A tale of 4 theories

Joint with:

- Ranganathan
- Battistella, Tseng, You.
- Battistella, Ranganathan (upcoming).

- $(X \mid D = D_1 + \dots + D_r)$

  smooth

  snc divisor.

- Setup enumerative geometry problem:

$g \geq 0$, $\beta \in H_2^+(X)$, $n \geq 0$

$\{1, \dots, r\}$

$\alpha_i^j \in \mathbb{N}$

$\{1, \dots, n\}$

"Tangent to D_j to order α_i^j at p_i." $(*)$

$\boxed{g = 0}$

$$\boxed{\mathrm{Nice}(X|D)} = \left\{ \mathbb{P}^1 \xrightarrow{f} X \ : \ f^* D_j \overset{(*)}{=} \sum_{i=1}^{n} \alpha_i^j \cdot p_i \right\} \subseteq \overline{M}(X)$$

Not compact.

How to compactify?

① Log stable maps.

Logarithmic structure:
sheaf of monomial
functions.

$U = \operatorname{Spec} \boxed{k[Q]}$ ← monoid $\sigma^\vee \cap M$.
Has preferred
functions.

$$\boxed{f^* s_j = \lambda \cdot \prod_{i=1}^{n} t_i^{\alpha_i^j}}$$

logarithmic
morphism.

Leqn for $D_j \subseteq X$

Leqn for $P_i \in C$.

$\rightsquigarrow \boxed{\mathrm{Log}(X|D)}$

$\cup\!|$ snc compactification

$\mathrm{Nice}(X|D)$.

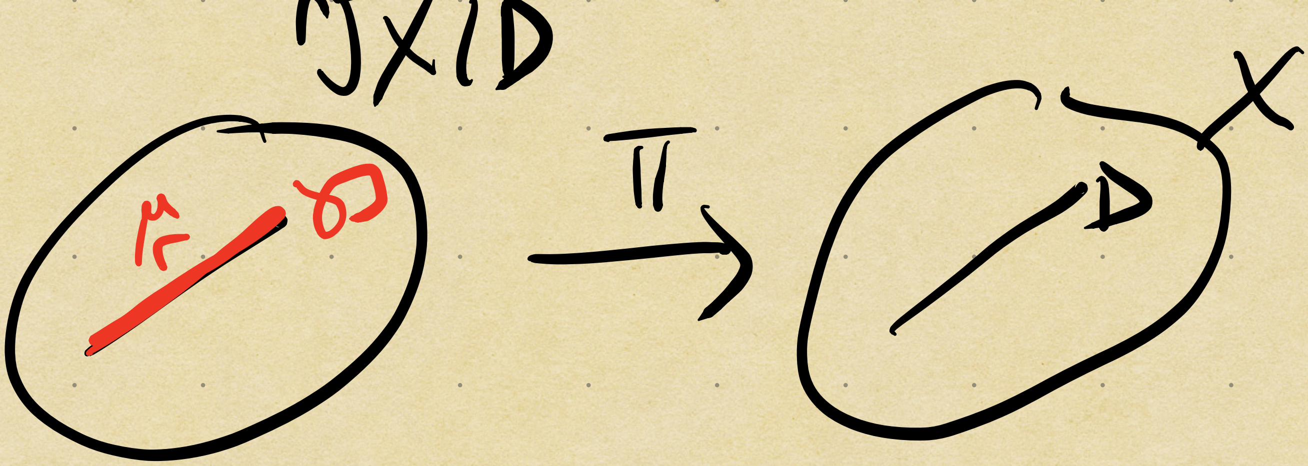
② orbifold structures

$\begin{cases} (X|D) & \text{smooth pair.} \\ r \in \mathbb{N} \end{cases}$

$\rightsquigarrow \sqrt[r]{X|D}$ root stack

$\Big\downarrow$ coarse moduli scheme

X



$$S_{\mathcal{D}}^{r} = \pi^* S_D.$$

$$\mathcal{D} = \frac{1}{r}\cdot \pi^* D.$$

<u>Idea</u>: Tangency condition becomes open in space of orbifold maps.

