On graded modules associated with permissible C^∞-divisors on tropical manifolds

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Mirror Symmetry and Related Topics, Kyoto 2023, 2023-12-18

Tropical Geometry = "Geometry over tropical semifield $\mathbb{T}$"

$$\mathbb{T} := (\mathbb{R} \cup \{-\infty\}, \max, +) \simeq (\mathbb{R}_{\geq 0}, \max, \cdot)$$

$$\mathbb{T}^n = (\mathbb{R} \sqcup \{-\infty\})^n = \bigsqcup_{I \subset \{1,\dots,n\}} \mathbb{R}_I$$

where $\mathbb{R}_I := \{(x_1, \dots, x_n) \in \mathbb{T}^n \mid x_i = -\infty \text{ iff } i \in I\} \simeq \mathbb{R}^{n-\sharp I}$

Definition 1 (Gross–Shokrieh'23, Mikhalkin–Zharkov'14)

A *rational polyhedral set* in $\mathbb{T}^n$ is a finite union of the closures of rational conv. polyhedra in some $\mathbb{R}_I$.

V: an open subset of a rational polyhedral set in $\mathbb{T}^n$.
$\mathcal{O}_V^\times$: the sheaf of continuous fcns f s.t.

$$f(x) = m_1x_1 + \cdots + m_nx_n + a, \quad (m_i \in \mathbb{Z}, a \in \mathbb{R}) \tag{1}$$
$$= \langle m, x\rangle + a \tag{2}$$

for some open nbd of x under the convention $0 \cdot (-\infty) := 0$.

Definition 2 (Gross–Shokrieh'23, (cf. Mikhalkin–Zharkov'14))

A *rational polyhedral space* is a pair $(S, \mathcal{O}_S^\times)$ where

(i) S is a 2nd-countable loc. cpt T_2-space

(ii) $\mathcal{O}_S^\times$ is a sheaf of abelian groups on S

s.t. for any $x \in S$, there exists an open nbd U of x and an isomorphism $(U, \mathcal{O}_S^\times|_U) \simeq (V, \mathcal{O}_V^\times)$ for some open subset V of a rational polyhedral set.

Example 3

(i) Every metric finite graph with no 1-valent vertex has a canonical model of a rational polyhedral space.

(ii) Integral affine manifolds (e.g. real tori $\mathbb{R}^n/\Lambda$).

Definition 4

A n-dimensional *integral affine manifold* is a pair of n-dimensional real manifold B and an altas $\{(U_i, \psi_i : U_i \to \mathbb{R}^n)\}_{i\in I}$ s.t. $\psi_j \circ \psi_i^{-1} \in \mathrm{GL}_n(\mathbb{Z}) \ltimes \mathbb{R}^n$ locally.

Each example is a *tropical manifold*, i.e., a rational polyhedral space S which is locally isomorphic to a direct product of the Bergman fan Σ_M for some loopless matroid M and $\mathbb{T}^n$ for some n.

D: a divisor on a cpt tropical curve C,
$r(D)$: the (Baker–Norine) rank of D,
K_C: the canonical divisor of C,
$\chi_{\mathrm{top}}(C)$: the topological Euler characteristic of C.

Theorem 5 (Tropical Riemann–Roch theorem)

Under the setting above,

$$r(D) - r(K_C - D) = \deg(D) + \chi_{\mathrm{top}}(C). \tag{3}$$

Theorem 6 (Hirzebruch–Riemann–Roch–theorem)

Let $\mathcal{L}$ be a line bundle on a cpt complex mfd X. Then,

$$\chi(H^\bullet(X;\mathcal{L})) = \int_X \mathrm{ch}(\mathcal{L})\,\mathrm{td}(X). \tag{4}$$

How about for cpt tropical mfds ?

RHS of HRR

S: a cpt tropical mfd of dimension n.
$H^{p,q}(S;\mathbb{Z})$: the (p,q)-th *tropical cohomology*.
The *tropical exponential sequence*

$$0 \to \mathbb{R}_S \to \mathcal{O}_S^\times \to \Omega^1_{\mathbb{Z},S} \to 0$$

defines the *1st Chern map* $c_1\colon H^1(S;\mathcal{O}_S^\times) \to H^1(S;\Omega^1_{\mathbb{Z},S})$ and

$$\mathrm{ch}(\mathcal{L}) := \exp(c_1(\mathcal{L})) \in H^{\bullet,\bullet}(S;\mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{Q}.$$

$\int_S\colon H^{n,n}(S;\mathbb{Z}) \to \mathbb{Z}$: the trace map of S.
$\mathrm{td}(S)$: the *Todd class* of S defined by
de Medrano–Rincón–Shaw (arXiv:2309.00229).

$$\mathrm{RR}(S;\mathcal{L}) := \int_S \mathrm{ch}(\mathcal{L})\,\mathrm{td}(S). \tag{5}$$

It is difficult to define the Euler characteristic of line bundles on tropical manifolds:

Reason: $\mathbb{T}$ is *not* an Abelian group.

