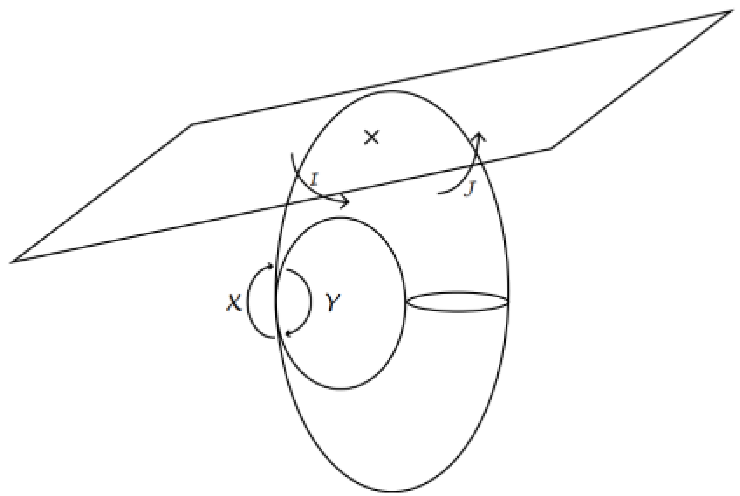
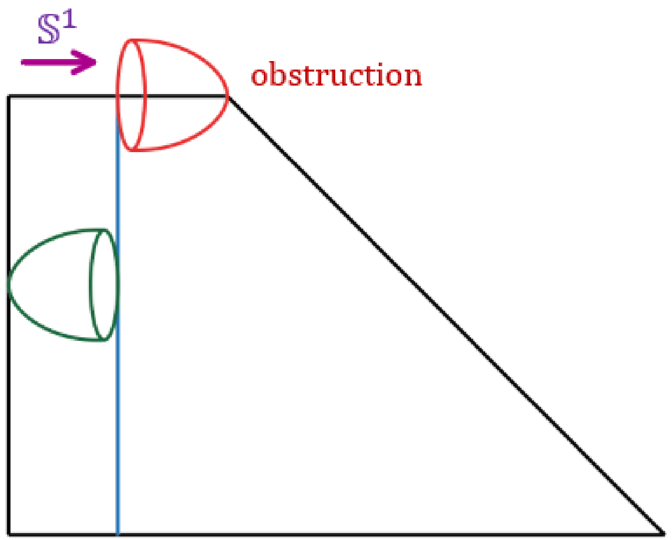
Teleman's conjecture and equivariant Lagrangian Floer theory

[L., Naichung Leung and Yan-Lung Li] Teleman's conjecture and equivariant correspondence (ArXiv:2312.13926)
[Jiawei Hu, L. and Ju Tan] Mirror Construction for Nakajima Quiver Varieties (ArXiv:2404.16172)
[Yoosik Kim, L. and Xiao Zheng] T-equivariant disc potential and SYZ mirror construction (Adv. Math.)





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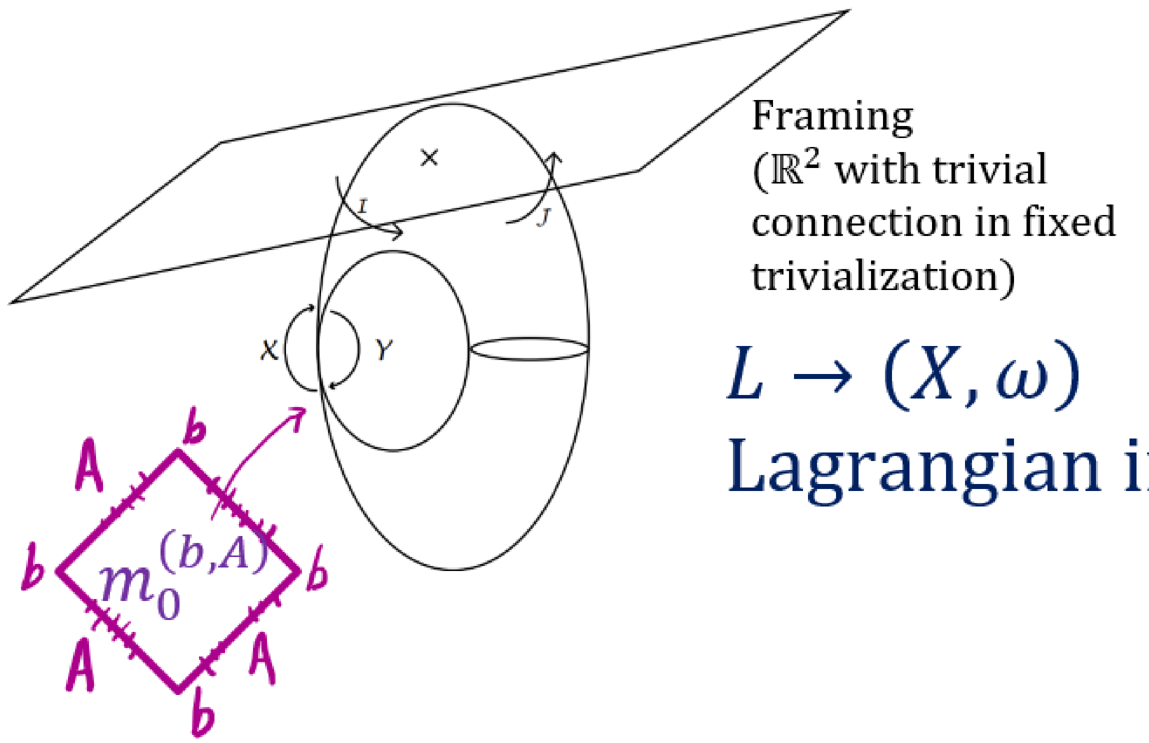
Floer curvature is moment map for nodal surface

