IRREGULAR HODGE STRUCTURES IN LANDAU-GINZBURG MODELS AND IN ARITHMETIC

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A. Variety with (super)potential

Y, Fano variety ....

$\updownarrow$ Mirror

(X, f), Landau-Ginzburg model

X sm, quasi-proj.

$f : X \to \mathbb{A}^1$ morphism

Specialization of Homological Mirror Symmetry :

$$H^{\cdot}(Y; \mathbb{C}) \simeq H^{\cdot}(X, f^{-1}(a); \mathbb{C}) \qquad (a \ll 0)$$

cf. Katzarkov-Kontsevich-Pantev, JDG 2017

E.g. $\quad Y = \mathbb{P}^n \quad ; \quad X = \mathbb{G}_m^n$

$$f = x_1 + \cdots + x_n + \frac{1}{x_1 \cdots x_n}$$

$Y = \mathbb{P}(w_0, \ldots, w_n)$ weighted proj. sp. with $(w_0, \ldots, w_n) = 1$.

$$\begin{tikzcd} \mathbb{G}_m^n \simeq X = \ker\{ \mathbb{G}_m^{n+1} \arrow[r, "x^w"] \arrow[d, "x_0 + \cdots + x_n"] & \mathbb{G}_m \} \\ \mathbb{A}^1 & \end{tikzcd}$$

with $f : X \to \mathbb{A}^1$.

Cf. Douai-Mann, The small quantum cohom. ...., 2013.

Hodge/de Rham aspects for (X, f)

an isomorphism

$$H^a(X, f^{-1}(a); \mathbb{C}) \simeq H^a_{dR}(X, f) := \mathbb{H}^a(X, (\Omega_X^\cdot, d + df))$$

By KKP, Esnault-Sabbah-Yu, M. Saito, T. Mochizuki.
one has a filtration on $H^\cdot_{dR}(X, f)$ with

$$F^\lambda / F^{\lambda+1} = H^\cdot(\bar{X}, \mathcal{F}_\lambda)$$

(I) $(\bar{X}, \bar{f})$ non-degenerate compactification of (X, f)

$\lambda = -\alpha + p$, $0 \le \alpha < 1$

$$\mathcal{F}_\lambda = \ker\left\{ \Omega^p(\log \bar{X} \setminus X)(\lfloor \alpha P \rfloor) \xrightarrow{d\bar{f}} \Omega^{p+1}(*) \big/ \Omega^{p+1}(\log)(\lfloor \alpha P \rfloor) \right\}$$

(II) We have $\mathbb{C}[u]$-module, Brieskorn lattice,

$$\mathbb{H}^{\cdot}\left(X, \left(\Omega^{\bullet} \otimes u^{-\bullet}\mathbb{C}[u],\ d + \tfrac{1}{u}df\right)\right) \qquad \text{(say } f \text{ proper)}$$

- It is free with (fiber at $u=1$) $= H_{dR}(X, f)$

$\rightsquigarrow$ trivial sheaf $\mathcal{K}^*$ on $\mathbb{A}^1$

- It is equipped with action of ∂_u, with pole of order 2 at $u=0$
- It is regular singular at $u=\infty$; monodromy is quasi-unipotent

Let $\mathcal{K}^{\lambda}$ = loc. free on $\mathbb{P}^1$, extending $\mathcal{K}^*$ s.t.

eigenvalues of $\mathrm{Res}_{u=\infty}\nabla \subset [\lambda, \lambda+1)$

Then $F^{\lambda} = \mathrm{image}\left\{ \Gamma(\mathbb{P}^1, \mathcal{K}^{\lambda}) \to H_{dR}(X, f) \right\}$

B. Toric case

$$X = \mathbb{G}_m^n \ , \quad f \ \text{non-degenerate}$$

(1) $$H^a_{dR}(X, f) = \begin{cases} \left[\mathbb{C}[x_1, \dots] \big/ (x_1 \partial_{x_1} f, \dots) \right] \cdot \frac{dx_1}{x_1} \cdots & \text{if } a = n \\ 0 & \text{if } a \neq n \end{cases}$$

$$x^\delta \in F^\lambda \quad \text{if} \quad \delta \in (n - \lambda) \cdot NP(f)$$

