Closed String Mirror Symmetry for Punctured Riemann Surfaces

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Based on the joint work (in progress) with

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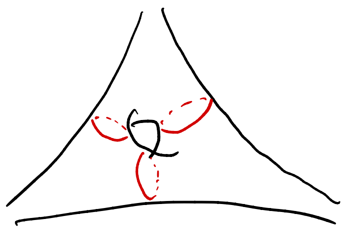
Known mirror constructions for punctured Riemann surfaces

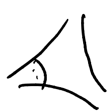
$\{f=0\} \subset \mathbb{C}^{\times}$ for a Laurent poly f w/ ≥ 3 terms

A] X

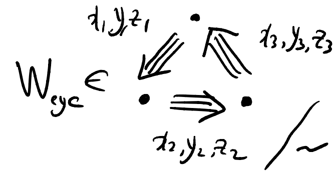


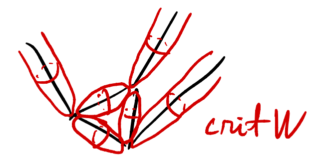
$f = z_1 + z_2 + \frac{1}{z_1 z_2}$

$\big\downarrow /G$



B] $\check{X} \; (\xrightarrow{W} \mathbb{C})$



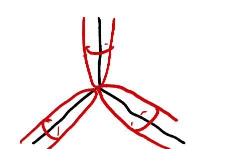


center $\searrow$ $\swarrow$

$\mathbb{C}^3/\mathbb{Z}_3, W$

$/\hat{G} \; \big\uparrow$

$\mathbb{C}^3, \; W = xyz$

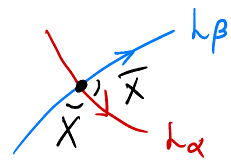


$\boxed{\text{Goal}}$ KS : $SH(X) \longrightarrow$ "$HH^*(MF(W))$"

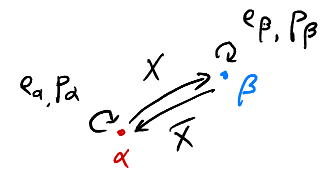
Noncommutative mirror

- $\mathbb{L} \overset{\text{Lag}}{\subset} X$, $\quad \mathbb{L} = \bigcup_\alpha L_\alpha$ (determined by some combinatorial data on X)

$\Longrightarrow$ generators of $CF(\mathbb{L},\mathbb{L})$:

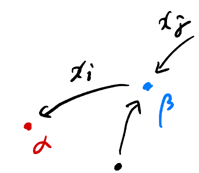


$\rightsquigarrow$



- $CF(\mathbb{L},\mathbb{L}) \ni \overset{0}{e_\alpha}, \overset{1}{X_i}, \overset{2}{\bar{X}_j}, \overset{3}{p_\beta}$

- For $\quad b = \sum_i x_i X_i \quad$ where $x_i \in \mathbb{C}\{x_1, \dots, x_\ell\}$ dual to X_i $\qquad Q$:

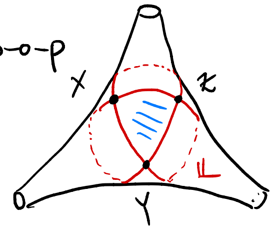


(weak) Maurer-Cartan eqn: $\quad m_1(b) + m_2(b,b) + \cdots = W(b)\cdot 1_{\mathbb{L}} + \sum_i f_i(x)\cdot \bar{X}_i$

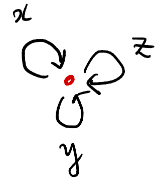
- Taking $x_i \in Jac(Q) := \Lambda Q \big/ \langle f_1, \dots, f_\ell \rangle$, $\quad b$ solves $\; m_1(b) + m_2(b,b) + \cdots = W(b)\cdot 1_{\mathbb{L}}$.