Atiyah-Bott: For a compact smooth surface L, curvature function $F(A) = dA + A \wedge A$ on

$$\left(\mathcal{A} = \{(\text{unitary}) \text{ connections } A\}, \int_L \mathrm{Tr}(- \wedge -)\right)$$

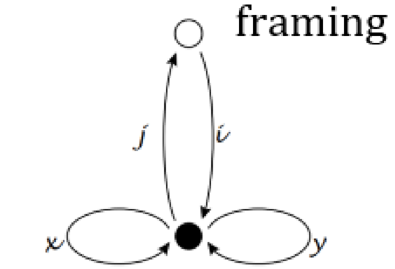
is the moment map of the gauge group $\mathcal{G}$.



$L \to (X, \omega)$

Lagrangian immersion

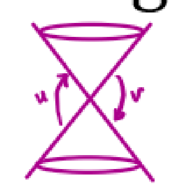
Strominger-Yau-Zaslow: T-duality
Fukaya-Oh-Ohta-Ono, Akaho-Joyce: Lagrangian deformation theory
Fukaya, Tu, Abouzaid, Yuan: family Floer theory
Cho-Hong-L.: Localized mirror functor



$$\mathcal{M}_L^{loc} = \left\{(b, A) : m_0^{(b,A)} = W(b, A) \cdot \mathbf{1}_L\right\} / \mathcal{G}_{\mathbb{C}}$$

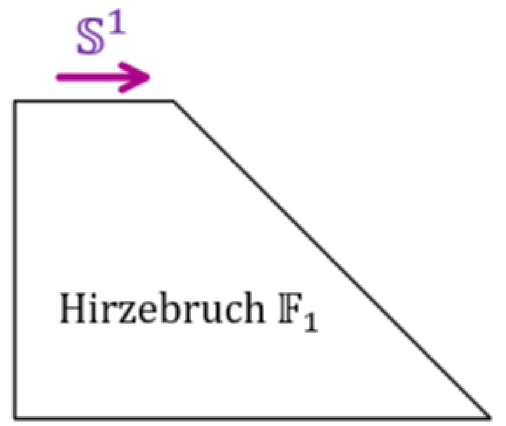
$$\subset \left(\bigoplus_{(s,t):\ \text{degree 1 immersed sector}} \mathrm{Hom}(\mathcal{E}_s, \mathcal{E}_t) \times \Omega^1(L, \mathfrak{gl}) \right) / \mathcal{G}_{\mathbb{C}}.$$

- $\mathcal{M}_L^{loc}$ is generally a stack and encoded by a quiver algebra with relations.
- Holomorphic symplectic form $\omega_{\mathbb{C}} = \int_L \mathrm{Tr}\left(m_{2,\text{const}}^{(b,A)}(-,-)\right)$ by counting constant polygons.

  - Skew-symmetric: involutive symmetry at each immersed sector. 

