Log BPS numbers and contributions of degenerate log maps

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Overview

Joint work with Jinwon Choi, Michel van Garrel, Sheldon Katz.

[1] *Local BPS invariants: enumerative aspects and wall-crossing*, Int. Math. Res. Not. IMRN 2020, no. 17, 5450–5475.

[2] *Log BPS numbers of log Calabi-Yau surfaces*, Trans. Amer. Math. Soc. 374 (2021), no. 1, 687–732.

[3] *Sheaves of maximal intersection and multiplicities of stable log maps*, Selecta Math. (N.S.) 27 (2021), no. 4, Paper No. 61.

Today:

- Topics related to log BPS numbers of log Calabi-Yau surfaces
- Contribution of degenerate maps, in particular $\mathbb{A}^1$-curves

$\mathbb{A}^1$-curves

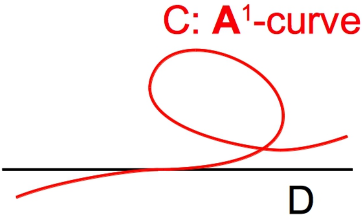
X: a projective algebraic variety, $D \subset X$: a normal crossing divisor.

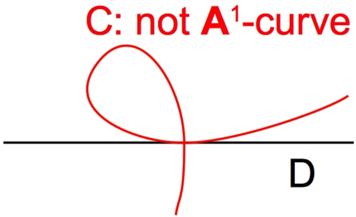
Definition

An $\mathbb{A}^1$-curve on (X, D) is an integral curve C in X such that the normalization of $C \setminus D$ is isomorphic to $\mathbb{A}^1$.

Another characterization: An $\mathbb{A}^1$-curve is an integral rational curve C which is maximally contact to D:

If $\nu : \mathbb{P}^1 \to C$ denotes the normalization, then $\#\nu^{-1}(D) = 1$.





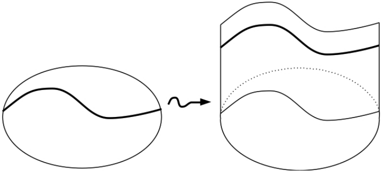
$\mathbb{A}^1$-curves and relative/log GW invariants

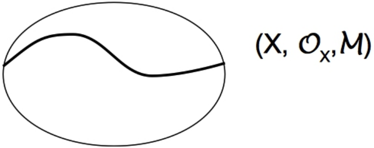
Enumeration of $\mathbb{A}^1$-curves (and other curves with conditions on the contact with D) is related to —

- Enumeration of curves on proj. var., via degeneration formula.
- Local GW invariants of the total space of $\mathcal{O}_X(-D)$.

Moduli theoretic approaches to the enumeration:

- Relative Gromov-Witten invariants: Based on the moduli space of "relative stable maps", where the target space is allowed to degenerate (expand).
- Log Gromov-Witten invariants: Based on the moduli space of "stable log maps", formulated in terms of "log geometry."





$\mathbb{A}^1$-curves and relative/log GW invariants (2)

There are also "orbifold" and "hybrid" approaches.
In the situation of this talk, all these invariants coincide.
(So I will refer to them as "log GW invariants")

As with usual GW invariants, log GW invariants are rational numbers: maps that are not generically one-to-one (e.g. multiple covers) give fractional contributions.

In dimension 2, reducible curves are also inevitable.

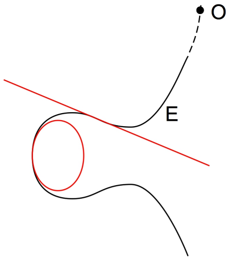
Problem

How can we relate the concrete enumeration of $\mathbb{A}^1$-curves and log GW invariants?

$\mathbb{A}^1$-curves on $(\mathbb{P}^2, E)$

Let $E \subset \mathbb{P}^2$ be a smooth cubic.

Take an inflection point $O \in E$ as the zero element of addition on E.



C: $\mathbb{A}^1$-curve of degree d on $(\mathbb{P}^2, E)$

$\Rightarrow \quad C \cap E = \{P\}$, where P is a $3d$-torsion.

We say a $3d$-torsion P is primitive (with respect to d) if

- $3|d$, and P is of order $3d$,
- $3 \not| d$, and P is of order d or $3d$ (depends on the choice of O).

$\mathbb{A}^1$-curves on $(\mathbb{P}^2, E)$, (2)

For a fixed P, how many $\mathbb{A}^1$-curves C of degree d with $C \cap E = \{P\}$?

