Topological SYZ fibrations with discriminant in codimension 2

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Strominger-Yau-Zaslow conjecture (1996):

X is an n-dimensional Calabi-Yau manifold, and $\check{X}$ is its mirror. Then there are two special Lagrangian "dual" torus fibrations

$$\begin{array}{ccc} X & & \check{X} \\ & \searrow \quad \swarrow & \\ & B & \end{array}$$

Big question: What about the singularities?

Kontsevich-Soibelman (2000):

X_t is a maximally degenerate CY family. Pick a Kähler class and let w_t be the CY metric in it.

$(X_t, w_t/(\text{diam X})^2) \to (B, g)$ is the Gromov-Hausdorff limit.

Conjecture:
$B_0 := B(\cong S^n) \setminus D$ is a $\mathbb{Z}$-affine manifold, i.e TB_0 has a flat connection with holonomy in $\mathrm{SL}_n(\mathbb{Z})$. D has codimension ≥ 2.
g is a Monge-Ampère metric: in affine coordinates $g_{ij} = \frac{\partial^2 K}{\partial y_i \partial y_j}$ with $\det g_{ij} = 1$.

$\Lambda \subset TB_0$, $\check{\Lambda} \subset T^*B_0$, integral lattices of flat sections.
The mirror pair: $X, \check{X}$ compactifications of $TB_0/\Lambda, T^*B_0\check{\Lambda}$.

Gross-Siebert: starting with (B, D) reconstruct $X, \check{X}$.

Input:

- B is a PL manifold and a regular CW-complex.
- $D \subset \operatorname{bsd} B \setminus$ (stars of vertices $\cup$ facets of B) a codimension 2 subcomplex in $\operatorname{bsd} B$.
- Monodromy assumptions (**semi-simple polytopal** singularities) at $y \in B_0$ a point near a face σ of D.

$L_\sigma \subset \Lambda_y, \check{L}_\sigma \subset \check{\Lambda}_y$ are invariant sublattices.

1. $\Delta_1, \ldots, \Delta_r$ convex lattice polytopes, they spans $L_i := \langle \Delta_i \rangle$ linearly independent sublatiices in L_σ.
   $\check{\Delta}_1, \ldots, \check{\Delta}_r$ convex lattice polytopes, they spans $\check{L}_i := \langle \check{\Delta}_i \rangle$ linearly independent sublattices in $\check{L}_\sigma$ orthogonal to all L_i's.
   Compatible: $\tau \prec \sigma$, $r_\sigma \leq r_\tau$, and under $L_\sigma \hookrightarrow L_\tau$, $\check{L}_\sigma \hookrightarrow \check{L}_\tau$ $\Delta_i^\sigma \preceq \Delta_i^\tau$ and $\check{\Delta}_i^\sigma \preceq \check{\Delta}_i^\tau$.

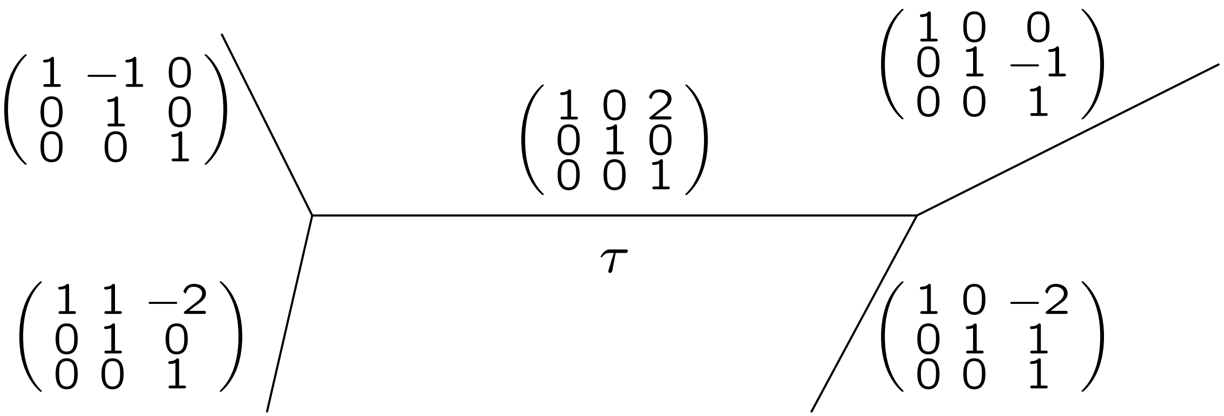
2. Y_i is the codimension one skeleton of the normal fan to Δ_i (AKA tropical hyperplane). Then D locally is homeomorphic to the union of $\mathbb{R}^s \times Y_i \times \check{Y}_i \times \mathbb{R}^{\ell - k_i + \check{\ell} - \check{k}_i}$.