  $m_0^{(b,A)}$ is the complex moment map with respect to $\mathcal{G}_{\mathbb{C}}$. Thus $\mathcal{M}_L^{loc}$ is holomorphic symplectic.
- $\mathcal{M}_{L^{fr}}^{loc}$ is a Nakajima quiver variety if L^{fr} is a framed nodal union of two-spheres (in a suitable stability condition for the quotient, with a coordinate change).
- The mirror functor $F^L(L^{fr})$ produces Yang-Mills instantons over ALE spaces: $0 \to (E, h_E) \to (F, h_F) \to (G, h_G) \to 0$ monadic complex.

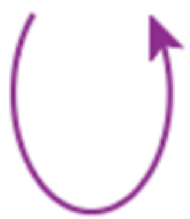
Teleman's conjecture: from 2D to 3D



$$W = z + w + \frac{1}{w} + \frac{1}{zw}$$

Symplectic (X, ω) ⟷ (Mirror symmetry) ⟷ **Complex** (Y, W)

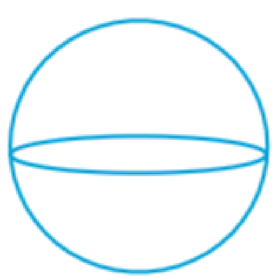
X_G → T^*BG



Hamiltonian G

Teleman's conjecture ⟷

F: (Y, W) → $\check{G}_{\mathbb{C}}/\mathrm{Ad}$



$X /\!/_* G$ ⟷ (mirror) ⟷ $F^{-1}\{c\}$ $\quad \overline{W} = c + w + \left(1 + \frac{1}{c}\right)\frac{1}{w} \sim \boldsymbol{w + \frac{1}{w}}$

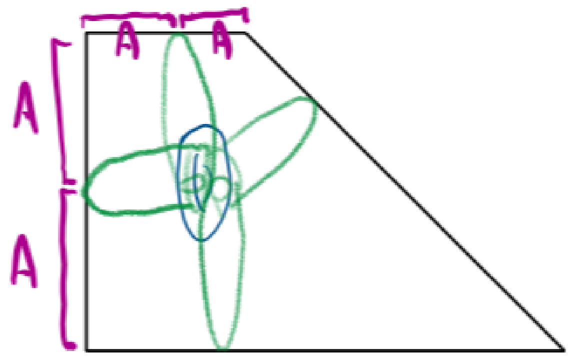
holomorphic Lagrangian brane in $[pt///G]$

3D mirror symmetry for pure gauge theory ⟷

holomorphic Lagrangian brane in BFM space

[**Chan-Leung**]: 3D mirror symmetry is 2D mirror symmetry.
[**Gonzalez-Iritani**], [**Gonzalez-Mak-Pomerleano**], [**Pomerleano-Teleman**]:
Closed-string construction via Seidel representation on equivariant quantum cohomology.

Equivariant localized SYZ mirror and correspondence



$L \subset (X, \omega)$

Strominger-Yau-Zaslow: T-duality
Fukaya-Oh-Ohta-Ono:
Lagrangian deformation theory
Cho-Hong-L.:
Localized mirror functor

$$W_L = T^A\left(z + w + \frac{1}{w} + \frac{T^A}{zw}\right)$$

$$\mathcal{M}_L^{loc} = \{b : m_0^b = W(b) \cdot \mathbf{1}_L\} \subset H^1(L, \mathbb{C})$$

