Computations and Locality in Relative Symplectic Cohomology

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Floer equation

- (M, ω) symplectic manifold, a compatible almost complex structure is a fiberwise linear map $J : TM \to TM$, which satisfies $J^2 = -Id$ and so that $\omega(\cdot, J\cdot)$ is a Riemannian metric.
- Take a smooth map $\mathcal{H} : S^1_t \times \mathbb{R}_s \to C^\infty(M, \mathbb{R})$, which is s independent near the ends.
- Floer equation for $u : S^1_t \times \mathbb{R}_s \to M$,

$$J\frac{\partial u}{\partial s} = \frac{\partial u}{\partial t} - X_{\mathcal{H}(s,t)}$$

- For $\mathcal{H} = 0$, we get pseudo-holomorphic curve equation
- If $\mathcal{H}$ only depends on s (resp. t), t-invariant (resp. s-invariant) solutions are continuation maps for Morse theory (resp. 1-periodic orbits)
- We will be counting solutions of the Floer equation which are asymptotic to 1-periodic orbits of the Hamiltonians at the ends

Novikov field

- More honestly, we will have to count Floer solutions with certain weights, which encode their topological energy

$$topE(u) := \int u^*\omega + \int_{S^1} \gamma_{out}^* H_{out} dt - \int_{S^1} \gamma_{in}^* H_{in} dt$$

- As a result, we define our invariants over the non-archimedean valued field (called the Novikov field)

$$\Lambda = \{\sum_{i \in \mathbb{N}} a_i T^{\alpha_i} \mid a_i \in \mathbb{Q}, \alpha_i \in \mathbb{R}, \text{ and for any } R \in \mathbb{R},$$
$$\text{there are only finitely many } a_i \neq 0 \text{ with } \alpha_i < R\}$$

- Can work over $\Lambda_{\geq 0}$ but I want to simplify
- Except in certain cases using Novikov parameters is forced on us for technical reasons, but it is also a feature!

Hamiltonian Floer theory

- (M, ω) geometrically bounded symplectic manifold such that $c_1(M) = 0$ with grading datum
- Given non-degenerate Hamiltonian H and time-dependent J such that (H, J) is dissipative and regular, we obtain a chain complex over Λ: $CF(H, J, \Lambda)$
    1. complete vector space over Λ generated by the 1-periodic orbits of X_H
    2. grading by Maslov type index
    3. self-map d by counting Floer solutions with weights $T^{topE(u)}$
    4. (Floer's theorem) $d^2 = 0$
- OK to omit J for what follows
- Can define chain maps $CF(H, \Lambda) \to CF(H', \Lambda)$ using the same idea of counting, which are isomorphisms on homology if M is closed! (continuation maps)

Acceleration data for compact sets

- Want to define an invariant of $K \subset M$ using Hamiltonian FT
- **Acceleration data** for compact $K \subset M$ is a family of S^1-dependent Hamiltonians H_τ, $\tau \in [1, \infty)$ such that:
  1. $H_\tau(t, x) < 0$, for every t, τ and $x \in K$.
  2. $H_\tau(t,x) \xrightarrow[\tau \to +\infty]{} \begin{cases} 0, & x \in K, \\ +\infty, & x \notin K, \end{cases}$ for every t
  3. $H_\tau(t, x) \geq H_{\tau'}(t, x)$, whenever $\tau \geq \tau'$
  4. For $n \in \mathbb{N}$, the flow of H_n satisfies non-degeneracy
- $\mathcal{C}(H_\tau) := CF(H_1, \Lambda) \to CF^*(H_2, \Lambda) \to \dots$
- The maps are given by continuation maps. Monotonicity requirement (3) implies that topological energies are all non-negative.

Definition of the invariant

- We will need to process $\mathcal{C}(H_\tau)$ to get a chain complex whose homology only depends on K: perhaps take homotopy colimit?
- Does not depend on K at all when M is closed!
- This is an infinite dimensional vector space equipped with a basis $v_1, v_2, \ldots,$ (up to ± 1) in each degree. We complete it degreewise by taking all sums

$$\sum_{i=1}^{\infty} a_i v_i$$

such that $a_i \in \Lambda$ so that $val(a_i) \to \infty$ as $i \to \infty$.
- Differential extends to the completion
- Resulting homology is independent of choices:

$$SH^*_M(K, \Lambda)$$

- Automatically get restriction maps for $K \subset K'$ with the presheaf property

Products and unit

- $SH^*_M(K, \Lambda)$ can be equipped with a unital BV algebra structure.
- Restriction maps are unital BV algebra homomorphisms.
- Vanishing is equivalent to $1 = 0$.