3. The local monodromy along the loop around a facet $(e, f) \subset D$, e edge in Δ_i and f edge in $\check{\Delta}_i$, some i is given by id $+ e \otimes f$.

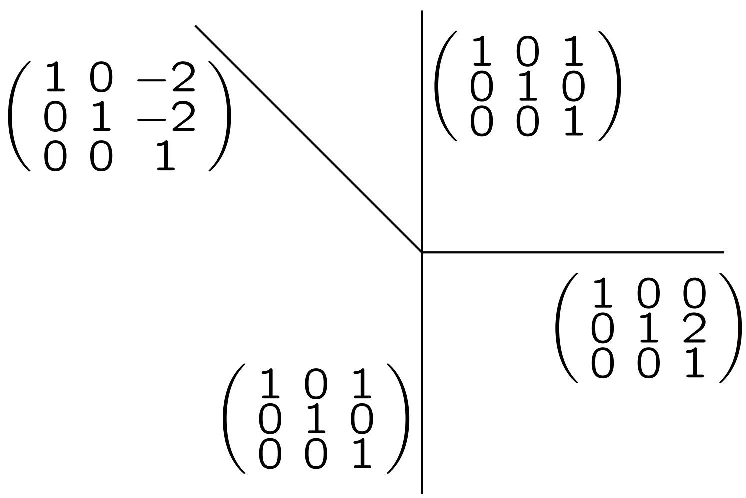
Local monodromy in a suitable basis (up to finite index):

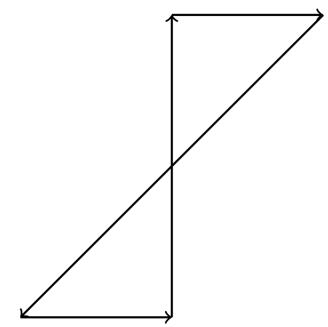
$$\begin{pmatrix} 1 & 0 & \cdots & 0 & \boxast & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 & 0 & \ddots & \ddots & \vdots \\ \vdots & 0 & \ddots & \vdots & \vdots & \ddots & \ddots & 0 \\ \vdots & \vdots & \ddots & 1 & 0 & \cdots & 0 & \boxast \\ 0 & \cdots & \cdots & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & 1 & \ddots & \vdots \\ 0 & \cdots & \cdots & 0 & \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}$$

A non-polytopal semi-simple affine structure:



Another non-polytopal semi-simple affine structure:





Theorem: Under the semi-simple simplicial assumptions both

$$X_0 = TB_0/\Lambda, \quad \check{X}_0 = T^*B_0/\check{\Lambda}$$

compactify to topological orbifolds $X, \check{X} \to B$ which are half-dimensional fibrations. X is a manifold if all simplices Δ's are unimodular, and similar for $\check{X}$.

Bonus: explicit description of the singular fibers.

2-dimensional case with focus-focus singularities $\begin{pmatrix} 1 & k \\ 0 & 1 \end{pmatrix}$: easy. Gross-Wilson (2000) did the CY metric version for K3.

Gross (2001): the quintic threefold.

Gross-Siebert program:

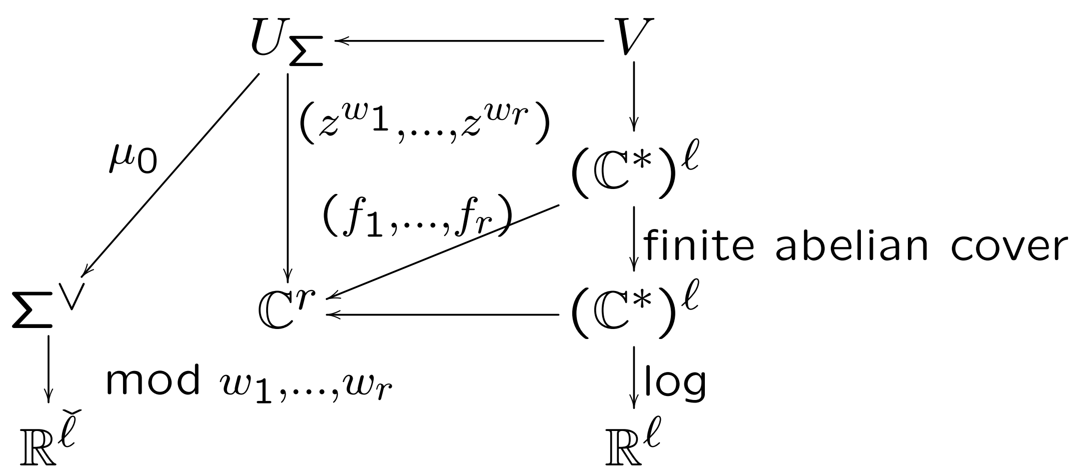
Let $(B, \mathcal{P}, \phi)$ be a $\mathbb{Z}$-affine manifold with semi-simple (elementary simplices) singularities. Then one can construct an algebraic scheme X_c with log-structure on it.

Sometimes (?) there is an analytic family $\mathcal{X}$ with X_c as a central fiber.

Theorem: The Kato-Nakayama space X_{KN} is homeomorphic to $\check{X}$.

The main problem: the natural map $X_{KN} \to B$ has discriminant in codimension 1.

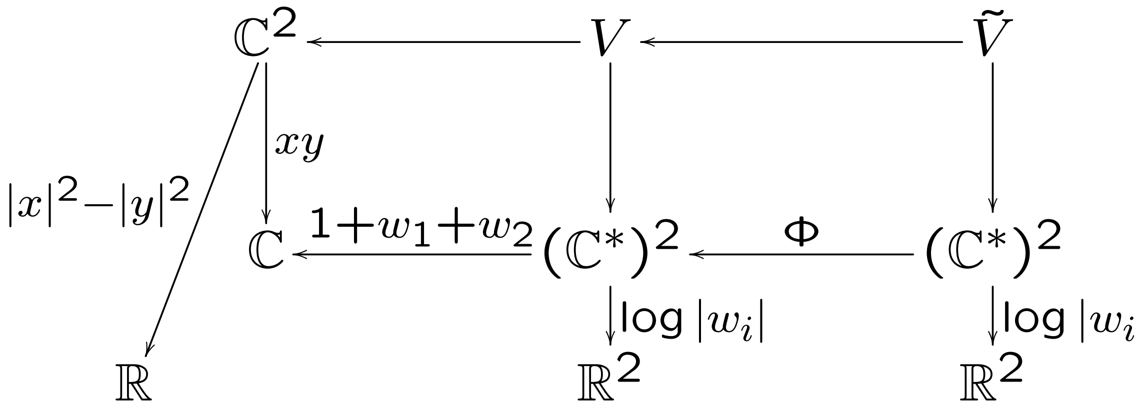
The local model near $\sigma \subset D$:

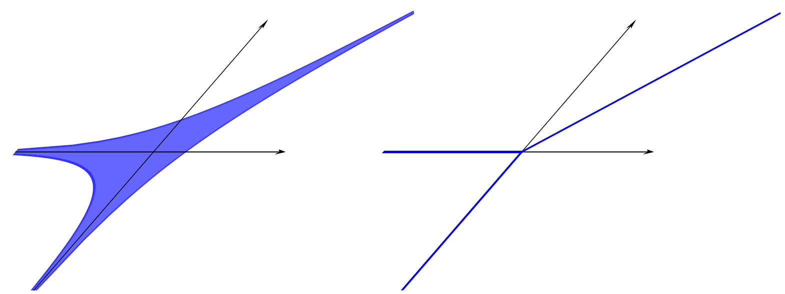


Each Δ_i defines $f_i = \sum_{v \in \mathrm{vert}\,\Delta_i} c_v z^v : (\mathbb{C}^*)^{l_i} \to \mathbb{C}$.

Σ is the cone over the convex hull $\mathrm{Conv}\{(\check{\Delta}_i, e_i)\} \subset \check{L}_\mathbb{R} \oplus \mathbb{R}^r$.
$\Sigma^\vee$ the dual cone and $U_\Sigma = \mathrm{Spec}\,\mathbb{C}[\Sigma^\vee_\mathbb{Z}]$ the associated affine toric variety.
$w_i(\check{\Delta}_i) = 1, \quad w_i(\check{\Delta}_j) = 0,\ j \neq i$, define the map $U_\Sigma \to \mathbb{C}^r$.