$L_G \subset X_G$

$BG \subset T^*BG$

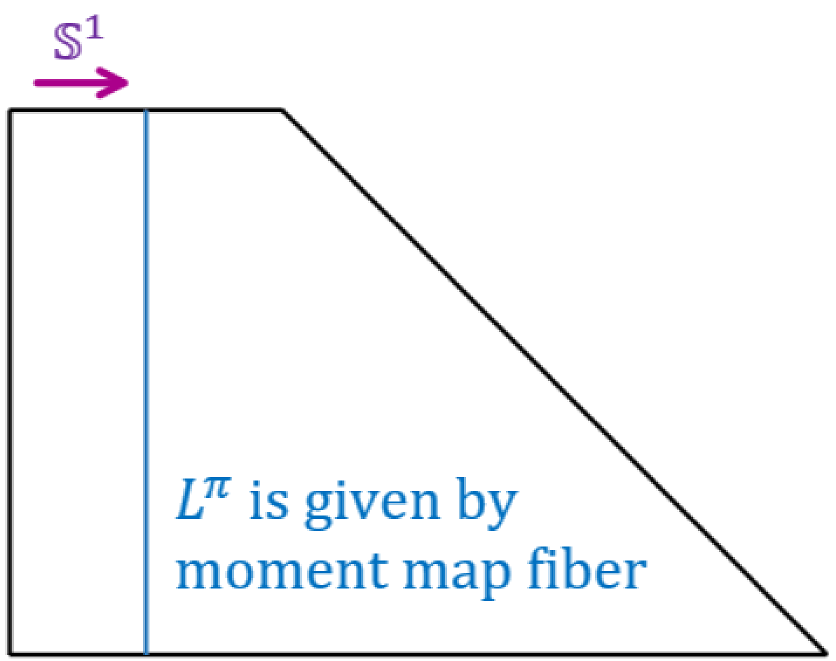
Hamiltonian G

Kim-L.-Zheng:
Floer theory for
Borel space

h_{L_G} $\boldsymbol{\lambda}$

$H^2(BG)$

$m_0^b(L_G) = W_L(b)\mathbf{1}_{L_G} + h_{L_G}(b)\boldsymbol{\lambda} \in H^2(L_G)$.
When L is a torus with Maslov index ≥ 2
and $G = \mathbb{S}^1$, $h_{L_G} = \log z$.



$L_G \subset X_G$

Localized mirror functor

$$\mathcal{M}_{L_G}^{loc} = h_{L_G}^{-1}\{0\}$$

Weirheim-Woodward
Fukaya: correspondence trimodule
L.-Leung-Li:
Equivariant correspondence
$L_G^\pi \subset X_G \times \bar{X}$

$$W_{L_G} = T^A\left(\mathbf{1} + w + \frac{1 + T^A}{w}\right)$$

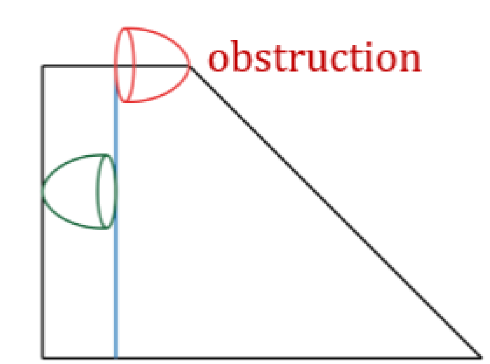
But: $\not\parallel$

$$\boldsymbol{W_{\bar{L} \subset \bar{X}} = T^A\left(w + \frac{1}{w}\right)}$$

$\bar{L} \subset \bar{X}$

Localized mirror functor

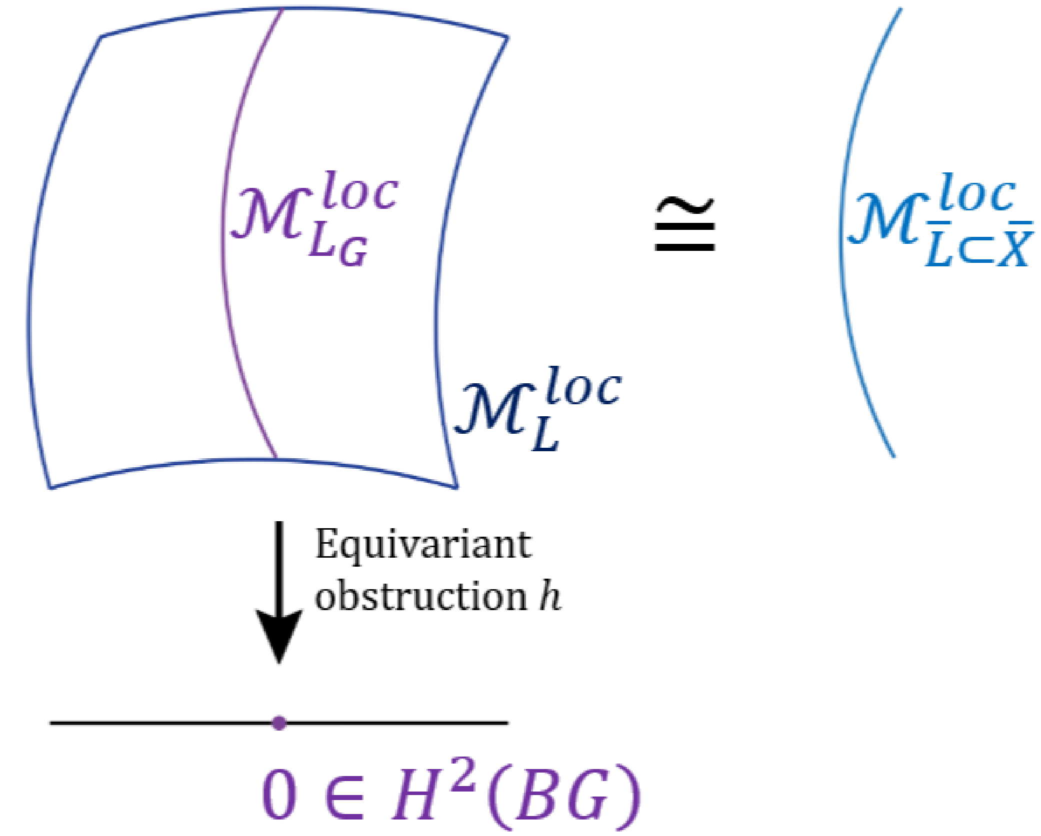
$\mathcal{M}_{\bar{L} \subset \bar{X}}^{loc}$