- $d = 1$: 1 for each inflection point P: Inflectional tangent line.
- $d = 2$: 1, if P is primitive,
  0, if P is an inflection point.
- $d = 3$: 3 nodal cubics, if P is primitive,
  2 nodal cubics or 1 cuspical cubic, if P is an infl. point.
- $d = 4$: Assuming that all $\mathbb{A}^1$-curves are nodal,
  16 if P is primitive,
  14 if P is of order 2 or 6,
  8 if P is an inflection point.
- $d = 5, 6, 7, 8$: Under certain technical assumptions,
  113, 948, 8974, 92840 if P is primitive.

Local/log correspondence

Observation

The numbers 1, 1, 3, 16, 113, ... for primitive P coincide with the BPS numbers of $\mathcal{O}_{\mathbb{P}^2}(-E)$ divided by $\pm 3d$.

This suggests a certain "local/log correspondence."

In the setting of total local/log(/orbifold) GW invariants, this was proven and generalized by Gathmann, van Garrel-Graber-Ruddat, Tseng-You, Nabijou-Ranganathan, Bousseau-Brini-van Garrel, ...

Taking the multiple cover formula for Gromov-Witten invariants into account, we define the log BPS numbers as follows:

Log BPS numbers

X: smooth rational surface, D: smooth anticanonical curve,
β: a curve class

$\overline{\mathrm{M}}_\beta(X,D)$: the moduli stack of genus 0, maximal contact (basic) stable log maps

$\mathcal{N}_\beta(X,D) = \deg\,[\overline{\mathrm{M}}_\beta(X,D)]^{vir}$
"genus 0, maximal contact log GW invariant"

We define log BPS numbers m_β by

$$\mathcal{N}_\beta(X,D) = \sum_{k|\beta} \frac{(-1)^{(k-1)\beta\cdot D/k}}{k^2} m_{\beta/k}.$$

By local/log correspondence, we have

$$m_\beta = (-1)^{\beta\cdot D-1}(\beta\cdot D)\cdot n_\beta(K_X),$$

where $n_\beta(K_X)$ is the BPS number of the total space of $\mathcal{O}_X(K_X)$.

Log BPS numbers (2)

For $P \in D$ s.t. $\beta|_D \sim (\beta \cdot D)P$, let $\mathcal{N}_\beta^P(X, D)$ denote the contribution of maps with contact at P to $\mathcal{N}_\beta(X, D)$, and define m_β^P by

$$\mathcal{N}_\beta^P(X, D) = \sum_{\substack{k|\beta \\ (\beta/k)|_D \sim ((\beta/k)\cdot D)P}} \frac{(-1)^{(k-1)\beta\cdot D/k}}{k^2} m_{\beta/k}^P.$$

For $X = \mathbb{P}^2$, $D = E$: smooth cubic, $H = [\text{line}]$, m_{dH}^P for primitive P is the number of $\mathbb{A}^1$-curves (with appropriate multiplicities): $1, 1, 3, 16, 113, \cdots$.

In this case (i.e. $d \leq 8$, P: primitive), we observe that

$$m_{dH} = (3d)^2 m_{dH}^P.$$

This looks natural, since the number of $3d$-torsions is $(3d)^2$ —
but why should all points contribute the same number?

Log BPS numbers (3)

In general, we conjecture:

Conjecture

Log BPS number m_β^P is independent of P such that $\beta|_D \sim (\beta \cdot D)P$.

In other words,

$$m_\beta^P = m_\beta/(\beta \cdot D)^2 \ (= (-1)^{\beta \cdot D - 1} n_\beta(K_X)/(\beta \cdot D)).$$

For $(X, D) = (\mathbb{P}^2, \text{(smooth cubic)})$, this was proven by Bousseau, using "tropical correspondence" by Gräfnitz.

We proved this for X: del Pezzo, D: smooth anticanonical, P: "primitive" and $p_a(\beta) \leq 2$.

Method: Explicit calculation of local/log side ([1]: local, [2]: log).

Quartics in $\mathbb{P}^2$

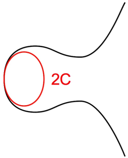
Let $E \subset \mathbb{P}^2$ be a general cubic, and $\beta = 4H$.
Let Z be the image cycle of a genus 0 stable log map of class β with maximal contact with E, and let $(\mathrm{Supp}\ Z) \cap E = \{P\}$.

P: primitive — Z is an (irreducible, reduced) $\mathbb{A}^1$-curve, and there are $\underline{16}$ such curves (counted with multiplicities).