3D example: Δ is the standard 2-simplex, $\check{\Delta} = [0,1]$.
$\{xy = 1 + w_1 + w_2\} \subset \mathbb{C}^2 \times (\mathbb{C}^*)^2$





Hopf-type S^1-fibration over $\mathbb{R} \times (\mathbb{C}^*)^2$.
The fibers collapse over the surface $\{0\} \times \{1 + w_1 + w_2 = 0\}$.

Want: A $\mathbb{T}^n$-fibration $(\mathbb{C}^*)^n \to \mathbb{R}^n$ such that the image of $P := \{1 + w_1 + \cdots + w_n = 0\}$ is the tropical hyperplane $Y \subset \mathbb{R}^n$.

Instead we will introduce two new tropical objects, both fiber over Y:

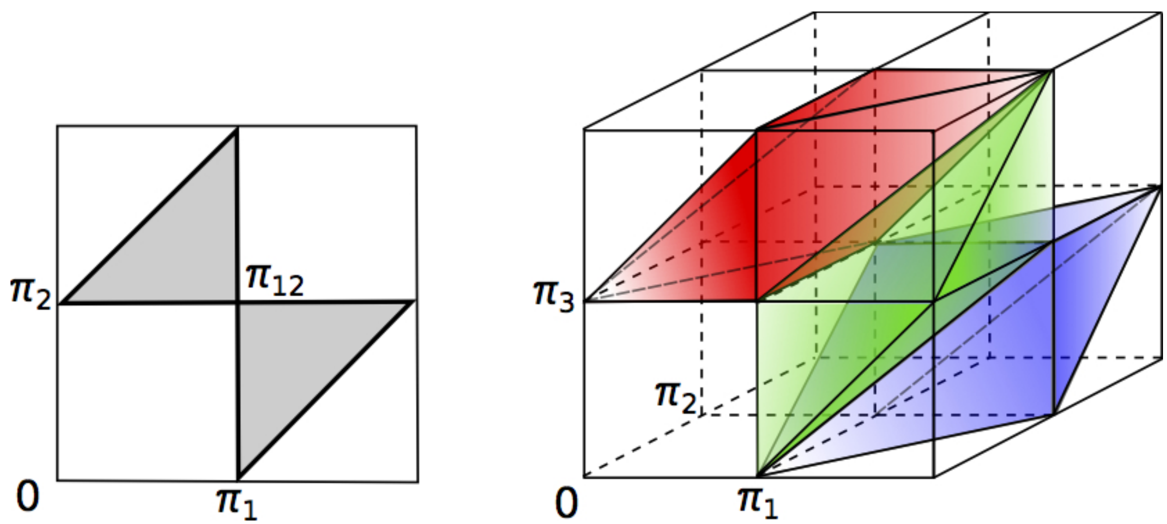
- Phase tropical pair-of-pants $\mathcal{TP}$.

- Ober-tropical pair-of-pants $\mathcal{OP}$.

Theorem: All three subspaces $P, \mathcal{TP}, \mathcal{OP} \subset (\mathbb{C}^*)^n$ are (ambient) isotopic.

The $(n-1)$-dimensional pair-of-pants P is the complement of $n+1$ generic hyperplanes in $\mathbb{P}^{n-1}$.
In homogeneous coordinates: $z_0 + z_1 + \cdots + z_n = 0$ in $(\mathbb{C}^*)^{n+1}/\mathbb{C}^*$.

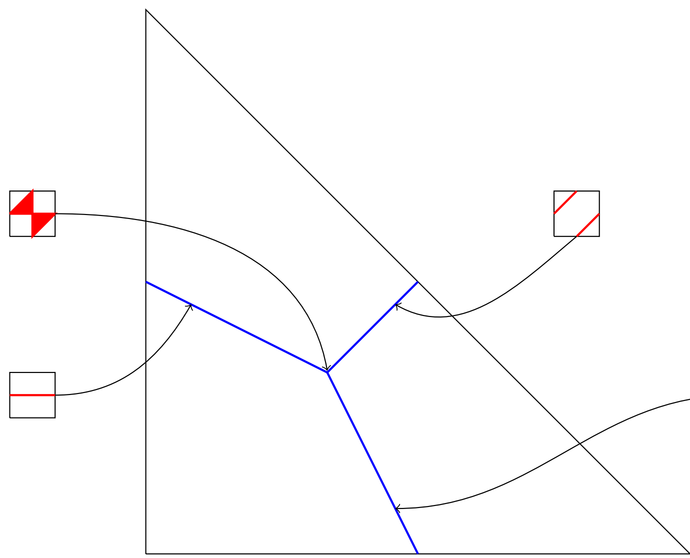
The amoeba $\mathcal{A}$ is the image of the Log map. It's convenient to compactify $\mathbb{R}^n$ to Δ, then $\mathcal{A} \subset \Delta$ is the hypersimplex.
The coamoeba is the image of the argument map: $\mathcal{C} := \overline{\text{Arg}(P)} \subset \mathbb{T}^n$.



