{array} \arrow[r, shift left, "\begin{array}{c}\text{Relative}\\\text{Mirror Symmetry}\end{array}"] & \begin{array}{c}\text{Complex geometry of} \\ (X_1,D)^\vee \text{ and } (X_2,D)^\vee\end{array} \arrow[l, shift left]
\end{tikzcd}
$$



Enumerative mirror symmetry

$$\text{A-model} \overset{\text{Mirror Symmetry}}{\rightleftarrows} \text{B-model}$$

$$\begin{tikzcd}
\text{Gromov–Witten theory} \arrow[r, shift left, "\text{Mirror Symmetry}"] \arrow[d, leftrightarrow, "\text{Degeneration}"'] & \text{Periods} \arrow[l, shift left] \arrow[d, leftrightarrow, "\text{Gluing, fibration}"] \\
\text{Relative Gromov–Witten theory} \arrow[r, shift left, "\text{Relative Mirror Symmetry}"] & \text{Relative Periods} \arrow[l, shift left]
\end{tikzcd}$$



What are the mirrors?

Recall,

Mirror symmetry for Calabi–Yau varieties

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Mirror symmetry for Fano toric varieties

Following Givental, the mirror of an n-dimensional Fano toric variety X is a Landau–Ginzburg model $((\mathbb{C}^\times)^n, f)$, where $f : (\mathbb{C}^\times)^n \to \mathbb{C}$ is called the super-potential.

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In general, it is expected the following

Mirror symmetry for Fano varieties

Mirror symmetry for Fano varieties

The mirror of a Fano variety X is a Landau–Ginzburg model $(X^\vee, W)$ consisting of a Kähler manifold $X^\vee$ satisfying $h^1(X^\vee) = 0$ and a proper map $W : X^\vee \to \mathbb{C}$, where W is called the superpotential.
Furthermore, the generic fiber of W should be mirror to the smooth anticanonical divisor of X.

Therefore, a proper LG model $(X^\vee, W)$ is actually mirror to a smooth log Calabi–Yau pair (X, D).
Givental's LG model is actually mirror to a toric variety with its toric boundary.

Mirror symmetry for smooth log Calabi–Yau pairs

For a smooth log Calabi–Yau pair (X, D)

$$\text{Log Calabi–Yau } (X, D) \quad \underset{}{\overset{\text{Mirror}}{\rightleftarrows}} \quad \text{LG model } (X^\vee, W)$$

$$\text{Noncompact Calabi–Yau } X \setminus D \quad \overset{\text{Mirror}}{\rightleftarrows} \quad X^\vee \text{ (without } W)$$

$$\text{Smooth anticanonical divisor } D \quad \overset{\text{Mirror}}{\rightleftarrows} \quad W^{-1}(t) \text{ for generic } t \in \mathbb{C}$$

Mirror symmetry for snc log Calabi–Yau pairs

For a smooth log Calabi–Yau pair (X, D), e.g. $D = D_1 + D_2$.

Log Calabi–Yau $(X, D_1 + D_2)$ $\overset{\text{Mirror}}{\rightleftarrows}$ LG model $(X^\vee, (W_1, W_2))$

Noncompact Calabi–Yau $X \setminus D$ $\overset{\text{Mirror}}{\rightleftarrows}$ $X^\vee$ (without W)

$$(D_1, D_1 \cap D_2) \quad \overset{\text{Mirror}}{\rightleftarrows} \quad W_1 : W_2^{-1}(t_2) \to \mathbb{C}$$

$$(D_2, D_1 \cap D_2) \quad \overset{\text{Mirror}}{\rightleftarrows} \quad W_2 : W_1^{-1}(t_1) \to \mathbb{C}$$

Smooth anticanonical divisor D $\overset{\text{Mirror}}{\rightleftarrows}$ $W_1^{-1}(t_1) \cap W_2^{-1}(t_2)$

$W_1 + W_2 : X^\vee \to \mathbb{C}$ is a non-proper LG model.

Examples

The mirror of $\mathbb{P}^1$ is $W = x + 1/x : \mathbb{C}^\times \to \mathbb{C}$, where $\mathbb{C}^\times$ is mirror to

$$\mathbb{P}^1 \setminus \{\infty, 0\} = \mathbb{C}^\times.$$

Similarly for other toric Fano varieties.