E.g. $f = x_1 + \dots + x_n + \frac{1}{x_1 \cdots x_n}$

$$H^n = \langle \underset{\substack{\Uparrow \\ F^{n-k}}}{x_1^k} \rangle_{k=0}^{n} \ , \quad \text{jumps are } \lambda = 0, 1, \dots, n$$

E.g. Douai - Sabbah, 2004

$$X = \ker\{ \mathbb{G}_m^{n+1} \xrightarrow{x^w} \mathbb{G}_m \}, \quad f = \text{restr. of } \sum w_i x_i$$

- Explicit description of ∂_u on $\mathcal{K}^*(X,f)$ of rk $\mu = \sum w_i$.

Hodge Jumps are $n + s_k - k$:

arrange $\mu \cdot \dfrac{a}{w_i}$ $\quad \begin{matrix} 0 \le i \le n \\ 0 \le a \le w_i - 1 \end{matrix}$ as $s_0 \le s_1 \le \cdots \le s_{\mu-1}$

Confluent hypergeom $\quad {}_0F_{\mu-1}\left(\underbrace{1, \cdots, 1}_{n}, \frac{1}{w_0}, \cdots, \frac{w_n - 1}{w_n} ; u\right)$

$\rightsquigarrow$ connection ∇ of rk μ.

$$[v \mapsto v^\mu]^* \nabla \simeq \mathcal{K}^*(X,f)|_{\mathbb{G}_m}$$

(2) Kim - Sabbah 2008

Further assume

- f convenient
- $\mathrm{crit}(f)$ simple
- critical values are distinct

Then $K^*(X, f)$ can be upgraded into a

Frobenius manifold. (via Hodge theory)

Question. Generalize the constr. of Frob. mfds ?

Under what conditions for (X, f) ?

E.g. Setting in KKP ?

C. Arithmetic applications.

Irreg. Hodge / exp. motives may

- provide flexibility to do arithmetic
- bridge exp. int and classical periods

Example (moments of Bessel functions)

Kloosterman connection Kl on $\mathbb{G}_{m,z}$

$$Kl = R[\mathbb{G}_m^2 \xrightarrow{\pi} \mathbb{G}_m]_* \left(d + d\left(x + \frac{z}{x}\right)\right) \qquad rk = 2$$

Symmetric power moments

$$H^1_{dR}(\mathbb{G}_m, \mathrm{Sym}^k Kl)$$

- Periods contains $\int_0^\infty I_0^a(t)\, K_0^{k-a}(t)\, t^{2b} \frac{dt}{t}$.

  $I_0(t)$, $K_0(t)$ are modified Bessel functions, ann. by $(t\partial_t)^2 - t^2$.

- irreg. Hodge jumps :

  $0, 2, 4, \dots, k-1$ (k odd)

- $$\begin{aligned} H^1_{dR}(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl}) &\subset H^1_{dR}(\mathbb{G}_m, \mathrm{Kl}^{\otimes k}) \\ &= H^{k+1}_{dR}\left(\mathbb{G}_m^{k+1}, \sum x_i + \frac{z}{x_i}\right) \\ &\subset H^{k+1}_{dR}\left(\mathbb{G}_m^{k+1}, t\sum y_i + \frac{1}{y_i}\right) \\ &\simeq H^{k+1}_{dR}\left(\mathbb{A}^1_t \times \mathbb{G}_m^k, t\sum y_i + \frac{1}{y_i}\right) \\ &= H^{k-1}_{dR}(\mathcal{K})(-1) \qquad \text{(k odd)} \end{aligned}$$

  $$\mathcal{K} = \left(\sum y_i + \frac{1}{y_i}\right)$$

- The Bessel moments are classical periods
- The (pure part of) the motive $H^1(\mathbb{G}_m, \mathrm{Sym}^k \mathrm{Kl})$ is potentially automorphic
  - Traces of Frobenius on étale cohom $\sim$ moment of Kloosterman sums

$$\sum_{x \in \mathbb{F}_p^\times} \exp\left(\frac{2\pi i}{p}\left(x + \frac{z}{x}\right)\right)$$

  - L-function has nice shapes and good properties ....