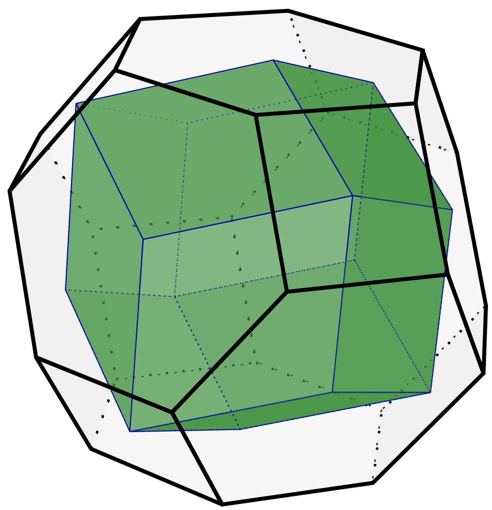
Y is the spine (the skeleton) of the amoeba $\mathcal{A}$.
The cones in Y are labeled by subsets $J \subseteq \{0, 1, \ldots, n\}$, $|J| \geq 2$.

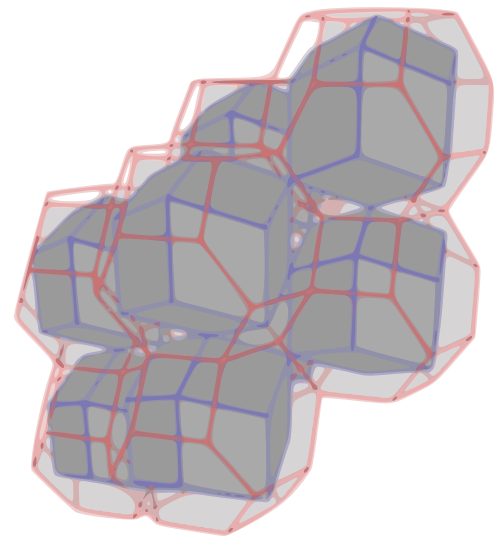
Partial coamoebas: $\mathcal{C}_J := \overline{\mathrm{Arg}(\{\sum_{j \in J} z_j = 0\})}$.

The phase tropical pair-of pants: $\mathcal{TP} := \bigcup_J Y_J \times \mathcal{C}_J \subset \Delta \times \mathbb{T}^n$.



The coamoeba for $n = 3$: the zonotope (its complement) and the permutahedron (its skeleton S)