Dependence on K

- $SH_M(\varnothing, \Lambda) = 0$
- $SH^*_M(M, \Lambda) = H^*(M, \Lambda)$ if M is closed
- If $K \subset M$ is displaceable from itself by a Hamiltonian diffeomorphism, then $SH_M(K, \Lambda) = 0$.
- If $K \times (S^1 \times \{0\}) \subset M \times (T^*S^1)$ is displaceable from itself by a Hamiltonian diffeomorphism, then $SH_M(K, \Lambda) = 0$.
- Invariance under symplectomorphisms
- It can be infinite dimensional and quite hard to compute.

A sample computation

- $\mathbb{R}^2$ has symplectic structure $dx \wedge dy$, which descends to $T^2 = \mathbb{R}^2/(2\pi\mathbb{Z})^2$
- Consider map $\pi : T^2 \to S^1 := \mathbb{R}/2\pi\mathbb{Z}$ which projects to x coordinate and consider translation invariant grading data
- We can compute the 1-periodic orbits of function $H = H(x)$, they occur whenever $H'(x)$ is an integer multiple of 2π.
- If $I \subset S^1$ is an interval of length $r < 2\pi$, we get

$$SH^0_M(\pi^{-1}(I), \Lambda) \simeq \Lambda < x, y > /(xy - T^{2\pi r})$$

- The RHS isomorphic to formal series $\sum_{n\in\mathbb{Z}} a_n x^n$ with coefficients in Λ which converge on I in the following sense:

$$val(a_n) + nb \to \infty \text{ as } |n| \to \infty,$$

for any $b \in \tilde{I} \subset \mathbb{R}$

Polytopal domains

- $\mathbb{R}^{2n}$ has symplectic structure $\sum dp_i \wedge dq_i$, which descends to $M = \mathbb{R}^n \times T^n$
- Consider the projection $\pi : M \to \mathbb{R}^n$ and standard grading datum
- Let $P \subset \mathbb{R}^n$ be a compact polytope with rational slope faces. Define $KS(P)$ as the completion of $\Lambda[(\mathbb{Z}^n)^\vee]$ with respect to the valuation

$$val(\sum a_\alpha z^{n_\alpha}) = \min_{\alpha, p \in P}(val(a_\alpha) + n_\alpha(p)).$$

- Isomorphic to Kontsevich-Soibelman's convergent functions on P and can be defined independently of coordinates
- Theorem: $SH^0_M(\pi^{-1}(P), \Lambda)$ is canonically isomorphic to $KS(P)$ compatibly with restriction maps.

Sketch proof of theorem

- By the Mayer-Vietoris property below suffices to consider convex P
- $SH^0_M(\pi^{-1}(P), \Lambda)$ is isomorphic to the completion of the Viterbo symplectic cohomology $SH^0(M, \Lambda)$ with respect to the action filtration defined by $\pi^{-1}(P)$ and primitive $\sum p_i dq_i$
- This method of computation should generalize to interiors of positive log CY varieties and allow us to import results from Viterbo symplectic cohomology
- In these cases one shows that $H^0(\widehat{tel}(\mathcal{C}(H_s), \Lambda_{\geq 0}))$ and $H^1(tel(\mathcal{C}(H_s), \Lambda_{\geq 0}))$ has finite torsion, which is special ($n = 2$ covered by Pascaleff)

Mayer-Vietoris property

Theorem (V.)

K_1, K_2 compact subsets of M. If K_1 and K_2 admit barriers, then there is an exact sequence

$$\begin{tikzcd} SH^*_M(K_1 \cup K_2) \arrow[r] & SH^*_M(K_1) \oplus SH^*_M(K_2) \arrow[dl] \\ SH^*_M(K_1 \cap K_2) \arrow[u, "{[1]}"] & \end{tikzcd} \qquad (1)$$

where the degree preserving maps are the restriction maps (up to sign)

For domains admitting barriers is a slightly weaker condition than the existence functions f_1, f_2 such that $K_i = \{f_i \leq 0\}$ and the Hamiltonian flows of f_i commute.