P: of order 6 or 2

- $Z = 2C$, where C is an $\mathbb{A}^1$-curve. Contribution to log GW: $9/4$, to log BPS: $\underline{2}$ (by Gross-Pandharipande-Siebert).
- $\underline{14}$ $\mathbb{A}^1$-curves (counted with multiplicities).

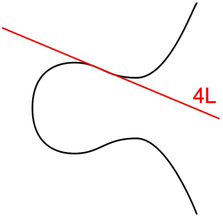
Total: $\underline{16}$.

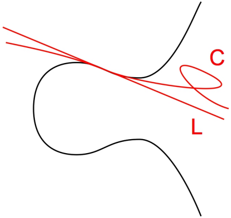


P: an inflection point,

- $Z = 4L$: Multiple cover contribution
  to log GW: $35/16$, to log BPS: $\underline{2}$.
- $Z = L + C$ ($\underline{2}$ C's): Each contributes $\underline{3}$ ... $\underline{3 \times 2 = 6}$.
- $\underline{8}$ $\mathbb{A}^1$-curves (counted with multiplicities).

Total: $\underline{16}$.





Reducible curves are unavoidable on a surface.

Theorem ([3])

For immersed $\mathbb{A}^1$-curves C_1, C_2 meeting D at P in a "general" way, $C_1 + C_2$ contributes $\min\{C_1 \cdot D, C_2 \cdot D\}$.

The case $(\#$ of components$) \geq 3$: Unknown.

Non-immersed $\mathbb{A}^1$-curves

Cubic $\mathbb{A}^1$-curves C on $(\mathbb{P}^2, E)$ at an inflection point P:

- 1 cuspidal cubic, if E is isomorphic to $y^2 = x^3 + 1$.
- 2 nodal cubics, otherwise.

If an $\mathbb{A}^1$-curve C is nodal, or more generally immersed, it is easily seen to have multiplicity 1 (infinitesimally rigid).

So, a cuspidal $\mathbb{A}^1$-curve should have multiplicity 2.

How can we calculate the multiplicity?

For a general E, it is not unreasonable to expect that all $\mathbb{A}^1$-curves are nodal, but the method is also interesting (analogy with K3 case).

Fantechi-Göttsche-van Straten's theorem (1)

Let C be a rational (integral) curve,
$\mathrm{M}_0(C,[C])$: moduli space of genus 0 stable maps to C of class $[C]$.

Set theoretically, $\mathrm{M}_0(C,[C]) = \{\nu\}$, where $\nu : \mathbb{P}^1 \to C$ is the normalization map, but it is not necessarily reduced.

Let

$$l(C) := \operatorname{length} \mathrm{M}_0(C,[C]),$$

$$m(C) := \begin{pmatrix}\text{the multiplicity of the genus } 0 \text{ locus} \\ \text{in the versal deformation space of } C\end{pmatrix}.$$

Fantechi-Göttsche-van Straten proved the following:

Theorem (Fantechi-Göttsche-van Straten)

If C has only planar singularities,

$$l(C) = m(C) = e(\overline{\mathrm{Pic}}^0(C)).$$

Fantechi-Göttsche-van Straten's theorem (2)

Theorem (Fantechi-Göttsche-van Straten)

Let S be a K3 surface, and C a rational (integral) curve on S. Then $\mathrm{M}_0(C, [C])$ coincides with $\mathrm{M}_0(S, [C])$ in a neighborhood of the normalization map $\nu : \mathbb{P}^1 \to C \subset S$.

The proof uses the relative compactified Jacobian:
Let $\mathcal{C} \to |C|$ be the universal curve over the complete linear system, and $\bar{J}\mathcal{C} \to |C|$ the associated relative compactified Jacobian.

A key fact is the following "unobstructedness":

Theorem (Mukai)

The total space $\bar{J}\mathcal{C}$ of the relative compactified Jacobian is nonsingular.

Logarithmic case: A moduli space $\mathcal{MMI}$

Back to our log setting:

- X: smooth projective surface,
- $D \subset X$: a smooth curve,
- β: a curve class. $w := \beta \cdot D$.

Definition

Let $\mathcal{MMI}_\beta(X, D)$ (modules with max. intersection) be the functor

$$(\mathrm{Sch}/\mathbb{C}) \to (\mathrm{Set}); T \mapsto \{\text{coh. sh. } \mathcal{F}/X \times T \text{ satisfying (a), (b)}\}/\cong$$

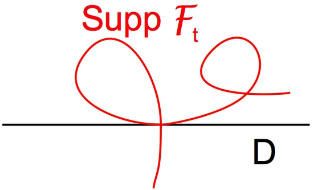
where

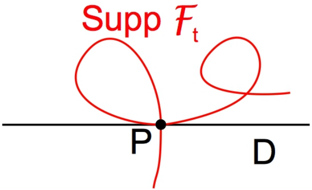
(a) $\mathcal{F}$ is flat over T, and for any geometric point t of T, $\mathcal{F}_t$ is a torsion-free sheaf of rank 1 on an integral curve C_t of class β, not contained in D.