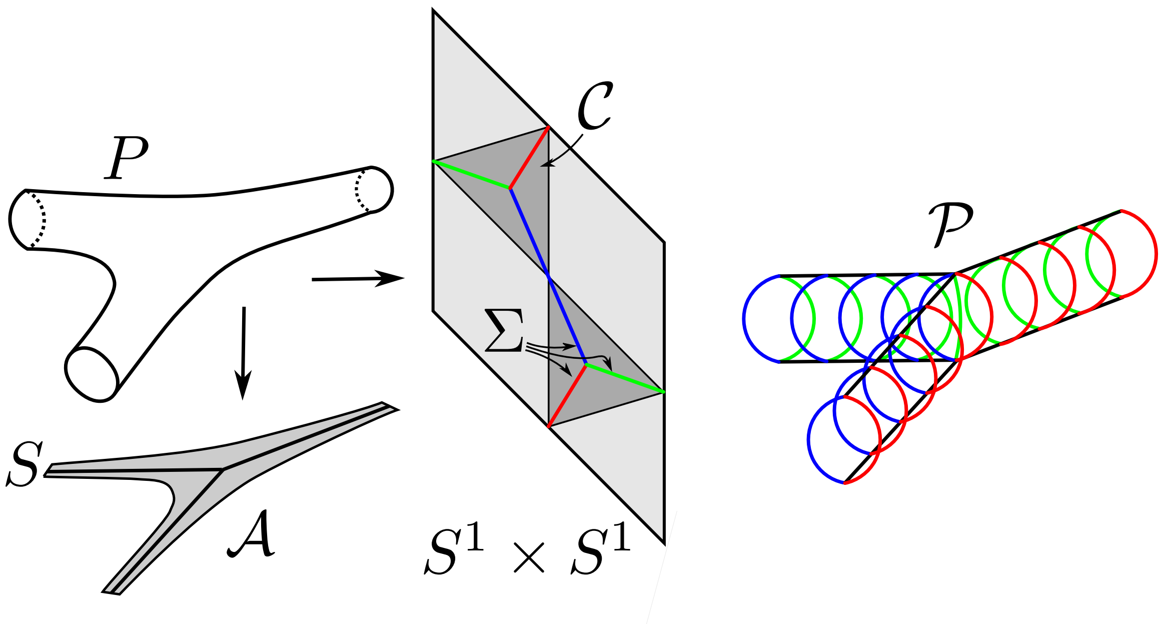




The faces of S are labeled by cyclically ordered partitions $\sigma = \langle I_1, \ldots, I_k \rangle$ of $\{0, 1, \ldots, n\}$.

The ober-tropical pair-of pants:
$\mathcal{TP} := \bigcup_J Y_J \times S_\sigma$ such that J is not in a single part of σ.

The $n = 2$ ober-tropical pair-of-pants:

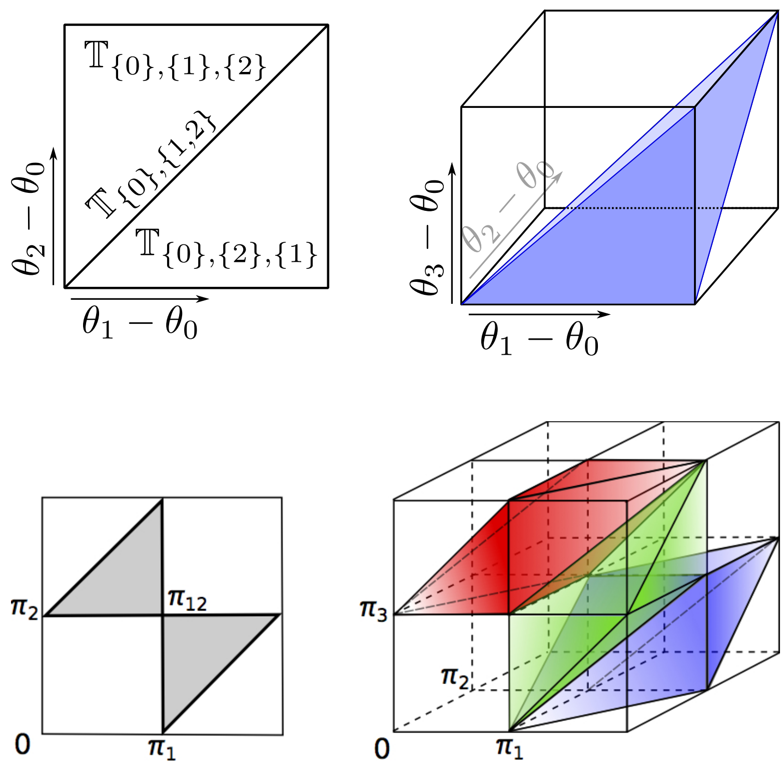


Symplectic geometer's Dream:

Fibers over generic points of S are homeomorphic to $\mathbb{R}^{n-1}$.

Fibers over generic points of Y are homeomorphic to $\mathbb{T}^{n-1}$.

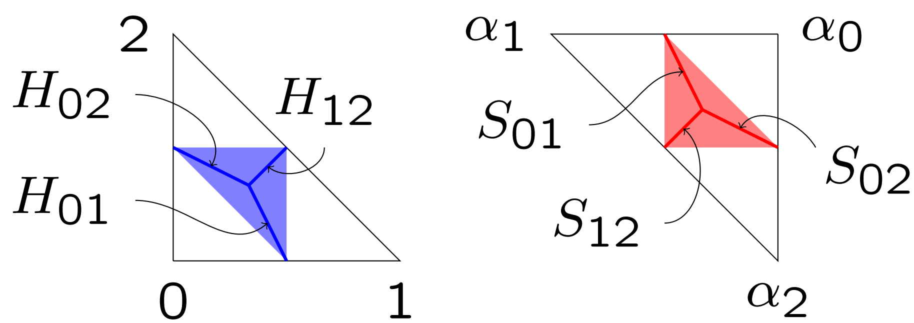
Proof of the Isotopy Theorem:
Break $\mathbb{T}^n$ into $n!$ simplices by ordering the arguments of z_i.