Mirror constructions

There are many mirror symmetry constructions, for example

- Greene-Plesser
- Givental/Hori–Vafa
- Batyrev–Borisov
- Strominger–Yau–Zaslow
- ...

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Question

Can the mirrors be constructed using Gromov–Witten invariants?

Absolute Gromov–Witten invariants

Moduli Space of Stable Maps

$\overline{M}_{g,n}(X,d)$: the moduli space of stable maps of degree d from genus g nodal curves with n-markings to a smooth projective variety X. It consists of

$$(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X,$$

where

- C is a projective, connected, nodal curve of genus g;
- $p_1, \dots, p_n$ are distinct non-singular points of C;
- $f_*[C] = d \in H_2(X)$;
- stable: automorphisms of the maps are finite

Definition

- For each marking p_i, there is an evaluation map:

$$\mathrm{ev}_i : \overline{M}_{g,n}(X,d) \to X$$
$$\{(C, \{p_i\}_{i=1}^n) \xrightarrow{f} X\} \mapsto f(p_i).$$

- For each marking p_i, there is a tautological line bundle L_i over $\overline{M}_{g,n}(X,d)$ whose fiber is the cotangent space of the curve at p_i. Let

$$\psi_i = c_1(L_i).$$

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Definition

Given cohomological classes $\gamma_i \in H^*(X)$, and $a_i \in \mathbb{Z}_{\geq 0}$, one can define the Gromov–Witten invariant

$$\left\langle \prod_{i=1}^n \tau_{a_i}(\gamma_i) \right\rangle_{g,n,d}^X := \int_{[\overline{M}_{g,n}(X,d)]^{\mathrm{vir}}} \prod_{i=1}^n (\mathrm{ev}_i^* \gamma_i) \psi_i^{a_i}.$$

Quantum Cohomology

The quantum cohomology ring $QH^*(X)$ is a deformation of the usual cohomology ring using Gromov–Witten invariants.

Quantum Product

Given $\alpha, \beta \in H^*(X)$, the quantum product is defined using three-point Gromov–Witten invariants.

$$\alpha \circ \beta = \sum_{d \in H_2^{\mathrm{eff}}(X)} \sum_k Q^d \langle \alpha, \beta, \phi_k \rangle_{0,3,d}^X \phi^k$$

where $\{\phi_k\}$ and $\{\phi^k\}$ are dual basis of $H^*(X)$.

Remark

We can write

$$\alpha \circ \beta = \alpha \cup \beta + H.O.T.$$

where $H.O.T. = \sum_{d>0} \cdots$

Quantum Cohomology

Big Quantum Product

Given $\tau \in H^*(X)$, one can define the big quantum product of $\alpha, \beta \in H^*(X)$

$$\alpha \circ_\tau \beta = \sum_{d \in H_2^{\mathrm{eff}}(X)} \sum_{n \geq 0} \sum_k \frac{Q^d}{n!} \langle \alpha, \beta, \tau, \cdots, \tau, \phi_k \rangle_{0,n+3,d}^X \phi^k$$

Enumerative mirror symmetry

- The A-model data is a generating function of genus zero Gromov–Witten invariants called the J-function $J_X(\tau, z)$:

$$J_X(\tau, z) = z + \tau + \sum_{\substack{(\beta,l)\neq(0,0),(0,1)\\ \beta\in \mathrm{NE(X)}}} \sum_{\alpha} \frac{q^\beta}{l!} \left\langle \frac{\phi_\alpha}{z-\psi}, \tau, \ldots, \tau \right\rangle^X_{0,1+l,\beta} \phi^\alpha,$$

  where $\tau = \tau_{0,2} + \tau' \in H^*(X)$; $\tau_{0,2} = \sum_{i=1}^r p_i \log q_i \in H^2(X)$; $\tau' \in H^*(X) \setminus H^2(X)$; NE(X) is the cone of effective curve classes in X; $\{\phi_\alpha\}$ is a basis of $H^*(X)$; $\{\phi^\alpha\}$ is the dual basis under the Poincaré pairing.
- The B-model data is period integrals called the I-function $I_X(y, z)$

Mirror theorem (Givental 1996, Lian–Liu–Yau 1997, ...)

$$J_X(\tau(y), z) = I_X(y, z)$$

where $\tau(y)$ is called the mirror map.