↳ Tangency orders are encoded in twisted sectors.

$$\begin{tikzcd} \sqrt[r]{C|P_1} \arrow[r] \arrow[d] \arrow[dd, bend right] & \sqrt[r]{X|D} \arrow[d] \\ C \arrow[r, "f"] & X \\ \boxed{\mathcal{M}_r \to \mathcal{M}_r} & \end{tikzcd}$$

$P_1 \longmapsto D$

$\rightsquigarrow$ $\boxed{\mathcal{M}_r \to \mathcal{M}_r}_{m}$

$\rightsquigarrow$ $\underbrace{\mathrm{Orb}(X|D)}$ $\longrightarrow \sqrt[r]{X|D}$

$\cup$

$\mathrm{Nice}(X|D)$.

$\underline{\mathrm{Th}^{m}}$ (Abramovich–Cadman–Wise, Tseng–You):

D smooth, $g=0$

$$\mathrm{Log}(X|D) \underset{\text{v. bir.}}{\simeq} \mathrm{Orb}(X|D).$$

<u>Q</u>: what about

$$\boxed{D \text{ snc},\ g=0}\ ?$$

$$(X|D) = (X|D = D_1 + D_2).$$

<u>Rank reduction</u>:

$$\boxed{GW(X|D) = \mathrm{Process}\big(GW(X|D_1),\ GW(X|D_2)\big)}$$

$\ldots(X|D_2))$.

$$(X|D) = (\mathbb{P}^n \mid H_1 + H_2).$$

③ Naive space:

Has a VFC.

$$\begin{tikzcd}
\boxed{\mathrm{Naive}(X|D_1+D_2)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathrm{Log}(X|D_1) \arrow[d, hook] \\
\mathrm{Log}(X|D_2) \arrow[r, hook] & \overline{\mathcal{M}}(X)
\end{tikzcd}$$

Smooth.

$$\mathrm{Naive}(X|D_1+D_2) = \mathrm{Log}(X|D_1) \cap \mathrm{Log}(X|D_2)$$
$$\subseteq \overline{\mathcal{M}}(X).$$

Thm (Battistella–NN–Tseng–You):

$$\mathrm{Naive}(X/D) \underset{\text{v.bir.}}{\simeq} \mathrm{Orb}(X/D).$$

$$\simeq \mathrm{Local}(X/D).$$

Pf: orbifold satisfies a product formula over $\pi(X)$. $\square$

Reduce to comparing

$$\underline{\log} \xleftrightarrow{?} \underline{\text{naive}}. \quad [\text{BNTY}]$$

$$\begin{gathered} \text{naive} \\ \| \\ \text{orbifold} \\ \| \\ \text{local.} \end{gathered}$$

Observation (NN-Ranganathan):

$$\mathrm{Log}(X|D) \underset{\text{b.bir.}}{\not\simeq} \mathrm{Naive}(X|D)$$

$(\mathbb{P}^n | H_1 + H_2)$: log convex

irreducible

contains nice locus as dense open.

Has many irreducible components, of excess dimension

Closure of Naive(X|D).

$$\mathrm{Log}(X|D) \subsetneq \mathrm{Naive}(X|D)$$

bigger.

<u>Expect:</u>

$$\boxed{GW^{\mathrm{Naive}}(X|D) = GW^{\log}(X|D) + \text{corrections}}$$

---

①

not transverse.

$$\mathrm{Naive}(X|D) = \underline{\mathrm{Log}(X|D_1)} \cap \underline{\mathrm{Log}(X|D_2)}$$

$$\subseteq \overline{M}(X).$$

Thm (NN-Ranganathan):
$\exists$ explicit sequence of blowups of $\overline{M}(X)$ which transversalizes the intersection.

$\downarrow$

Fulton's blowup formula relates strict transforms and total transforms.

$\rightsquigarrow$ get comparison.

---

② Log GW inv'ts satisfy birational invariance

$$(C, P_1, \ldots, P_n) \to (X, D)$$

$$\rightsquigarrow \boxed{C \setminus \{P_1, \ldots, P_n\} \to X \setminus D}$$

Naively: counts don't change if you blowup a locus in D.

Correct: don't change if you perform a log blowup $(X|D)$.

(Abramovich-wise)

<u>Issue</u>: naive/orbifold invts don't satisfy birational invariance.

<u>Thm</u> (Battistella-NN-Ranganathan):

$$\cdots \to (X|D)^{\dagger\dagger} \to (X|D)^{\dagger} \to \underset{\text{snc}}{(X|D)}$$

invariant stabilises after fin. many blowups, and then have:

$$\log = \text{orbifold/naive}.$$