A summary of what I wrote on the subject

(foundational and mirror symmetry related ones, symplectic topology mostly ignored)

2018 (PhD thesis, published in 2021)

- (M,ω) closed

x d

homotopy coherent presheaf of chain complexes over $\Lambda_{\geq 0} = \mathbb{K}[[T^{\mathbb{R}_{\geq 0}}]]$.

on compact subsets:

well-defined up to homotopy

$$K \subset M \mapsto SC^*_M(K) = \widehat{\mathrm{tel}}\,(CF(H_1) \to CF(H_2) \to \dots)$$

$H_i : S^1 \times M \to \mathbb{R}$ cofinal sequence for K.

- Main result: Assume that $M \xrightarrow{\pi} B$ is an involutive map: $f, g \in C^\infty(B) \Rightarrow \{\pi^* f, \pi^* g\} = 0$.

e.g. Lagrangian torus fibr.

Then, $P \subset B \mapsto SC^*_M(\pi^{-1}(P))$ forms a homotopy coherent sheaf with respect to finite covers by compact subsets. ("G-topology")

- (displaceability): If $K \subset M$ is displaceable from itself by Hamiltonian diff, then $SH^*_M(K)$ is torsion. We obtain an energy capacity inequality.

- Good example: $(S^2, \int\omega = 1)$, then for a disk $\mathbb{D} \subset S^2$:

$$SH_{S^2}(\mathbb{D}) \underset{\Lambda_{\geq 0}}{\otimes} \Lambda = \begin{cases} 0, & \text{area}(\mathbb{D}) < \frac{1}{2} \\ QH(S^2), & \geq \frac{1}{2} \end{cases}$$

This follows from the results above unless area $(\mathbb{D}) = \frac{1}{2}$.

2020 (w. D. Tonkonog)

- unital product structure on $SH^*_M(K)$.
- $L \subset M$ Lagrangian $\rightsquigarrow HF^*_M(K; L)$

2022 (w. M. Abouzaid & Y. Groman)

- canonical (big) models of $SC^*_M(K)$.
- full genus 0 structure ("framed E_2")

on $SC^*_M(K)$ using homotopy left Kan extension

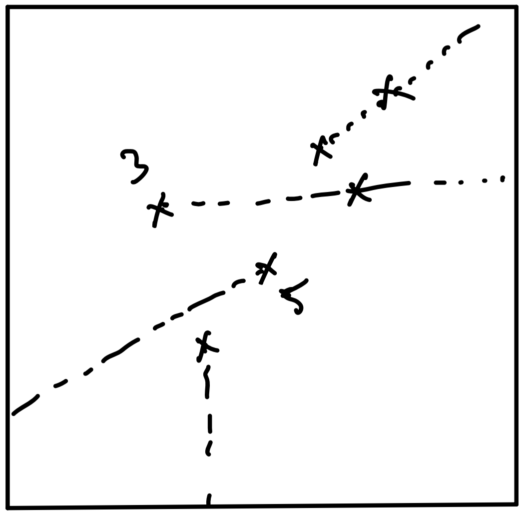
2022 (w. Y. Groman) "Locality for complete embeddings"

- extend the theory to the case where M is geometrically bounded
  - ↳ includes complete Liouville manifolds
  - ↳ if non-compact, infinite volume

a class of examples ↓

Eigenray diagram:

disjoint embedded rays with rational slope & some multiset of "nodes" on them



$\mathbb{R}^2$

$\mathcal{R}$ eigenray diagram $\rightsquigarrow$ $(M_{\mathcal{R}}, \omega) \to B_{\mathcal{R}}$

geometrically bdd but not exact nor convex & conical at ∞ (symplectic cluster manifolds)

Lagrangian torus fib with only focus-focus singularity (involutive)