I-function

For quintic threefold the I-function is

$$I_X(y) = \sum_{d\geq 0} y^{H+d} \frac{\prod_{a=1}^{5d}(5H+a)}{\prod_{a=1}^{d}(H+a)^5}$$
$$= \sum_{d\geq 0} \frac{(5d)!}{(d!)^5} y^d H^0 + O(H).$$

Remark

Such a mirror theorem has been proved in many cases including toric stacks, Grassmannians, partial flag varieties etc.

Relative Gromov–Witten invariants

Relative Gromov–Witten theory is the enumerative theory of counting curves with tangency condition along a divisor (a codimensional one subvariety).

- X: a smooth projective variety.
- D: a smooth divisor of X.
- For $d \in H_2(X, \mathbb{Q})$, we consider a partition $\vec{k} = (k_1, \ldots, k_m)$ of $\int_d [D]$. That is,

$$\sum_{i=1}^{m} k_i = \int_d [D], \quad k_i > 0$$

- $\overline{M}_{g,\vec{k},n,d}(X, D)$: the moduli space of $(m+n)$-pointed, genus g, degree $d \in H_2(X, \mathbb{Q})$, relative stable maps to (X, D) such that the relative conditions are given by the partition $\vec{k}$.

Relative Gromov–Witten invariants

Evaluation Maps

There are two types of evaluation maps.

$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X, D) \to D, \quad \text{for } 1 \leq i \leq m;$$
$$\mathrm{ev}_i : \overline{M}_{g,\vec{k},n,d}(X, D) \to X, \quad \text{for } m+1 \leq i \leq m+n.$$

The first m markings are relative markings with contact order k_i, the last n markings are interior markings.

Data

- $\delta_i \in H^*(D, \mathbb{Q})$, for $1 \leq i \leq m$.
- $\gamma_{m+i} \in H^*(X, \mathbb{Q})$, for $1 \leq i \leq n$.

Definition

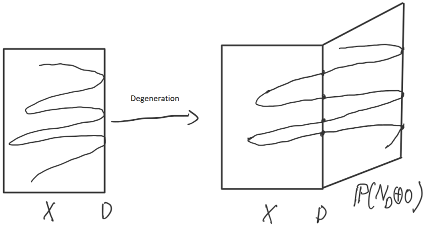
Definition

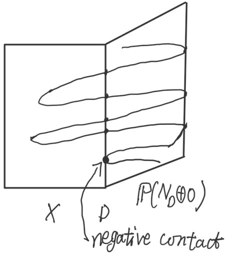
Relative Gromov–Witten invariants of (X, D) *are defined as*

$$\left\langle \prod_{i=1}^{m} \tau_{a_i}(\delta_i) \middle| \prod_{i=1}^{n} \tau_{a_i}(\gamma_{m+i}) \right\rangle_{g,\vec{k},n,d}^{(X,D)} := \tag{1}$$

$$\int_{[\overline{M}_{g,\vec{k},n,d}(X,D)]^{vir}} \prod_{i=1}^{m} \mathrm{ev}_i^*(\delta_i)\bar{\psi}_i^{a_i} \prod_{i=1}^{n} \mathrm{ev}_{m+i}^*(\gamma_{m+i})\bar{\psi}_i^{a_i}.$$

Relative invariants with negative contact orders (Fan–Wu–Y, 2020)





Structure of relative Gromov–Witten theory (Fan–Wu–Y)

Relative Gromov–Witten invariants with (possibly) negative contact orders have the following properties:

- Relative quantum cohomology ring
- Topological recursion relation
- WDVV equation
- Givental's formalism: Givental's symplectic vector space, Lagrangian cone etc.
- Virasoro constraint (in genus zero).

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Remark

For those who are familiar with Gross–Siebert program, similar invariants also appear in their program where they are called punctured Gromov–Witten invariants.

Relative mirror theorem

We have the following Givental-style mirror theorem.

Theorem (Fan–Tseng–Y, 2019)

Assume that D is nef, the I-function for the pair (X, D) is

$$I_{(X,D)}(t,z) = \sum_{d \in \overline{\mathrm{NE}}(X)} J_{X,d}(t,z) y^d \left(\prod_{0 < a \leq D \cdot d - 1} (D + az) \right) [\mathbf{1}]_{-D \cdot d}.$$

When $-K_X - D$ is nef, we have

$$J_{(X,D)}(\tau(t), z) = I_{(X,D)}(t,z),$$

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Question

What is the mirror of a pair?