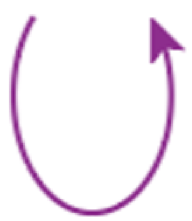
Equivariant Lagrangian correspondence L_G^π has non-trivial obstruction $m_0^{L_G^\pi}$.
Remove the obstruction by taking bulk deformation on $\bar{X}$ (or X_G).

Theorem [L.-Leung-Li]: ($L_G \subset X_G$ versus $L/G \subset X$)
Suppose $L_G^\pi \subset X_G \times \bar{X}$ is weakly unobstructed.
For G-invariant Lagrangian $L \subset X$, if G acts freely on L, we have:

1. $\mathcal{M}^{loc}_{\bar{L}\subset\bar{X}} \cong \mathcal{M}^{loc}_{L_G \subset X_G} = h^{-1}\{0\} \subset \mathcal{M}^{loc}_{L\subset X}$ where
   $h(b) = h_{L_G}(b) + h_{L_G^\pi}: \mathcal{M}^{loc}_{L\subset X} \to H^2(BG)$.
2. $W_L(b) + W_{L^\pi} = W_{\bar{L}}(\bar{b})$ under the above isomorphism.
3. $HF(\bar{L}, \bar{b}) \cong HF_G(L, b)$ under the above isomorphism.



Ex. $L \subset (X, \omega)$ $\qquad$ $\mathcal{M}^{loc}_{L\subset X} = \{b : m_0^b = W(b, A) \cdot \mathbf{1}_L\} \subset H^1(L, \mathbb{C})$



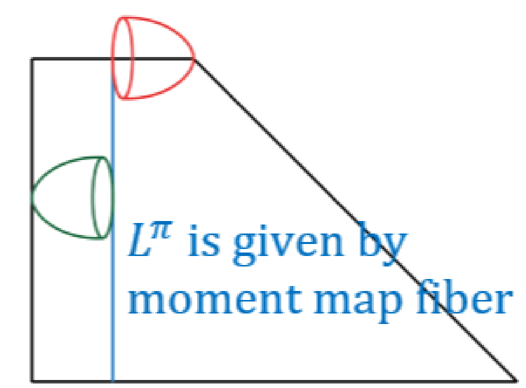
Hamiltonian $G = \mathbb{S}^1$

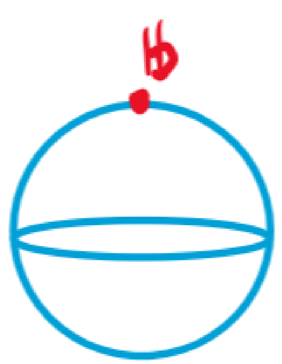
$\downarrow h = \log z$

$H^2(B\mathbb{S}^1) \cong \mathbb{C}$

$$W_L = T^A\left(z + w + \frac{1}{w} + \frac{T^A}{zw}\right).$$

$$W_{L^\pi} = -T^A.$$





$\bar{L} \subset \bar{X}^{\mathbb{b}}$

$$\boldsymbol{W_{\bar{L}\subset\bar{X}^{\mathbb{b}}} = T^A\left(w + \frac{1 + T^A}{w}\right)}.$$

Equivariant Lagrangian theory for connected Lie group:
Woodward-Xu: gauged Floer theory; **Daemi-Fukaya and Xiao**: equivariant transversality.