$\rightarrow$ We cannot apply homological algebra directly.

On the other hands, tropical geometers expect the following:

Conjecture 7 (de Medrano–Rincón–Shaw'23)

For any cpt tropical mfd S,

$$\chi_{\mathrm{top}}(S) = \int_S \mathrm{td}(S) = \mathrm{RR}(S;0). \tag{6}$$

In particular, $\chi_{\mathrm{top}}(S)$ behaves like an analog of the Euler characteristic of the structure sheaf of a cpt complex mfd.

Our approach: Consider analogs of Floer complexes of Lagrangian submanifolds (without differentials).

S: a rational polyhedral space.
$\exp\colon \mathbb{T}^n \to \mathbb{R}^n_{\geq 0}\colon (x_1, \ldots, x_n) \mapsto (\exp(x_1), \ldots, \exp(x_n))$

Definition 8 (cf. Mikami(arXiv:2303.09809))

A continuous fcn $f : S \to \mathbb{R}$ is *weakly-smooth* if for any $x \in S$ there exists a chart $\psi\colon U_x \to \mathbb{T}^n$ and a positive C^∞-function h on an open subset of $\mathbb{R}^n$ s.t.

$$f|_{U_x} = \log \circ h \circ \exp \circ \psi. \tag{7}$$

Remark 9

If S is an integral affine manifold, then f is weakly-smooth if and only if f is a C^∞-function on S.

$\mathcal{A}_S^{\mathrm{weak}}$: the sheaf of weakly-smooth fcns. on S.
The exact sequence

$$0 \to \mathcal{O}_S^\times \to \mathcal{A}_S^{\mathrm{weak}} \to \mathcal{A}_S^{\mathrm{weak}}/\mathcal{O}_S^\times \to 0 \tag{8}$$

gives a surjection $H^0(S; \mathcal{A}_S^{\mathrm{weak}}/\mathcal{O}_S^\times) \to H^1(S; \mathcal{O}_S^\times); s \mapsto [s]$.
$\mathrm{Div}^\infty(S) := H^0(S; \mathcal{A}_S^{\mathrm{weak}}/\mathcal{O}_S^\times)$: the group of C^∞-divisors.

Remark 10 (C^∞-divisors are analogs of Lagrangian sections)

If B is an integral affine manifold, then every element $s \in \mathrm{Div}^\infty(B)$ defines a Lagrangian section $\check{s}$ of the standard Lagrangian torus fibration $\check{f}_B \colon T^*B/T^*_{\mathbb{Z}}B \to B$.

We will construct a graded module $\mathrm{LMD}^\bullet(S; s)$ for a *permissible* C^∞-divisor s as an analog of Floer complexes (without differentials).

S: a rational polyhedral space,
$\Omega_S^1 := \Omega_{\mathbb{Z},S}^1 \otimes_{\mathbb{Z}_S} \mathbb{R}_S = \mathcal{O}_S^\times / \mathbb{R}_S \otimes_{\mathbb{Z}_S} \mathbb{R}_S$,
$T_x S := \mathrm{Hom}_{\mathbb{Z}}((\Omega_{\mathbb{Z},S}^1)_x, \mathbb{R})$,
$\mathrm{LC}_x S (\subset T_x S)$: the local cone of S at x,
$\phi\colon \mathrm{LC}_x S \to T_x S$: the canonical inclusion,
Under the identification $(\Omega_S^1)_x \simeq T_0^*(T_x S)$, we set

$$\mathrm{SS}(S)_x := \mathrm{SS}(\phi_! \phi^{-1} \mathbb{Z}_{T_x S}) \cap (\Omega_S^1)_x, \tag{9}$$

where $\mathrm{SS}(\phi_! \phi^{-1} \mathbb{Z}_{T_x S})$ is the micro-support of $\phi_! \phi^{-1} \mathbb{Z}_{T_x S}$.