Gromov–Witten theory of snc pairs

There are two types of invariants associated with a simple normal crossing (snc) pair

- (Punctured) logarithmic Gromov–Witten invariants: Abramovich–Chen–Gross–Siebert.
- Orbifold Gromov–Witten invariants of root stacks: Tseng–Y.

Remark

log and orbifold invariants are not equal in general.

Root stacks

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Multi-root stacks

For $D = D_1 + D_2 + \cdots + D_m$ and pairwise coprime $r_1, r_2, \ldots, r_m$. The multi-root stack is

$$X_{D,\vec{r}} = X_{D_1,r_1} \times_X X_{D_2,r_2} \times_X \ldots \times_X X_{D_m,r_m}.$$

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Tseng–Y

Genus zero Gromov–Witten invariants of $X_{D,\vec{r}}$ is independent of $\vec{r}$ when r_i are sufficiently large. So, we can consider the large $\vec{r}$-limit of the Gromov–Witten theories of $X_{D,\vec{r}}$ as a Gromov–Witten theory for the pair (X, D).

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Intrinsic mirror construction

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A general construction of the variety $X^\vee$ is through intrinsic mirror symmetry in the Gross–Siebert program. One considers a maximally unipotent monodromy (MUM) degeneration $\pi : \mathcal{X}_{\text{mum}} \to S$, where S is an affine curve, of X (or $X \setminus D$).

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The mirror

$$X^\vee := \operatorname{Proj}(QH^0(\mathcal{X}_{\text{mum}}, \mathcal{D}_{\text{mum}}))$$

the projective spectrum of the degree zero part of the relative quantum cohomology $QH^0(\mathcal{X}_{\text{mum}}, \mathcal{D}_{\text{mum}})$ of $(\mathcal{X}_{\text{mum}}, \mathcal{D}_{\text{mum}})$, where $\mathcal{D}_{\text{mum}}$ is a certain divisor that contains $\pi^{-1}(0)$.

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Example: Tyurin degeneration

Let X be a Calabi–Yau manifold, a Tyurin degeneration $\pi : \mathcal{X}_{\text{tyu}} \to S$ of X is a semi-stable degeneration and the central fiber has two irreducible components

$$\pi^0 = X_1 \cup_D X_2.$$

$$B := \operatorname{Proj}(QH^0(\mathcal{X}_{\text{tyu}}, \mathcal{D}_{\text{tyu}}))$$

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For example, the degeneration of the quintic threefold: $Q_5 \rightsquigarrow Q_4 \cup \mathrm{Bl}_C \mathbb{P}^3$.

The Doran–Harder–Thompson Conjecture

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If a degeneration $\pi : \mathcal{X} \to S$ can be further degenerated to a maximal degeneration $\pi_{\mathrm{mum}} : \mathcal{X}_{\mathrm{mum}} \to S'$, then

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Remark

To prove the DHT conjecture, we need to relate theta functions of different degenerations.

Theta function

What are theta functions?

Theta functions satisfy the following multiplication rule

$$\vartheta_{p_1} \star \vartheta_{p_2} = \sum_{r \in B(\mathbb{Z}),\beta} N^{\beta}_{p_1,p_2,-r} \vartheta_r, \quad p_1, p_2 \in B(\mathbb{Z}). \tag{2}$$

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Structure constants

The structure constants $N^{\beta}_{p_1,p_2,-r}$ are defined as 3-pointed invariants of (X, D) with contact orders given by $p_1, p_2, -r$ and a point constraint for the punctured point. Namely,

$$N^{\beta}_{p_1,p_2,-r} = \langle [1]_{p_1}, [1]_{p_2}, [\mathrm{pt}]_{-r} \rangle^{(X,D)}_{0,3,\beta}. \tag{3}$$

Gross–Siebert, 2022

When D has zero dimensional strata, theta functions are defined using punctured logarithmic Gromov–Witten invariants of broken line types.

Theta functions for smooth pairs

Theta function

Let $QH^0_{\log}(X,D)$ be the degree zero subalgebra of the relative quantum cohomology ring $\mathrm{QH}^*_{\log}(X,D)$ of a smooth log Calabi–Yau pair (X,D). The set

$$\{\vartheta_p\}, p \in \mathbb{Z}_{\geq 0}$$

of theta functions form a canonical basis of $\mathrm{QH}^0_{\log}(X,D)$.