Borel construction (with Floer theory pioneered by Seidel-Smith for $\mathbb{Z}_2$) has advantages of:

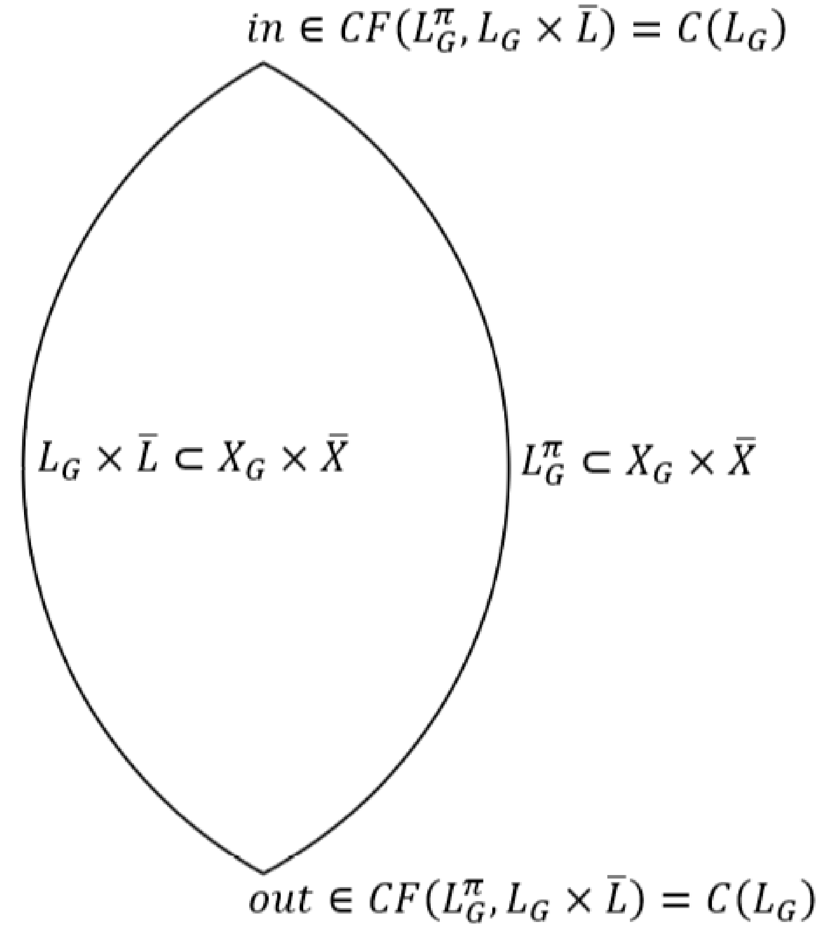
1. Desingularize L/G;
2. Gain module structure over $H^*(BG)$.

Theorem [L.-Leung-Li]:
Suppose $L_G^{\pi} \subset X_G \times \bar{X}$ is weakly unobstructed.
For G-invariant Lagrangian $L \subset X$ where G acts freely on L, we have:

1. $\mathcal{M}^{loc}_{\bar{L}\subset\bar{X}} \cong \mathcal{M}^{loc}_{L_G\subset X_G} = h^{-1}\{0\} \subset \mathcal{M}^{loc}_{L\subset X}$ where $h(b) = h_L(b) + h_{L^{\pi}}: \mathcal{M}^{loc}_{L\subset X} \to H^2(BG)$.
2. $W_L(b) + W_{L^{\pi}} = W_{\bar{L}}(\bar{b})$ under the above isomorphism.
3. $HF(\bar{L}, \bar{b}) \cong HF_G(L, b)$ under the above isomorphism.

Key ingredient:
Equivariant Lagrangian correspondence L_G^{π};
consider $HF(L_G^{\pi}, L_G \times \bar{L})$:

1. Solve $(b, \bar{b})$ for $m_1^{b+\bar{b}}(1_{L_G}) = 0$.
2. $m_1^2 = \left(W_{L^{\pi}} - (W_{\bar{L}} - W_L)\right)\mathrm{Id} + h\,\lambda$. Apply to 1_{L_G}.
3. $m_2\left(1_{L_G}, \pi^*_{L_G\times\bar{L}\to\bar{L}}(-)\right)$ and $m_2\left(1_{L_G}, \pi^*_{L_G\times\bar{L}\to L_G}(-)\right)$ provide chain maps that give $HF(\bar{L}) \underset{\cong}{\longrightarrow} HF(L_G^{\pi}, L_G \times \bar{L}) \underset{\cong}{\longleftarrow} HF(L_G)$.