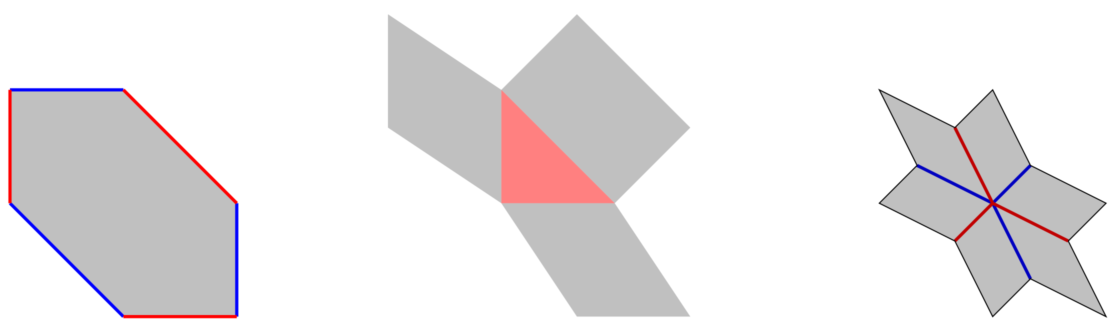


Each $(\Delta_J \times \mathbb{T}^n_\sigma, \mathcal{P}_{J,\sigma})$ is the standard ball pair.

Equal rights: Both amoeba and coamoeba are hypersimplices in Δ and they have very similar skeleta.

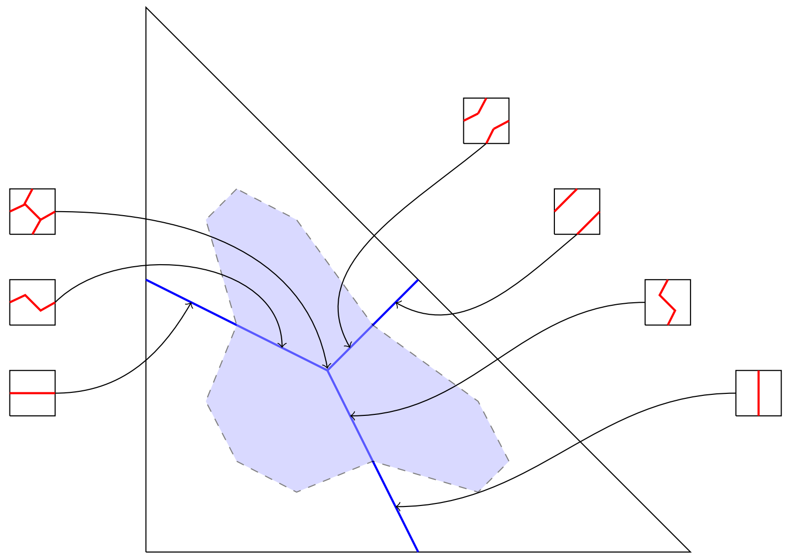
Example: the three balls in $\Delta^2 \times \Delta^2$:





Advantages and disadvantages: phase vs. ober

– Fading off the wiggling of red circles along $Y \subset \Delta$ for the ober-tropical model.



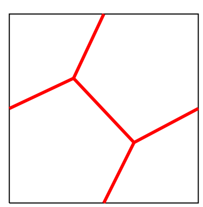
– No wiggling in the phase tropical model.

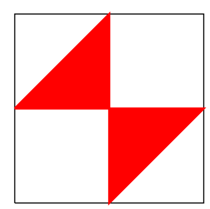
Singular fibers:

– Ober-tropical: fibers are equi-dimensional.

– Phase tropical: fibers are not equi-dimensional.

Example: The most degenerate fiber for the local model $xy = 1 + w_1 + w_2$ is the S^1 fibration over $\mathbb{T}^2$ where the circle collapses over the coamoeba (in the phase tropical case) or over its skeleton (in the ober-tropical case)





An application to mirror symmetry: lifting tropical cycles C in B to holomorphic cycles in X and to Langrangians in $\check{X}$.