The construction

The base of the Landau–Ginzburg mirror of (X,D) is $\operatorname{Spec} QH^0_{\log}(X,D) = \mathbb{A}^1$ and the superpotential is $W = \vartheta_1$, the unique primitive theta function of $QH^0_{\log}(X,D)$.

Structure constants

We have the following identity for structure constants.

Proposition (Y, 2022 & Yu Wang, 2022)

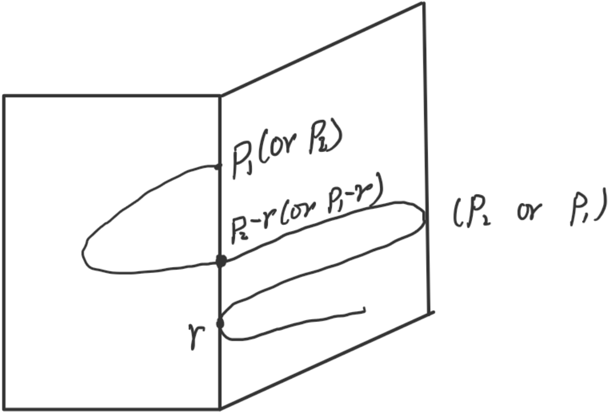
The structure constants $N^{\beta}_{p_1,p_2,-r}$ can be written as two-point relative invariants (without negative contact):

$$N^{\beta}_{p_1,p_2,-r} = (p_1 - r)\langle [\mathrm{pt}]_{p_1-r}, [1]_{p_2}\rangle^{(X,D)}_{0,2,\beta} + (p_2 - r)\langle [\mathrm{pt}]_{p_2-r}, [1]_{p_1}\rangle^{(X,D)}_{0,2,\beta}$$

Proof of the proposition

$$N^{\beta}_{p_1,p_2,-r} = (p_1 - r)\langle [\mathrm{pt}]_{p_1-r}, [1]_{p_2}\rangle^{(X,D)}_{0,2,\beta} + (p_2 - r)\langle [\mathrm{pt}]_{p_2-r}, [1]_{p_1}\rangle^{(X,D)}_{0,2,\beta}$$

Proof by picture:



Theta functions

We define the theta function in terms of two-point relative Gromov–Witten invariants.

Definition

For $p \geq 1$, the theta function is

$$\vartheta_p := x^{-p} + \sum_{n=1}^{\infty} n N_{n,p} t^{n+p} x^n, \tag{4}$$

where

$$N_{n,p} = \sum_{\beta} \langle [\mathrm{pt}]_n, [1]_p \rangle_{0,2,\beta}^{(X,D)}.$$

Theta functions

Proposition (Y, 2022)

The above definition of the theta functions satisfy the multiplication rule

$$\vartheta_{p_1} \star \vartheta_{p_2} = \sum_{r \geq 0, \beta} N^{\beta}_{p_1, p_2, -r} \vartheta_r. \tag{5}$$

Proof.

By the WDVV equation for relative Gromov–Witten theory and the above identity for structure constants. □

Theta functions for snc pairs

We are interested in general snc pairs. The simplest example is when $D = D_1 + D_2$ has two irreducible components.

Question

Can theta functions be defined using two-pointed orbifold invariants as in the smooth divisor case:

$$\langle [\mathrm{pt}]_{\vec{k}}, [1]_{\vec{p}} \rangle_{0,2,\beta}^{(X,D)}?$$

Theta functions for snc pairs

We are interested in general snc pairs. The simplest example is when $D = D_1 + D_2$ has two irreducible components.

Question

Can theta functions be defined using two-pointed orbifold invariants as in the smooth divisor case:

$$\langle [\mathrm{pt}]_{\vec{k}}, [1]_{\vec{p}} \rangle_{0,2,\beta}^{(X,D)}?$$

Answer

No. The virtual dimension is not correct in general. And it is not of the broken line type.

Broken line types

To define orbifold theta functions, we need to define orbifold invariants that are similar to log invariants of the broken line types. In particular, we need

- One marking p_1 with insertion $[1]_{\vec{p}}$, $p_i \geq 0$.
- One marking p_2 with contact order $\vec{k}$ and the irreducible component C_0 containing p_2 maps into the stratum $D_{\vec{k}} := \cap_{i:k_i \neq 0} D_i$.