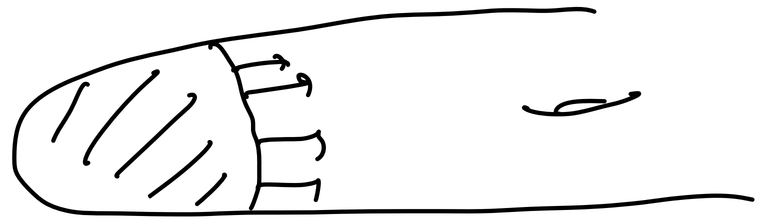
"nodal" integral affine structure with $\mathbb{R}^2 \setminus$ rays as a chart, rays $\to$ monodromy invariant

- (main result) let $U \subset M$ open be geometrically bounded (with restricted ω). For $K \subset U$, we have canonical locality isomorphisms (slightly weaker)

$$SH^*_U(K) \cong SH^*_M(K).$$

compatible with restriction maps and nested inclusions $K \subset U \subset V \subset M$.

Examples : •



Completion of Liouville subdomain

complete Liouville domain

• M_1 (complete Liouville manifold with nodal S^2 skeleton)

$\downarrow \pi_1$

B_1

Lagrangian tail

P_1

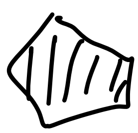
There are 2 symplectic embeddings

$$E_{1,2} : T^*T^2 \hookrightarrow M_1$$

with $\pi_1^{-1}(P_1)$ the image of $\pi_0^{-1}(P_0)$

T^*T^2

$\downarrow \pi_0$



P_0

$$\begin{array}{ccc} & SH^{*}_{M_1}(\pi_1^{-1}(P_1)) & \\ E_1 \ \wr\!\!\| & & \|\!\wr \ E_2 \\ SH^{*}_{M_0}(\pi_0^{-1}(P_0)) & \overset{(*)}{\cong} & SH^{*}_{M_0}(\pi_0^{-1}(P_0)) \end{array}$$

an incarnation of wall crossing iso's.

- in general removing "Lagrangian tails" give complete embeddings $M_{R'} \hookrightarrow M_R$ where R' is a sub-eigenray diagram of R.

<u>2023</u> (w. Y. Groman)

"closed string mirrors of M_R's"

- constructed rigid analytic space Y_R with non-archimedean SYZ fibration $Y_R \xrightarrow{p_R} B_R$ - s.t.

$$SH^{0}_{M_R}(\pi_R^{-1}(P); \Lambda) \cong \mathcal{O}_Y(p_R^{-1}(P))$$

for $P \subset B_R$ a rational slope polygon.

Rmk: Another way to state the result is to say that $SH^0_{M_R}(\pi_R^{-1}(-); \Lambda)$ is the sheaf constructed by Kontsevich-Soibelman for a particular choice of "lines".

This extends an observation of Seidel

$$SH^0_{T^*T^n}(\pi_0^{-1}(-1); \Lambda) \cong \mathcal{O}_{(\Lambda^*)^n}(\mathrm{trop}^{-1}(-)) \quad (**)$$

from "Speculation on pair of pants decomposition"

- (main computation) The map $(*)$ is really given by restricting the wall crossing transformation

$$x \longmapsto x(1+y) \quad \swarrow \text{ action rescaled generators.}$$

$$y \longmapsto y$$

Onwards: HMS and closed string for $\dim \geq 6$ (ideally for closed symplectic manifolds)

For HMS, we need to construct the mirror Y so that there $\exists$ a cohomologically full and faithful embedding

$$\mathrm{Fuk}^{bc}(M) \longrightarrow \mathrm{Coh}^{dg}(Y)$$

by construction. This requires doing the construction of $\mathcal{O}_Y$ using a "Lagrangian section" instead of SH^0. This was what Seidel really considered for $(**)$.