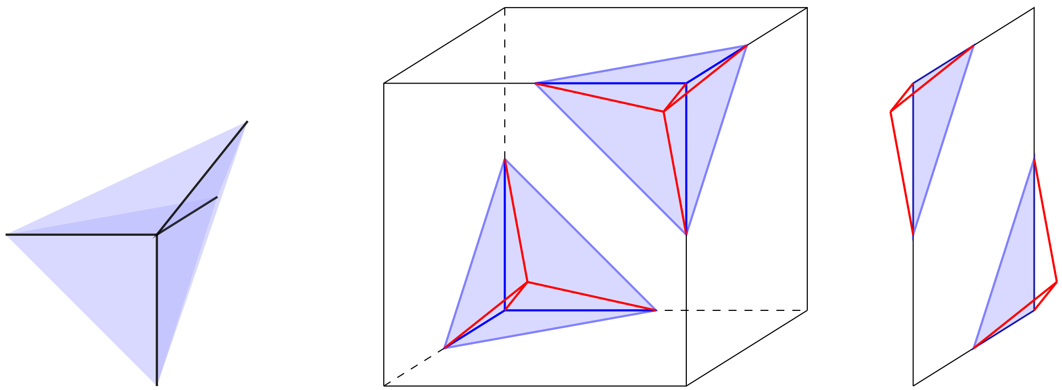
Matessi [2018], Mikhalkin [2019], Abouzaid-Ganatra-Iritani-Sheridan [2018], Ruddat-Siebert [2019], Wang [2020].

Two main issues:
(1) local lifts over the smooth part $B_0 \subset B$ (no problem in codimension ≤ 1): geometry of C;
(2) analyzing the behavior at the discriminant (no problem in dimension ≤ 1): geometry of $X, \check{X}$.

– $\dim C = 0$ or n is easy.

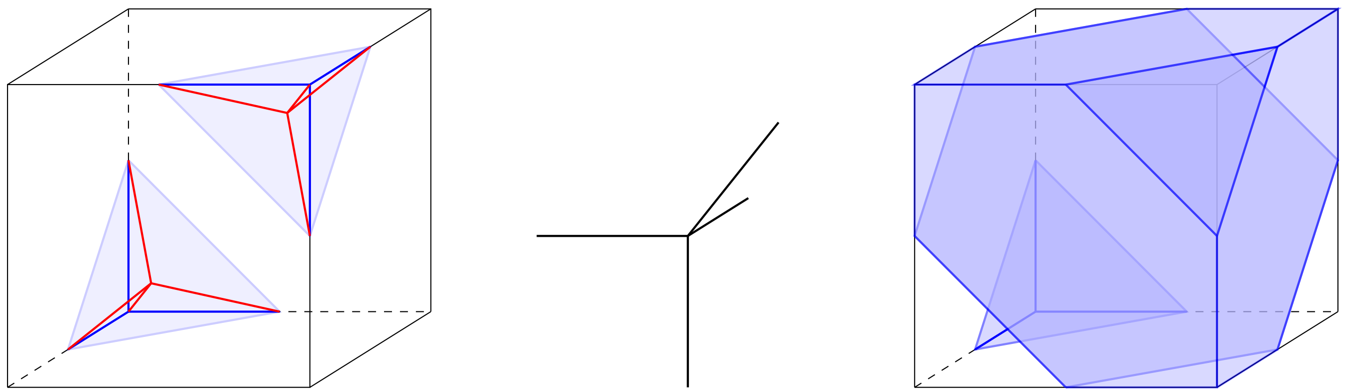
– C is (locally) a hyperplane.

Matessi's cocoamoeba and its skeleton: over the vertex and a ray in tropical hyperplane in $\mathbb{R}^3$.



Question: what happens at the discriminant?

– C is a curve. 3-valent vertices = hyperplane case.
Lifting the 4-valent vertex (the 4-punctured $\mathbb{P}^1$):



"Holomorphic" side is ok.
But "Lagrangian" is not a manifold : Link at each vertex is $\mathbb{RP}^2$!
Mikhalkin: The topology is different for each of the 3 resolutions.

Question: Is it possible to lift the 2 -skeleton of the tropical hyperplane in $\mathbb{R}^4$ to a manifold in either X or $\check{X}$? Holomorphic lift = the complement of 5 generic lines in $\mathbb{P}^2$. Lagrangian = ?

THANK YOU!