Example 11

If C is a metric finite graph with no 1-valent vertex, then

$$\mathrm{span}_{\mathbb{R}}(\mathrm{SS}(C)_x) = \begin{cases} (\Omega_S^1)_x \simeq \mathbb{R}^{\mathrm{val}(x)-1}, & \text{if } \mathrm{val}(x) \neq 2, \\ \{0\}, & \text{o.w.} \end{cases}$$

Definition 12

A weakly-smooth fcn f on a rational polyhedral space S is *prepermissible* at $x \in S$ if

$$df(x) \notin \operatorname{span}_{\mathbb{R}}(SS(S)_x) + (\Omega^1_{\mathbb{Z},S})_x \setminus (\Omega^1_{\mathbb{Z},S})_x (\subset (\Omega^1_S)_x). \quad (10)$$

A C^∞-divisor $s = \{(U_i, f_i)\}_{i \in I}$ on S is *prepermissible* if every f_i is prepermissible.

Example 13

(i) Every C^∞-fcn on an integral affine mfd is prepermissible.

(ii) Let C be a metric finite graph with no 1-valent and $x \in C_{\mathrm{sing}}$, then a weakly-smooth function f is prepermissible at x iff $df(x) \in (\Omega^1_{\mathbb{Z},S})_x \simeq \mathbb{Z}^{\mathrm{val}(x)-1}$.

If S is compact, $\{x \in S \mid \operatorname{span}_{\mathbb{R}}(SS(S)_x) = (\Omega^1_S)_x\}$ is finite.

For any $x \in S$, we set

$$\check{X}_0(S)_x := ((\Omega^1_{\mathbb{Z},S})_x / \operatorname{span}_{\mathbb{R}}(SS_x(S)) \cap (\Omega^1_{\mathbb{Z},S})_x) \otimes_{\mathbb{Z}} \mathbb{R}/\mathbb{Z} \quad (11)$$

$$\check{X}_0(S) := \bigcup_{x \in S} \check{X}_0(S)_x$$

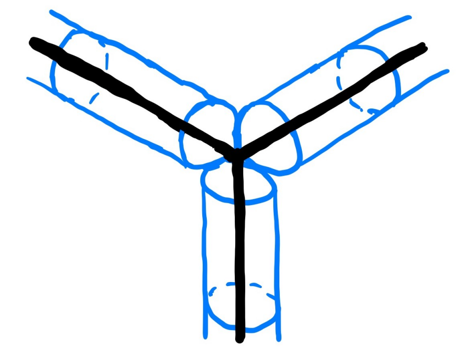


Figure: S and $\check{X}_0(S)$ (cf. Auroux–Efimov–Katzarkov'22)

Every prepermissible C^∞-divisor $s = \{(U_i, f_i)\}_{i\in I}$ defines a section $\check{s}\colon S \to \check{X}_0(S)$ of $\check{f}_S\colon \check{X}_0(S) \to S$ and the intersection $s_0 \cap s := \check{f}_S(\mathrm{Im}(\check{s}_0) \cap \mathrm{Im}(\check{s}))$ with the zero divisor.

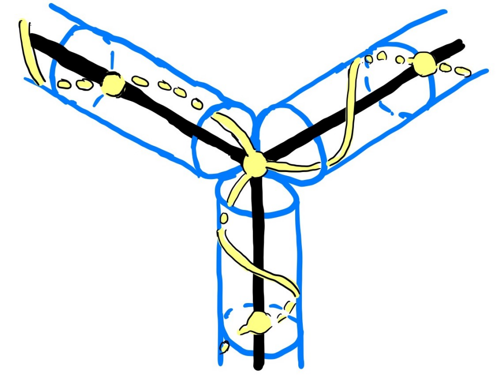


Figure: the zero section $\check{s}_0$ and a prepermissible section $\check{s}$

Definition 14

An *intersection data* of a prepermissible C^∞-divisor s is a family $\{f_p \colon U_p \to \mathbb{R}\}_{p \in s_0 \cap s}$ of weakly-smooth functions on an open nbd U_p of p s.t. $s|_{U_p} = (f_p)$ and $\mathrm{Crit}(f_p) = \{p\}$.

For a given intersection data of s, we set

$$\mathrm{LMD}^\bullet(S; s) := \bigoplus_{p \in s_0 \cap s} H^\bullet(R\Gamma_{\{f_p \geq f_p(p)\}}(\mathbb{Z}_{U_p})_p) \qquad (12)$$

The graded module $\mathrm{LMD}^\bullet(S; s)$ is independent of the choice of intersection data.