[Cho-H.-Lau] $\; W \in Jac(Q)$ is a noncommutative LG model. (W: central)

Ex1. p-o-p



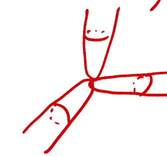
$\longleftrightarrow$

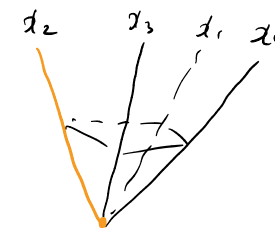


$$f_1 = xy - yx = 0$$
$$f_2 = yz - zy = 0$$
$$f_3 = zx - xz = 0$$

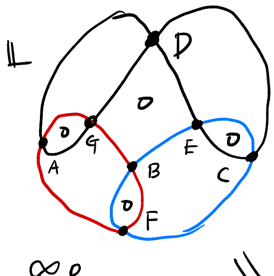
$\rightsquigarrow$ $\mathbb{C}[x, y, z] \ni W = xyz$

crit(W)

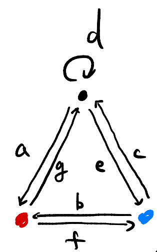




Ex2. X: 5-punctured sphere



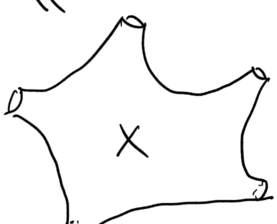
$\longleftrightarrow$

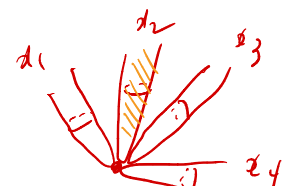


$$\Big/ \quad ec = ag \quad ebf = de$$
$$ceb = bga \quad bfc = cd$$
$$fce = gaf \quad afb = da$$
$$fbg = gd$$

$$\text{center} \rightsquigarrow Z \cong \mathbb{C}[x_1, x_2, x_3, x_4] \Big/ x_1 x_3 = x_2^2 x_4$$
$$\cup\!\!\!\,\,\,$$
$$W = x_2 x_4$$

||

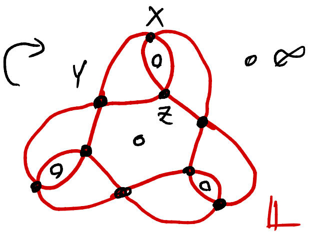




crit W

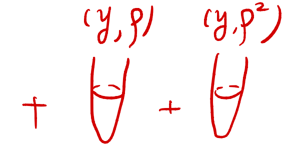
*Orbifold mirror

$G = \mathbb{Z}/3 \circlearrowright$



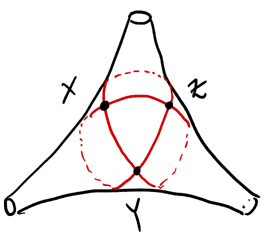
$\longleftrightarrow$

needs to add two more "twisted sectors"

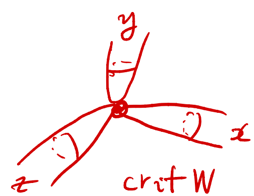


$\big/ G \downarrow$ $\qquad\qquad$ $\uparrow \big/ \hat{G}$

$W = xyz$



$\longleftrightarrow$



$\circlearrowleft \hat{G}, \quad (x,y,z) \longmapsto (\rho x, y, \rho^{-1} z)$

$\rho = e^{2\pi i/3}$

[H.-Jin-Lee] X : G-principal bundle / (pair of pants) $\xleftrightarrow{\text{mirror}}$ $\mathbb{C}^3/\hat{G} \xrightarrow{W} \mathbb{C}$

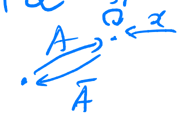
in the sense that $SH^*(X) \cong$ "orbifold" Koszul cohomology of W

Closed string B-model invariants

- $(\mathbb{L}, b)$ boundary deformed Lagrangian w/ $b = \sum x_i X_i$ weak MC $\quad (x_i \in \Gamma Q/\sim)$

$\Rightarrow \boxed{CF((\mathbb{L},b),(\mathbb{L},b))} \ni \vec{x}A$ for $\vec{x} \in \Gamma Q/\sim$ and A a Floer generator

each elt. is assigned w/ a path $\in \Gamma Q$ e.g.