Remark:
More generally for composition of correspondences, Fukaya defined A_∞-trimodule. In our case, the trimodule $\left(CF(L_G, L_G^{\pi}, \bar{L}), n_{p,q,r}\right)$ is captured by $\left(CF(L_G \times \bar{L}, L_G^{\pi}), m_{(p+r)+1+q}\right)$.

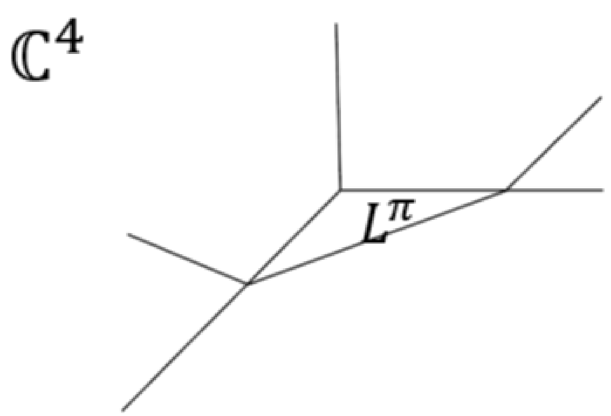
Equivariant obstruction and inverse mirror map for semi-Fano toric quotient

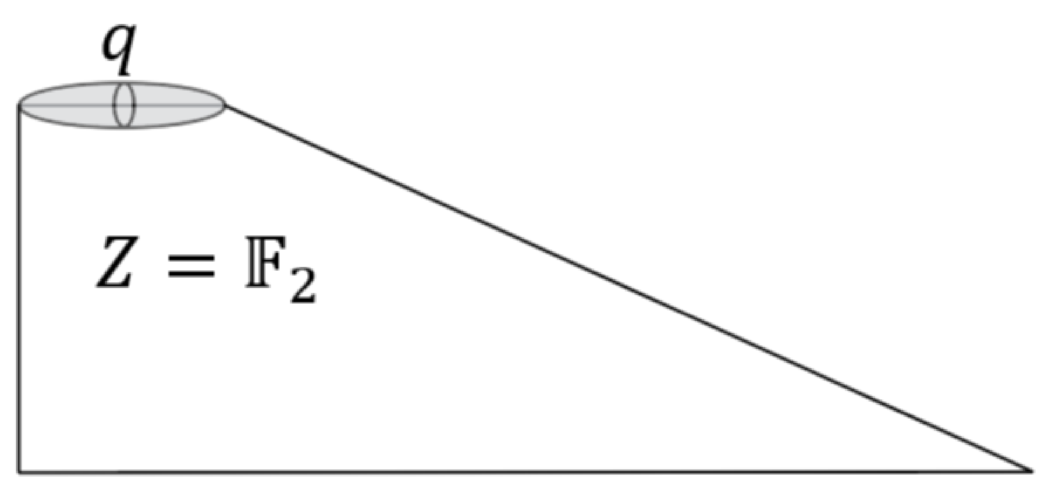
Theorem [L.-Leung-Li]:
Let $Z = \mathbb{C}^n//_* T^k$ be a compact toric semi-Fano manifold, where the moment map level set transversely intersects with all toric divisors .

Then the equivariant obstruction of the Lagrangian correspondence $L^\pi: Fuk(Z) \longrightarrow Fuk(\mathbb{C}^n)$ equals
$\boldsymbol{\lambda\, h(q) = \lambda\ \log\big(\textbf{inverse mirror map}(q)\big)}$
where q is the Kaehler parameter of Z corresponding to the equivariant parameter λ.

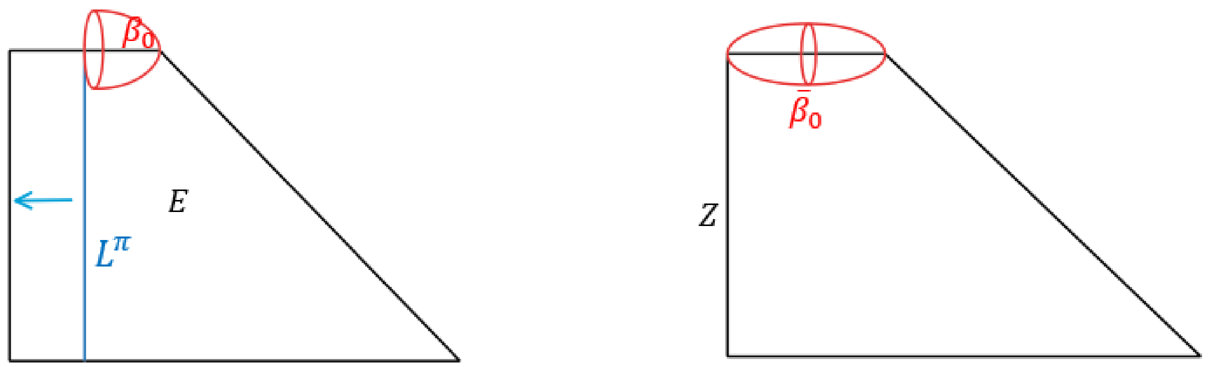
Ex. Hirzebruch surface $Z = \mathbb{F}_2 = \mathbb{C}^4//_* T^2$ where T^2 acts in the directions (0,1,1,0) and (1,0,−2,1) corresponding to fiber and section class respectively.