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Idea in the orbifold setting

Replace the second marking, by two markings: one with sufficiently large positive contact order $\vec{\mathbf{b}}$ and the other with sufficiently large negative contact order $-\vec{\mathbf{b}} + \vec{k}$.

Smooth pairs revisit

One can prove the following identity

$$n\langle [1]_p, [\mathrm{pt}]_n, \rangle_{0,2,\beta}^{(X,D)} = \langle [1]_p, [1]_{\mathbf{b}}, [\mathrm{pt}]_{-\mathbf{b}+n}\rangle_{0,3,\beta}^{(X,D)}, \text{ for } \mathbf{b} \gg n.$$

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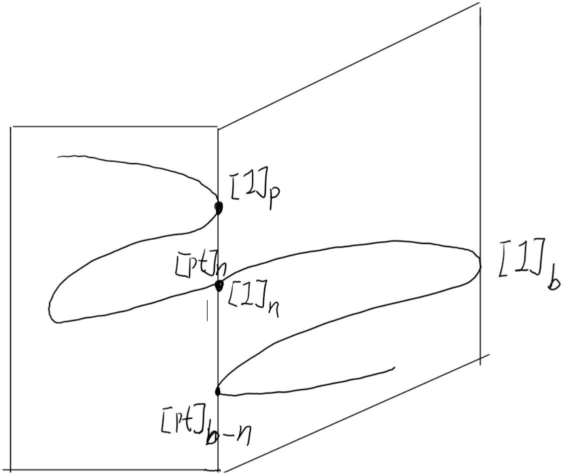

Smooth pairs revisit

Theorem

Y, 2021 Genus zero mid-age invariants does not depend on choice of $\mathbf{b}$ *when* $\mathbf{b}$ *is sufficiently large.*

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The theta function can be rewritten as

$$\vartheta_p := \sum_{k\in\mathbb{Z}} \sum_{\beta: D_i\cdot\beta=k+p} N^{\beta}_{p,\mathbf{b},-\mathbf{b}+k} t^{\beta} x^{k},$$

where

$$N^{\beta}_{p,\mathbf{b},-\mathbf{b}+k} = \langle [1]_p, [1]_{\mathbf{b}}, [\mathrm{pt}]_{-\mathbf{b}+n} \rangle^{(X,D)}_{0,3,\beta}.$$

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Theta functions for snc pairs

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Y, 2024 Genus zero mid-age invariants for multi-root stacks does not depend on choice of $\vec{\mathbf{b}}$ *when* $\vec{\mathbf{b}}$ *is sufficiently large.*

Theta functions for snc pairs

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Definition (Y, 2024)

For an snc log Calabi–Yau pair (X, D), the theta functions are defined as follows: Fix $\vec{p} \in B(\mathbb{Z}) \smallsetminus \{0\}$. Let $\sigma_{\max} \in \Sigma(X)$ be a maximal cone of $\Sigma(X)$ such that $p \in \sigma_{\max}$, then

$$\vartheta_{\vec{r}}(p) := \sum_{\vec{k} \in \mathbb{Z}^m} \sum_{\beta : D\cdot\beta = k_i + r_i} N^{\beta}_{\vec{r}, \vec{\mathbf{b}}, -\vec{\mathbf{b}}+\vec{k}} t^{\beta} x^{\vec{k}},$$

where the invariants are three-point genus zero orbifold invariants of (X, D) with contact orders (or ages) $\vec{r}, \vec{\mathbf{b}}$ and, $-\vec{\mathbf{b}} + \vec{k}$, where the last two markings are mid-age markings along D_i for $i \in I_\sigma$.

Application: DHT conjecture via intrinsic mirror symmetry

On going

Comparing theta functions for different degenerations.

More applications

The following results are proved for smooth pairs, and will be generalized to snc pairs in an upcoming work.

The work of Grafnitz–Ruddat–Zaslow, 2022

Conjecture & Title of the paper by [GRZ]:

"The proper Landau–Ginzburg potential is the open mirror map."

Theorem (GRZ)

For a toric del Pezzo surface X with a smooth anticanonical divisor D, the proper potential is

$$W = \textit{open mirror map of } K_X = \sum n_0^{\mathrm{open}}(K_X),$$

a generating function of genus zero open Gromov–Witten invariants of K_X.