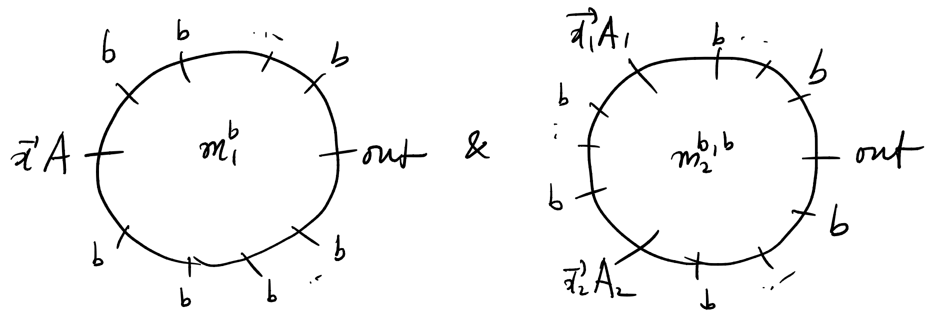


- It is a dga w/

$$m_1^b(\vec{x}A) := \sum m_k(b,\cdots,b,\vec{x}A, b\cdots b)$$

$$m_2^b(\vec{x}_1 A_1, \cdots, \vec{x}_k A_k) := \sum m_k(b,\cdots,b,\vec{x}_1 A_1, b,\cdots,b,\vec{x}_2 A_2, b,\cdots b)$$

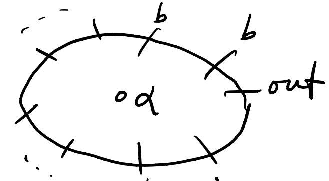
counting



<u>Convention</u> $m_k(\vec{x}_1 A_1, \cdots, \vec{x}_k A_k) := \vec{x}_k \vec{x}_{k-1} \cdots \vec{x}_1 m_k(A_1, \cdots, A_k)$

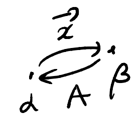
Kodaira-Spencer map (closed-open map)

$$
\begin{array}{ccc}
ks : SH^*(X) & \longrightarrow & HF((\mathbb{L},b),(\mathbb{L},b)) \quad \text{a ring homomorphism} \\
\rotatebox{90}{$\in$} & & \\
\alpha & \longmapsto &
\end{array}
$$



(w/ bdy $\subset \mathbb{L}$)

- observe $SH^*(X) \xrightarrow{ks} HF_{loop}((\mathbb{L},b),(\mathbb{L},b)) \ni \vec{x}A$

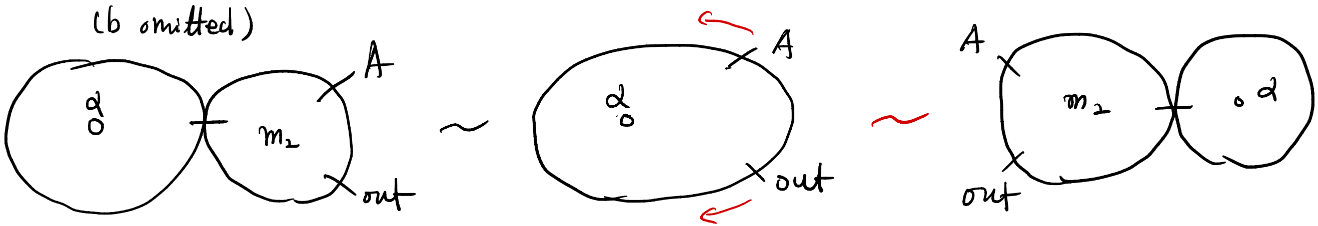


- Moreover

<u>Lemma</u> ks factors through $SH^*(X) \longrightarrow \underset{\text{center}}{Z} \subset HF_{loop}((\mathbb{L},b),(\mathbb{L},b))$