Remark 15

If S is a cpt integral affine manifold and the Lagrangian section L_s associated with s intersects with the zero section transversely, then the corresponding Floer complex is isomorphic to $\mathrm{LMD}^\bullet(S; s) \otimes_{\mathbb{Z}} \Lambda_{\mathrm{nov}}$ as graded modules.

Let f be a weakly-smooth function on S s.t. (f) is prepermissible and $s_0 \cap (f) = \{p\}$ for some $p \in S$. Then,

$$\begin{aligned} \mathrm{LMD}^\bullet(S;(f)) &= H^\bullet(R\Gamma_{\{f \geq f(p)\}}(\mathbb{Z}_S)_p) \\ &\simeq \varinjlim_{p \in U \subset_{\mathrm{open}} S} \tilde{H}^{\bullet-1}(U \cap \{f < f(p)\}; \mathbb{Z}) \end{aligned} \tag{13}$$

Example 16 (Morse function)

Let $f_{n,m} \colon \mathbb{R}^{n+m} \to \mathbb{R}; (x_1, \ldots, x_{n+m}) \mapsto \sum_{j=1}^{n} x_j^2 - \sum_{k=1}^{m} x_{n+k}^2$. Then, for a sufficiently small open nbd U of 0

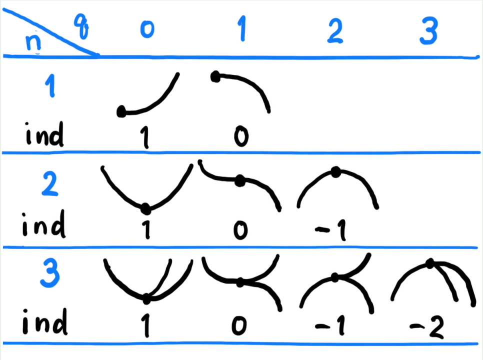
$$\mathrm{LMD}^\bullet(U;(f_{n,m})) \simeq \tilde{H}^{\bullet-1}(B^n \times S^{m-1}; \mathbb{Z}) \simeq \mathbb{Z}[-m] \tag{14}$$

under the convention $S^{-1} = \varnothing$.

Example 17

$\Gamma_n := \mathbb{R}_{\geq 0}(1, \ldots, 0) \cup \cdots \cup \mathbb{R}_{\geq 0}(0, \ldots, 1) \cup \mathbb{R}_{\geq 0}(-1, \ldots, -1)$
$f_n \colon \Gamma_n \to \mathbb{R}$: a weakly-smooth fcn s.t. $s_0 \cap (f) = \{0\}$

$$\chi(\mathrm{LMD}^\bullet(\Gamma_n, (f_n))) = 1 - \sharp(\pi_0(\{f(v) < f(0)\})) \qquad (15)$$
$$=: 1 - q \qquad (16)$$



Definition 18

A prepermissible C^∞-divisor s on S is *permissible* if $\sharp(s_0 \cap s) < \infty$ and $\mathrm{LMD}^\bullet(S; s)$ is finitely generated.

Conjecture 19 (T. (a modified version after [M–R–S'23]))

For any permissible C^∞-divisor on a cpt tropical manifold S, the following equation holds

$$\chi(\mathrm{LMD}^\bullet(S; s)) = \int_S \mathrm{ch}([s])\,\mathrm{td}(S). \tag{17}$$

Theorem 20 (T. (cf. Auroux–Efimov–Katzarkov'22))

Conjecture 19 is true when S is a cpt tropical curve or integral affine mfd admitting a Hessian metric.

P: a d-dimensional Delzant lattice polytope in $(\mathbb{R}^d)^\vee$.

$$f_P(x_1, \dots, x_d) := \log \left(\sum_{m \in P \cap (\mathbb{Z}^d)^\vee} \exp(\langle m, x \rangle) \right)$$

Every f_P naturally defines a permissible C^∞-divisor s_P on the tropical toric manifold X_P^{trop} of P.

Proposition 21 (T.)

$$\chi(\mathrm{LMD}^\bullet(X_P^{\mathrm{trop}}; s_P) = \sharp(P \cap (\mathbb{Z}^d)^\vee), \tag{18}$$

$$\chi(\mathrm{LMD}^\bullet(X_P^{\mathrm{trop}}; -s_P) = (-1)^d \sharp(\mathrm{int}(P) \cap (\mathbb{Z}^d)^\vee). \tag{19}$$