The proper LG potential is the relative mirror map

After extracting certain coefficients of the J-function and the I-function, take derivatives, and match coefficients, we have

Theorem (Y, 2022)

Let X be a smooth projective variety with a smooth nef anticanonical divisor D. Let $W := \vartheta_1$ be the mirror proper Landau–Ginzburg potential. Set $q^\beta = t^{D\cdot\beta}x^{D\cdot\beta}$. Then

$$W = x^{-1}\exp\left(g(y(q))\right),$$

where

$$g(y) = \sum_{\substack{\beta\in \mathrm{NE}(X)\\ D\cdot\beta\geq 2}} \langle [\mathrm{pt}]\psi^{D\cdot\beta-2}\rangle^X_{0,1,\beta}\, y^\beta (D\cdot\beta-1)!$$

and $y = y(q)$ is the inverse of the relative mirror map

$$\sum_{i=1}^r p_i \log q_i = \sum_{i=1}^r p_i \log y_i + g(y)D. \tag{6}$$

The proper LG potential is the relative mirror map

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X is not necessary toric or Fano or of dimension 2.

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This is a natural expectation from the point of view of relative mirror symmetry. Recall that the proper Landau–Ginzburg model $(X^\vee, W)$ is mirror to the smooth log Calabi–Yau pair (X, D). The relative mirror theorem [Fan–Tseng–Y] relates relative Gromov–Witten invariants with relative periods (relative I-functions) via the relative mirror map.

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In order to have a mirror construction with a trivial mirror map, the proper Landau–Ginzburg potential W should encode the instanton corrections and the instanton corrections should be the inverse relative mirror map.

Open mirror map

relative and open mirror maps

The open invariants of $\mathcal{O}_X(-D)$ encoding the instanton corrections are expected to be the inverse mirror map of the local Gromov–Witten theory of $\mathcal{O}_X(-D)$. We observe that the relative mirror map and the local mirror map coincide up to a sign. Therefore, the works of Chan, Cho, Lau, Leung, Tseng on open Gromov–Witten invariants of toric Calabi–Yaus imply the following theorem.

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Theorem (Y, 2022)

Let (X, D) be a smooth log Calabi–Yau pair, such that X is toric and D is nef. The proper Landau–Ginzburg potential of (X, D) is the open mirror map of the local Calabi–Yau manifold $\mathcal{O}_X(-D)$.

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Corollary

The open-closed duality implies the proper Landau–Ginzburg potential is the open mirror map.

SYZ mirror symmetry

Remark

We consider these results provide a complete story from algebro-geometric point of view. It provides a connection between the Gross–Siebert mirror construction and the relative version of the enumerative mirror symmetry (Givental-style mirror symmetry) of Fan–Tseng–Y.
For the SYZ mirror symmetry, one may naturally expect that the proper potential is a generating function of genus zero open Gromov–Witten invariants of $X \setminus D$.

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Remark

Open Gromov–Witten invariants of $\mathcal{O}_X(-D)$ appear here maybe because there exists an identity between open Gromov–Witten invariants of $\mathcal{O}_X(-D)$ and the open Gromov–Witten invariants of $X \setminus D$, similar to the local-relative correspondence for closed Gromov–Witten invariants. However, we do not know if such an identity will be true in general.

Connection to the Fanosearch program

For Fano varieties, we observed that the function

$$g(y) = \sum_{\substack{\beta \in \mathrm{NE}(X) \\ D\cdot\beta \geq 2}} \langle [\mathrm{pt}]\psi^{D\cdot\beta-2}\rangle^X_{0,1,\beta} y^\beta (D\cdot\beta - 1)!$$

is closely related to the regularized quantum periods in the Fano search program.

Theorem (Y, 2022)

The function $g(y)$ coincides with the anti-derivative of the regularized quantum period.

Connection to the Fanosearch program

Remark

It is expected that regularized quantum periods of Fano varieties coincide with the classical periods of their mirror Laurent polynomials. Therefore, as long as one knows the mirror Laurent polynomials, one can compute the proper Landau–Ginzburg potentials.
For example, the proper Landau–Ginzburg potentials for all Fano threefolds can be explicitly computed.
More generally, there are large databases of quantum periods for Fano manifolds to compute the proper Landau–Ginzburg potentials.

Remark

The Laurent polynomials are considered as the mirror of Fano varieties with maximal boundaries (or as the potential for the weak, non-proper, Landau–Ginzburg models). Therefore, we have an explicit relation between the proper and non-proper Landau–Ginzburg potentials.

The End

Thank you!