(b omitted)

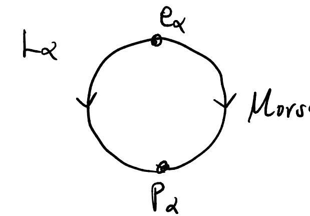
pf) Cobordism



$m_2^b(ks(\alpha), A)$ $\sim$ $\sim$ $m_2^b(A, ks(\alpha))$

$\square$

$HF_{loop}((\mathbb{L}, b), (\mathbb{L}, b))$ vs $(Jac(Q), W)$



$$CF_{loop}((\mathbb{L},b),(\mathbb{L},b)) \ni \begin{tikzcd} e_\alpha \arrow[r, bend left, "\delta_+"] & X_i \arrow[l, bend left, "\delta_-"] \arrow[r, bend left, "\delta_+"] & \bar{X}_j \arrow[l, bend left, "\delta_-"] \arrow[r, bend left, "\delta_+"] & p_\beta \arrow[l, bend left, "\delta_-"] \end{tikzcd}$$

$\circlearrowleft d := m_1^b = \delta_+ + \delta_-$

i) (CF_{loop}, δ_+) computes $HH^*(Jac(Q))$

$\;\;\; \cong Hom_{Jac(Q)^e}(\mathbb{P}^\bullet, Jac(Q))$ for a certain free resolution $\mathbb{P}^\bullet$ of $Jac(Q)$ [Ginzburg]

for "suitably chosen $\mathbb{L}$"

ii) $(H^*(CF_{loop}, \delta_+), \delta_-) = (HH^*(Jac(Q)), \{W, -\})$ [Wong]

RHS computes $HH^*_c(MF(W)) = HH^*((Jac(Q), W))$ [Caldararu-Tu]

Theorem [Dahye Cho, Hyungjun Jin, H., Sangwook Lee] $X = \{f=0\} \subset \mathbb{C}^X$. For suitably chosen $\mathbb{L}$,

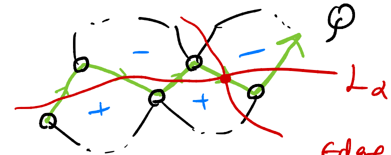
$$KS : SH^*(X) \xrightarrow{\cong} HF_{loop}((\mathbb{L},b),(\mathbb{L},b))$$

pf) · TQFT gives a ring hom.

· Directly compare underlying v.sp. using the above spectral seq. $\square$

Mirror symmetry via dimer duality (Bocklandt)

- A dimer $\mathcal{Q}$ on X is a oriented quiver embedded in X s.t.
  - vertices of $\mathcal{Q}$ = punctures in X
  - faces are either $(+)$ or $(-)$ oriented



Edge of $\mathcal{Q}$ $\leftrightarrow$ self-$\cap$ of $\mathbb{L}$

- A zigzag cycle $\subset \mathcal{Q}$ is a closed path in $\Gamma\mathcal{Q}$ s.t.

  $\updownarrow$
  $L_\alpha \subset \mathbb{L}$

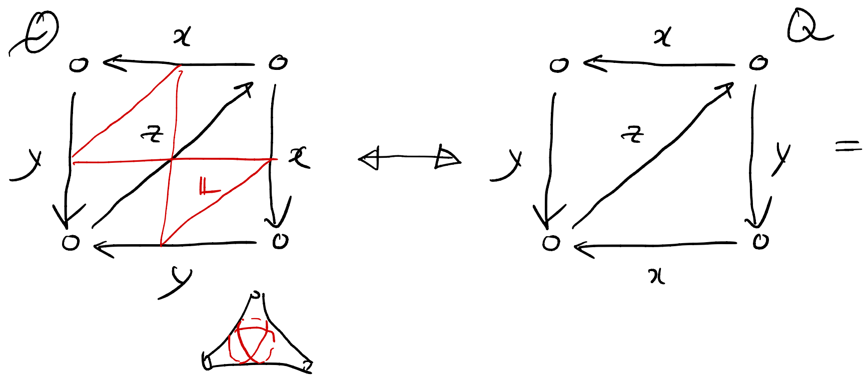
- The dual dimer Q is obtained by flipping $(-)$-faces and reversing their arrows