$W_{\mathbb{F}_2} = z + w + (1+q)w^{-1} + qz^{-1}w^{-2} = W_{\mathbb{C}^4} = Z_1 + Z_2 + Z_3 + Z_4.$
(Kaehler parameter of fiber class taken to be 1.)
$0 = h^{(0,1,1,0)} = h_L^{(0,1,1,0)} + h_{L^\pi}^{(0,1,1,0)} = \log Z_2 Z_3 + h_{L^\pi}^{(0,1,1,0)} = \log(1+q) + h_{L^\pi}^{(0,1,1,0)}.$
$\Rightarrow h_{L^\pi}^{(0,1,1,0)} = -\log(1+q)\,.$





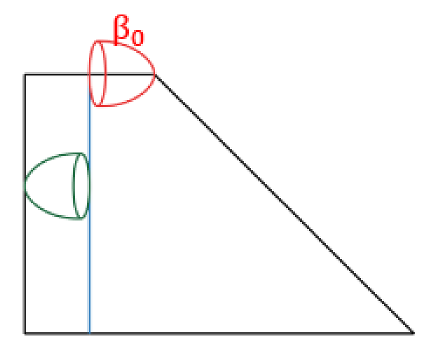
Correspondence for Seidel representation



For Hamiltonian $\mathbb{S}^1$-action on (Z,ω),
take $E = (Z \times \mathbb{C}^2) //_c \mathbb{S}^1 \to \mathbb{P}^1$.
Seidel element: $S := \sum_{s,a} \langle \iota_* \phi_a \rangle^E_{0,1,s} \, \phi^a q^s \in H(Z)$
counts sections s evaluated to a fiber $\iota: Z \to E$.
Multiplication by S defines an action on $QH(Z)$.

$\iota: Z \to E$ gives a correspondence $L^\pi \subset Z \times E$
(the same as that for quotient $E //_0 \mathbb{S}^1 \to Z$).
However, L^π is obstructed by Maslov-zero disc moduli $\mathcal{M}_1(\beta) \xrightarrow{ev} L^\pi$.

Ex.



$$W_{L^E} = T^A\left(w + \frac{1}{w}\right) + qz + \frac{T^A q}{zw}.$$

$$W_{L^\pi} = -q.$$

$$W_{\bar{L}\subset Z^\flat} = T^A\left(w + \frac{1+q}{w}\right).$$

Work in progress:
Assume semi-Fano: $c_1(C) \geq 0$ for any curve class C of Z.

1. Under the degeneration $L^\pi \to Z$, there is a cobordism between $\mathcal{M}_1^{open}(\beta_0 + \alpha) \times_{ev} L^\pi$ and $\pi^{-1}\left(\mathcal{M}_1^{closed}(\bar{\beta}_0 + \alpha) \times_{ev} \iota(Z)\right)$ where $\alpha \in H_2^{\text{eff,fiber}}(E) \subset H_2(E \times Z)$.
2. $m^{L^\pi}_{0,\mathbb{S}^1} \equiv qS \bmod 1_{L^\pi}$ and up to q^2.
3. Bulk deformation of Z by qS solves the obstruction of L^π up to q^2.
4. $CO_L(S) \equiv T^A \exp h_L \in \operatorname{Jac}(W_L)$ for an equivariant Lagrangian $L \subset Z$.

Remark:
4 is known for compact toric semi-Fano manifolds by combining:
[**Chan-L.-Leung-Tseng**]: $CO_L(S) = T^A w^{-1}$.
[**Kim-L.-Zheng**]: $h_L = \log w^{-1}$.