Floer theoretic invariants with support

Goal: Analyze global quantum invariants of symplectic manifolds using local-to-global principles.

Inspiration: SYZ mirror symmetry

Tool: Hamiltonian Floer theory.

Let's start with a toy case that involves Morse theory.

M closed smooth manifold

We can do Morse theory on M.

$f: M \to \mathbb{R}$ Morse function

$X \in \Gamma(TM)$ gradientlike vector field

If (f, X) satisfies the Morse-Smale condition.

$\rightsquigarrow$ $CM^*(f, X)$ Morse complex

This is filtered chain complex with valuation (filtration level) of a critical point p equal to $f(p)$.

Given a regular homotopy (f_s, X_s) between two choices (f_0, X_0) & (f_1, X_1), we get the continuation chain map.

$$CM^*(f_0, X_0) \to CM^*(f_1, X_1)$$

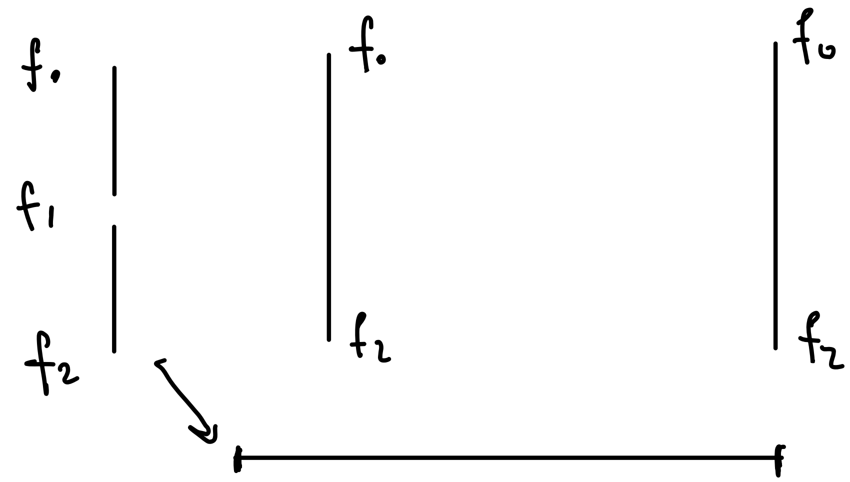
<u>Important</u>: If $\frac{\partial f_s}{\partial s} \geq 0$, then this map is filtered, i.e. valuation non-decreasing. We refer to this as the "monotonicity" of (f_s, X_s).

Assume that we have a family of continuation map data parametrized by a compact manifold with corners P. Crucially, we allow broken data.

Under a regularity assumption,

- rigid counts (over P) define a map of degree $-\dim P$
- one dimensional families define a null homotopy for the sum of the maps defined by the codimension 1 faces of P.

Example



This allows us to construct a dg-functor $\mathrm{Morse}_M \xrightarrow{CM^*} Ch^{dg}_{\mathbb{Z}}$

- obj: Morse-Smale pairs
- morphisms: cubical chains of monotone continuation data

Let $K \subset M$, we can consider the full subcategory $\mathrm{Morse}_M(K)$ consisting of objects (f, X) with $f|_K < 0$. We consider the bar complex model of the homotopy colimit of CM^* over this category (really, weighted colimit over canonical augmentation) $\bigoplus_{f_0<\dots<f_n} \mathrm{Cont}^*(f_0, f_1) \otimes CM^*(f_0)\ [n]$ val 0

This has an induced filtration and we define $CM^*_M(K)$ to be the degreewise completion. These form a presheaf of chain complexes

When K is closed, we can characterize cofinal sequences in $\mathrm{Morse}_M(K)$:

$$f_i(x) \longrightarrow \begin{cases} 0, & x \in K \\ \infty, & x \notin K \end{cases} \quad \text{as } i \to \infty.$$

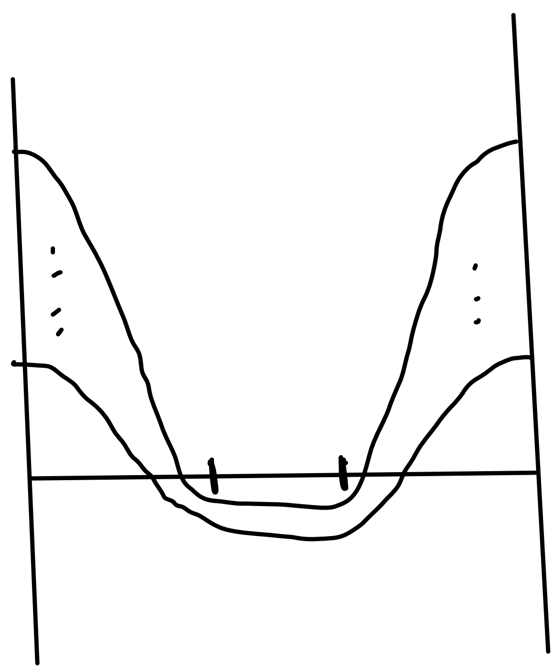
This gives a smaller telescope model that we can use for computations. For example, we can prove:

$$\mathrm{tel} \xrightarrow[\text{q-iso.}]{} CM^*_M(K)$$

- If $K \subset M$ is a codimension 0 compact submanifold with boundary (domain) $H^*(CM^*_M(K)) \cong H^*(K)$.