(M,ω) closed graded symplectic. jt w. Abouzaid-Groman

* L_1, L_2 tautologically unobstructed Lagrangian branes (for some almost complex structures)

We can define a presheaf over compact subsets

$$K \mapsto CF^*_M(L_1, L_2; K)$$

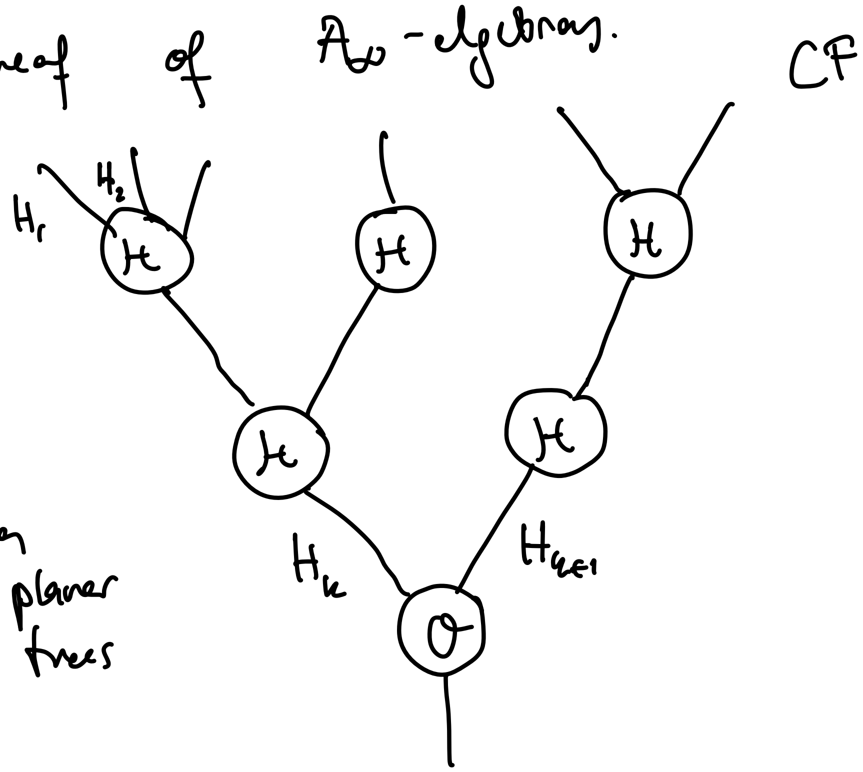
defined over $\Lambda = k((T^{\mathbb{R}}))$ to keep it simple

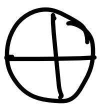
T-adic completion

$$\widehat{\operatorname{hocolim}}_{\substack{H|_K < 0 \\ \text{monotone}}} CF^*(L_1, L_2; H)$$

gen'd by 1-chords Floer solutions weighted by $T^{\text{top}E}$

* If $L_1 = L_2$, then we in fact obtain a presheaf of A_∞-algebras.





Hamiltonian labelled planar levelled trees

* We can also define Fukaya A_∞ categories with support.

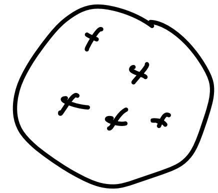
$$\mathrm{Fuk}_M(K)$$

extending the previous two structures.

We restrict to tautologically unobstructed version for simplicity.

We consider the following cases

$$\begin{tikzcd} M \arrow[d, "\text{Lag.}"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} (T^2,\omega) \arrow[d, "\pi"] \\ S^1 \end{tikzcd} \qquad \begin{tikzcd} K3,\omega \arrow[d, "\pi"] \\ S^2 \arrow[u, bend left, "\exists \text{ Lag section}"] \end{tikzcd}$$

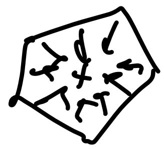


<u>Thm</u> (descent) $P \mapsto \mathrm{Fuk}_M(\pi^{-1}(P))$ is a homotopy sheaf of A_∞-cat's over compact G-topology of B.

Let $R \subset M$ be a Lagrangian section.

<u>Prop</u>: There exists a fibre preserving anti-symplectic involution fixing R pointwise.