Sheaf property

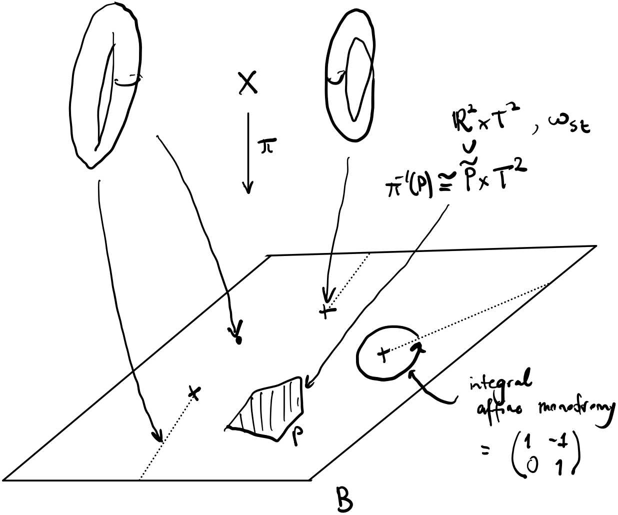
- $\pi : M^{2n} \to B^n$ a proper involutive map
- In many cases relevant to mirror symmetry $SH^*_M(\pi^{-1}(P), \Lambda)$ is non-negatively graded (as above)
- MV property implies sheaf property for

$$\mathcal{F}(\cdot) := SH^0_M(\pi^{-1}(\cdot), \Lambda)$$

- $\mathcal{F}$ behaves very much like a sheaf of functions (already seen this for $T^2 \to S^1$ and $T^*T^n \to \mathbb{R}^n$)
- Goal: Construct a Λ-analytic mirror space $\mathcal{Y}$, which also fibers over B such that the push-forward of the structure sheaf is $\mathcal{F}$.

Example: symplectic cluster manifolds

- Consider $X := M \setminus D$ a Looijenga interior with symplectic form $Im(\Omega)$ - there are other symplectic forms (see Lin et. al.)
- Choosing toric models, one can construct multiple complete nodal Lagrangian torus fibrations $\pi : X \to B$, where the singular integral affine structure on B can be explicitly computed as an eigenray diagram, which is how we think of $(X, Im(\Omega))$ from now on
- In particular X is geometrically of finite type: there exists almost complex structure J and exhaustion function $f : X \to \mathbb{R}$ which has finitely many critical points and whose Hamiltonian vector field is C^1 bounded
- We have many symplectic embeddings $X' \to X$ of simpler Looijenga interiors (e.g. cluster charts) compatible with Lagrangian fibrations along large open subsets


X
π
$\mathbb{R}^2 \times T^2$, ω_{st}
$\pi^{-1}(P) \cong \tilde{P} \times T^2$
P
B
integral affine monodromy
$= \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$

Locality

- We would like to be able to use our computations in simple Looijenga interiors (e.g. T^*T^2) for X

Theorem (Groman-V.)

Let M^{2n} be geometrically bounded, Y^{2n} be geometrically of finite type, and $\iota : Y \to M$ be a symplectic embedding. Then we can construct natural isomorphisms $SH^*_Y(K) \simeq SH^*_M(\iota(K))$ for each homologically finite torsion compact subset $K \subset Y$.

- This is good enough for our purposes but the torsion finite assumption can be lifted
- Proof uses dissipativity techniques developed by Groman in his thesis
- Different ideas needed for closed M

Locality for symplectic cluster manifolds

- Let $\mathcal{R}$ be an eigenray diagram.
- Take a compact convex polygon $\tilde{P}$ in $\mathbb{R}^2$ which is disjoint from all the rays of $\mathcal{R}$.
- Then there is an induced isomorphism

$$\mathcal{F}_{\mathcal{R}}(\psi_{\mathcal{R}}(\tilde{P}) = P) \to KS(\tilde{P}).$$

- The isomorphism in the statement is determined by an eigenray diagram representation. Representing symplectic cluster manifolds by different eigenray diagrams, we obtain distinct locality isomorphisms of the same form.

Wall-crossing

- Consider the completed Pascaleff manifold X_1 which has a nodal Lagrangian fibration with a single pinched torus fiber
- Choose a convex rational polygon in the base not intersecting the eigenline
- There are two non-Hamiltonian isotopic embeddings of T^*T^2 giving rise to two locality isomorphisms with Kontsevich-Soibelman function spaces
- The comparison map is not monomial - the simplest wall-crossing transformation
- This computation has not appeared anywhere yet