$\mathcal{Q}$	Q
L_α = zigzag cycles $(= L_\alpha)$	vertices α
$L_\alpha \cap L_\beta$ = intesecting edges bet'n zigzags	arrows $\alpha \cdot \longrightarrow \cdot \beta$
(weak) Maurer-Cartan relation	(combi) relation "R_a"
(Lagrangian Floer) potential	central element W

[Bocklandt] $\rightsquigarrow$ $Jac(Q), W$

Thm. [Bocklandt] $\quad WFuk(X) = mf(W) \; (\subset MF(W))$

Ex.

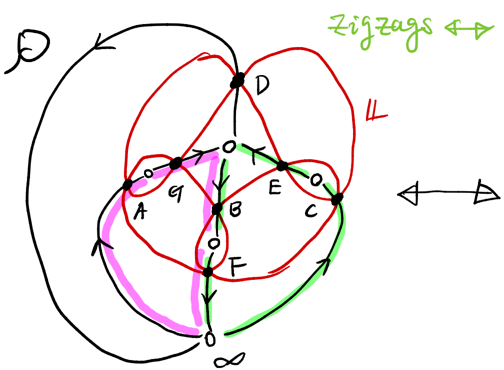


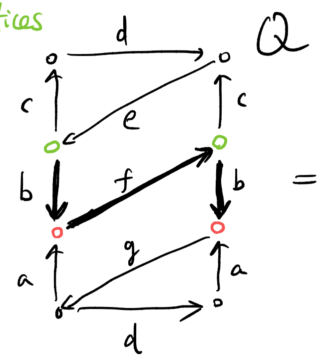
$R_x : yz - zy$
$R_y : zx - xz$
$R_z : xy - yx$

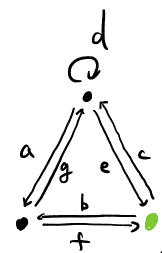
$W = xyz$

Ex.

Zigzags ⟷ Vertices







/

$ec = ag$ $\quad ebf = de$
$ceb = bga$ $\quad bfc = cd$
$fce = gaf$ $\quad afb = da$
$fbg = gd$

$W = agd + gafb + bfce + d$

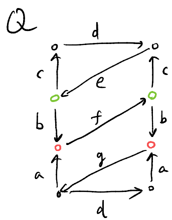
Noncommutative crepant resolutions

From now on, assume $Q \subset T^2$ (+ some consitency condition)

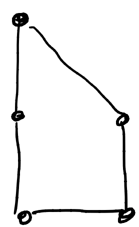
Def. A perfect matching on Q is a collection $\{a_1, \dots, a_l\}$ of edges s.t.

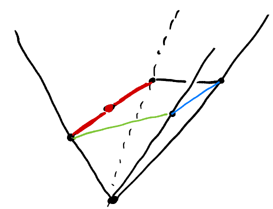
$$\left| \partial F \cap \{a_1, \dots, a_l\} \right| = 1 \quad \text{for } {}^{\forall}\text{face } F$$

For fixed P_0, $\{P - P_0 \,;\, P \text{ perf. mat.}\}$ form a lattice polygon $MP \subset H^1(T^2; \mathbb{R}) \cong \mathbb{R}^2$

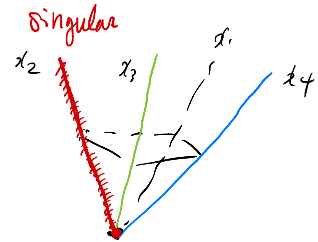


MP(Q)