P convex $\Rightarrow$ $CF^*_M(R;\pi^{-1}(P))$ supported in degree 0 and is essentially a commutative algebra A_P.

sufficiently small means convex, rational slope at most 1 X and  "shrink"

<u>Thm</u> (reference Lagrangian $\rightsquigarrow$ structure sheaf of a mirror analytic space)

suff small P $\rightsquigarrow$ $\mathcal{A}_P$ is affinoid algebra

$P \supset P'$ $\rightsquigarrow$ $\mathcal{M}(\mathcal{A}_{P'}) \subset \mathcal{M}(\mathcal{A}_P)$

sub-affinoid

+ cocycle condition $\blacksquare$

<u>Thm</u> (local generation) For sufficiently small convex polytope $P \subset B$, R split-generates $\mathrm{Fuk}_M(\pi^{-1}(P))$.

The last two are proved using deformation arguments. They hold locally. After shrinking, the actions $\to 0$ and by a monotonicity argument the non-local solution are higher order.

We obtain rigid analytic space $(Y, \mathcal{O}_Y)$ by gluing.

<u>HMS functor</u> $B = \bigcup_{i=1}^{N} P_i$ sufficiently fine

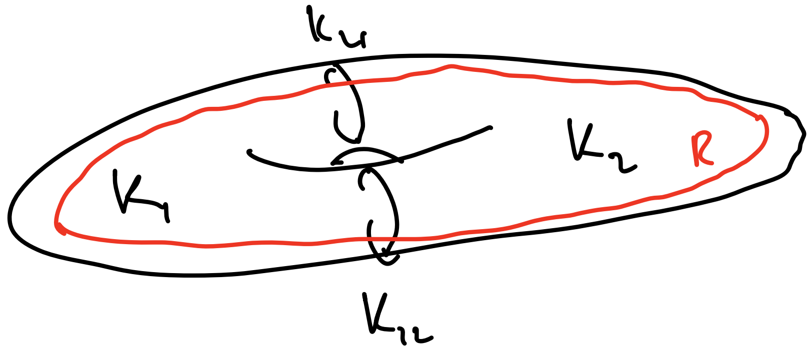
$$\mathrm{Fuk}(M) \xrightarrow{\text{descent}} \operatorname*{holim}_{\emptyset \neq I \subset [N]} {}_{A_\infty\text{-cat}} \mathrm{Fuk}_\mu(P_I)$$

$$\xrightarrow{\text{local gen}} \operatorname{holim} \mathrm{Perf}\left(CF^*_M(R; P_I)\right)$$

$$\xrightarrow{\text{str sheaf}} \mathrm{Coh}^{dg}\left(\mathrm{Glue}(M(A_i)) = Y\right),$$

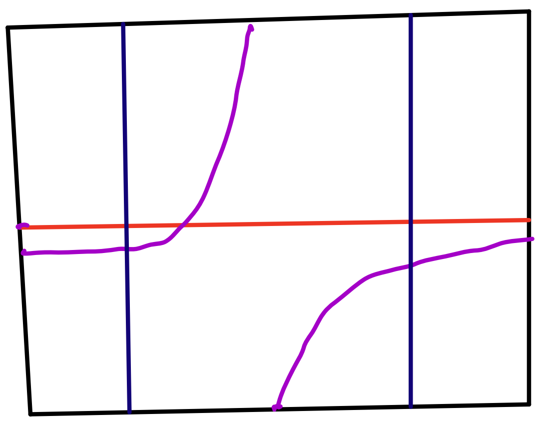
where all arrows are cohomologically full and faithful embeddings.

To get essential surjectivity, one needs an extra argument (not part of this theory) let's do an example on T^2



Y is constructed by gluing two non-archimedean annuli $M\left(\frac{\Lambda\langle x,y\rangle}{xy - T^{Area}}\right)$

$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d] \arrow[dr] & \mathcal{A}_2 \arrow[dl] \arrow[d] \\
\mathcal{A}_{12} & \mathcal{A}_{21}
\end{tikzcd}$$



$$\begin{tikzcd}
\mathcal{A}_1 \arrow[d, "\cdot x"'] \arrow[dr] & \mathcal{A}_2 \arrow[dl] \arrow[d] \\
\mathcal{A}_{12} & \mathcal{A}_{21}
\end{tikzcd}$$