<u>Idea in an example</u>:

$M = S^1$
$K =$ connected closed interval



$$\begin{tikzcd}
CM(f_1) & & CM(f_2) & & CM(f_3) & \\
CM(f_1)[1] \arrow[u, "id"] \arrow[urr, "K_1"] & & CM(f_2)[1] \arrow[u, "id"] \arrow[urr, "K_2"] & & CM(f_3)[1] \arrow[u] \arrow[ur] &
\end{tikzcd}$$

$$K_i : \begin{array}{l} min_i \longmapsto min_{i+1} \\ max_i \longmapsto max_{i+1} \end{array}$$

$val(min_i) \to 0$ vs. $val(max_i) \to \infty$

as $i \to \infty$.

$\Rightarrow$ can built a primitive of max_i,

min_i & min_{i+1} are cohomologous.

We get $\mathbb{Z}[1]$ as desired.

<u>Thm</u> (generalized Mayer-Vietoris property)

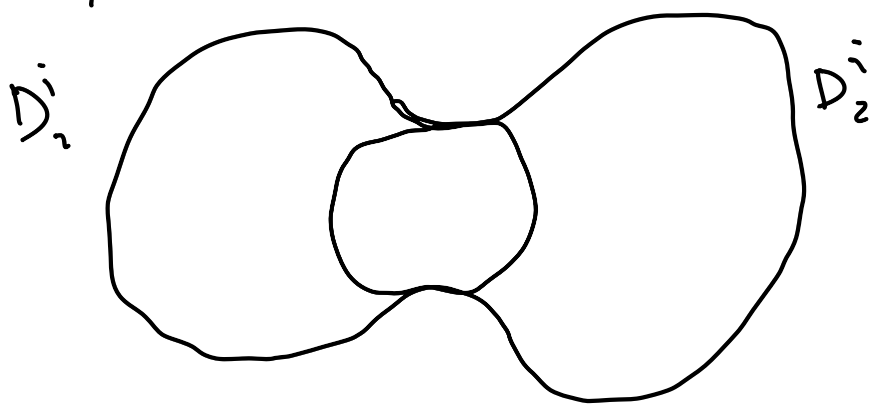
Let $K_1, .., K_n \subset M$ compact. Then

$$CM^*(K_1 \cup .. \cup K_n) \longrightarrow Cech(CM; K_1, .., K_n)$$

is a quasi-isomorphism

Pf: We reduce to $n=2$ using a simple induction.

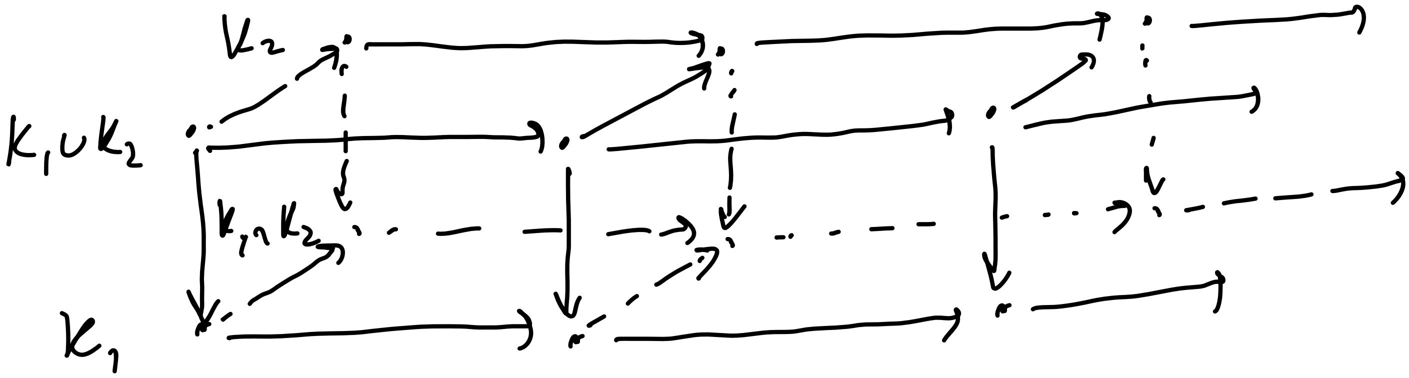
For $n=2$, one way to proceed is to simultaneously approximate K_1 & K_2 by domains whose union and intersection are also domains that approximate $K_1 \cup K_2$ and $K_1 \cap K_2$



We consider collar functions for D_1^i & D_2^i and extend to all M by constants. The max and min are collar functions for $\cap$ and $\cup$.