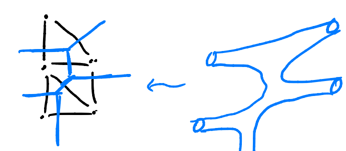
fan



moment polytope

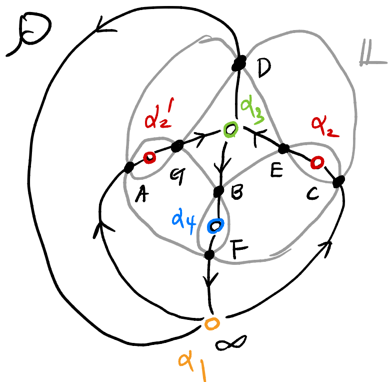
$2\,\mathrm{Area}(MP(Q)) = \#\text{ vertices of } Q$
$= \#\text{ zigzag paths in } Q$

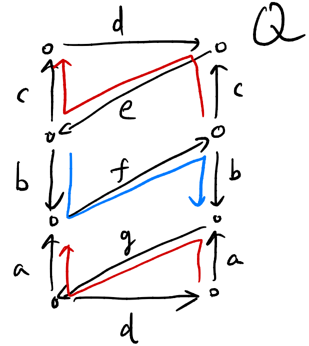
Rmk. $MP(Q)$ determines the topolgical type of X_Q:

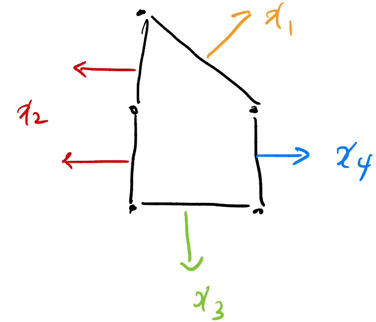


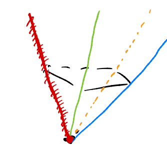
Features of Kodaira-Spencer map

(anti) zigzags in $Q \in H_1(T^2, \mathbb{Z})$ are normal to $MP(Q)$









$$SH^0(X) \xrightarrow[\cong]{KS} HF^{even}_{loop}((\mathbb{L},b),(\mathbb{L},b))$$

winding number=1 orbits

$$\cong$$

$$\frac{\mathbb{C}[x_1,x_2,x_3,x_4]}{(x_i x_j : i \neq j)} \oplus \langle \text{some } \bar{X}_i\text{-terms} \rangle$$

contains corrections for α_2 & α_2'

$$\mathbb{C}[x_1] \oplus \mathbb{C}[x_2] \oplus \mathbb{C}[x_3] \oplus \mathbb{C}[x_4]$$

$$\alpha_2 + \alpha_2' \longmapsto x_2$$

$$\alpha_1, \alpha_3, \alpha_4 \longmapsto x_1, x_3, x_4$$

Pick's theorem

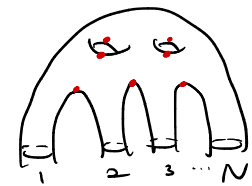
By explicit B-model calc

$$\#\text{ independent gen. of } HF^{odd}_{loop} \text{ over } HF^{even}_{loop} = \#\text{Vertices of } Q + 1$$

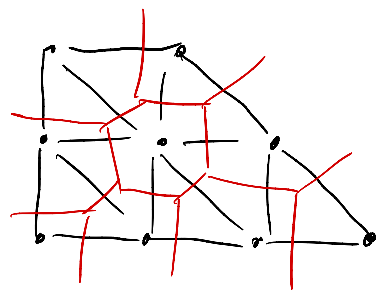
$$\updownarrow \qquad\qquad \|$$

$$\text{deg 1 contractible generators of } SH^{odd}(X) = 2g + N - 1$$

(top'l info of X_p)



MP(Q)



$$\Longrightarrow \therefore \begin{cases} g = \#\text{ interior lattice} =: I \\ N = \#\text{ boundary lattice} =: B \end{cases}$$

$$\begin{aligned} \therefore \ \text{Area}(MP(Q)) &= \frac{1}{2}(\#\text{vertices of } Q_0) \\ &= \frac{1}{2}(2g + N - 2) \\ &= I + \frac{1}{2}B - 1 \end{aligned}$$

Pick's formula.

Thank

You!