(b) There is a section $\sigma : T \to D \times T$ such that $\mathcal{F}|_{D \times T} \cong \mathcal{O}_{w \cdot \sigma(T)}$.

Logarithmic case: A moduli space $\mathcal{MMI}$ (2)

$\mathcal{MMI}_\beta(X, D)$ is represented by a (non-proper) scheme
(again denoted by $\mathcal{MMI}_\beta(X, D)$)





For a point P with $\beta|_D \sim wP$, let $\mathcal{MMI}^P_\beta(X, D)$ be subscheme of $\mathcal{MMI}_\beta(X, D)$ representing $\mathcal{F}$ s.t. $\mathcal{F}|_{D\times T} \cong \mathcal{O}_{w\cdot(\{P\}\times T)}$.

Let $|\mathcal{O}_X(\beta, P)|^{\circ\circ}$ denote the set

$$\{C \in |\beta| : \mathrm{Supp}\ C \not\supseteq D, C|_D = wP \text{ and } C \text{ is integral}\}$$

regarded as an open subvariety of a projective space.

Unobstructedness of $\mathcal{MMI}$

Now let X be a smooth rational projective surface,
and D an anticanonical curve.

Theorem ([3])

- *$\mathcal{MMI}_\beta^P(X, D)$ can be identified with an open subscheme of the relative compactified Picard scheme over $|\mathcal{O}_X(\beta, P)|^{\circ\circ}$.*
- *$\mathcal{MMI}_\beta^P(X, D)$ is an open and closed subscheme of $\mathcal{MMI}_\beta(X, D)$.*
- *$\mathcal{MMI}_\beta(X, D)$ and $\mathcal{MMI}_\beta^P(X, D)$ are nonsingular of dimension $2p_a(\beta)$.*

Thus, if $C \in |\mathcal{O}_X(\beta, P)|^{\circ\circ}$ and F is a rank 1, torsion-free sheaf on C invertible near P, the relative compactified Picard scheme over $|\mathcal{O}_X(\beta, P)|^{\circ\circ}$ is nonsingular at $[F]$.

Multiplicity of an $\mathbb{A}^1$-curve

In particular, if C is nonsingular at P, the relative compactified Picard scheme is nonsingular at any point over $[C]$.

From the arguments of [FGS], this implies the following:

Theorem ([3])

Let C be an $\mathbb{A}^1$-curve on (X, D) which is nonsingular at $P = C \cap D$.

Then the natural map $\mathrm{M}_0(C, [C]) \to \overline{\mathrm{M}}_\beta(X, D)$ is an isomorphism to a 0-dimensional connected component.

Thus the contribution of C to the log GW invariant $\mathcal{N}_\beta^P(X, D)$ (and the log BPS number m_β^P) is equal to

$$l(C) = e(\overline{\mathrm{Pic}}^0(C)).$$

An ingredient of the proof: Deformation theory

C: integral curve on X with $C|_D = wP$,
F: rank 1, torsion-free on C, invertible at P.

Tangent space to $\mathcal{MMI}_\beta^P(X, D)$ at $[F]$ is

$$\mathrm{Im}(\mathrm{Ext}^1_{\mathcal{O}_X}(F, F(-D)) \to \mathrm{Ext}^1_{\mathcal{O}_X}(F, F)).$$

Tangent space to $\mathcal{MMI}_\beta(X, D)$ at $[F]$ is

$$\mathrm{Im}(\mathrm{Ext}^1_{\mathcal{O}_X}(F, F(-(w-1)P)) \to \mathrm{Ext}^1_{\mathcal{O}_X}(F, F)),$$

where

$$F(-(w-1)P) = \mathrm{Ker}(F \to (F|_D)|_{(w-1)P}).$$

These spaces coincide.

Compactification?

How can we compactify $\mathcal{MMI}_\beta(X, D)$?

Use expansion of X:

- Maulik-Pandharipande-Thomas, *Curves in K3 surfaces and modular forms*
- Li-Wu, *Good degeneration of Quot-schemes and coherent systems*
- Maulik-Ranganathan, *Logarithmic Donaldson-Thomas theory*

Logarithmic structure on a coherent sheaf?

Unobstructedness?