We make functions Morse very carefully

Thus, we can cook up a diagram



whose slices look like a direct sum of $\mathbb{Z} \to \mathbb{Z}$ along one edge or

$$\begin{tikzcd} \mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Z} \arrow[d] \\ \mathbb{Z} \arrow[r] & \mathbb{Z} \end{tikzcd}$$

plus valuation increasing terms. $\blacksquare$

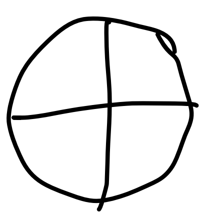
We can also construct even larger models of $CM^*(K)$ which admit A_∞ structures by working over the metrized planar trees operad. Descent works so as to take into account these structures

$\{\mathcal{O}(n)\}_{n=2,3,\dots}$ operad of metrized stable ribbon trees

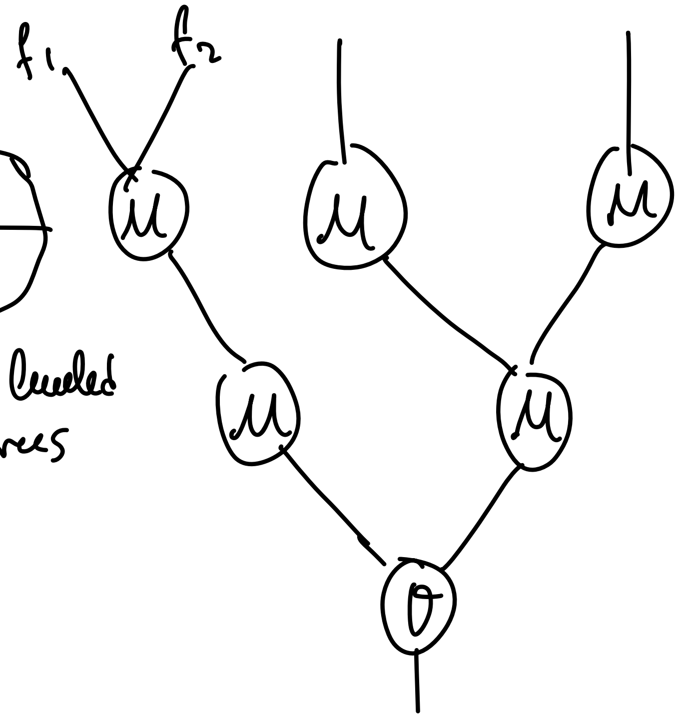
$\mathrm{Morse}^{A_\infty}_M$ - multicategory.

$\mathrm{Hom}(f_1,\dots,f_n\,;\,g)$ cubical chains of ribbon trees with Morse pairs at every pt.

$$\begin{tikzcd} & \mathrm{Morse}^{A_\infty}_M \arrow[r, "\text{Fukaya}"] \arrow[d] & \mathrm{Ch}^{dg} \\ A_\infty \arrow[r, hook, "\cong"] & C_*(\mathcal{O}) \arrow[ur, dashed] & \end{tikzcd}$$



labelled leveled ribbon trees



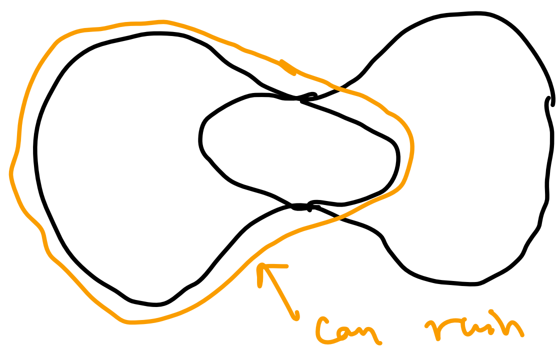
We can go thru the same procedure w/ Hamiltonian Floer complexes instead of Morse complexes.

We can do open string or closed string. Two main differences

1) Locality is a much bigger issue.

Assume that K is contained as a domain inside M and M'. The invariants of M & M' with support on K may not be the same. (Ex: $K = \mathbb{D}$, area 1 $\quad S^2$, area 2 / area 3)

2) Descent only holds under a Poisson commutativity assumption



We now turn to SYZ setup

$$\begin{array}{c} M \\ \downarrow \\ B \end{array} \quad \text{Special Lag fibration}$$

$\rightsquigarrow$ descent in the base

$\rightsquigarrow$ locality with abstract isomorphism for sufficiently small neighborhoods of fibers.