General Picture:

A-model $\qquad\qquad$ B-model

Y $\qquad\qquad$ X.

HMS: $\quad \mathrm{Fuk}(Y) \longleftrightarrow D^b\mathrm{Coh}(X)$

$\mathrm{Mir}(E) \longleftrightarrow E$

Central Charges: $\displaystyle\int_{\mathrm{Mir}(E)} \Omega_t = \int_X ch(E)\, w(t, \tfrac{D}{2\pi i})\, \mathrm{Todd}(X)$

(Gamma Conj.

(Hosono's version)) $\qquad \displaystyle \shortparallel$

$$\int_X t^{-\vec{D}}\, \hat{\Gamma}_X \cdot (2\pi i)^{\deg/2} \cdot ch(E) + O(t)$$

$\vec{D} = \{D_1, \cdots, D_k\}$ a basis of $\mathrm{Pic}(X)$

where $\hat{\Gamma}_X = \prod_i \Gamma(1+\delta_i)$ (δ_i Chern Roots of X)

$$= \exp\left(-\gamma\, c_1(X) + \sum_{k=2}^{\infty} (-1)^k \zeta(k)(k-1)!\, ch_k(TX)\right)$$

Remark: Formal solution to Picard–Fuchs eq.

$$w(t) = \sum_{n\in\mathbb{N}^k} C(n)\, t^n$$

$$\Rightarrow w(t, \tfrac{D}{2\pi i}) = \sum_{n\in\mathbb{N}^k} C(n+\tfrac{D}{2\pi i})\, t^{n+\frac{D}{2\pi i}}$$

Gross-Siebert Program:

$$\begin{tikzcd} \mathcal{Y} \arrow[dr] & & \mathcal{X} \arrow[dl] \\ & (B, \Delta, \mathscr{P}, \varphi) & \end{tikzcd}$$

Tropical Geometry:

$$\begin{tikzcd} \text{Lagrangian} & & \text{Coherent Sheaves.} \\ & \text{Tropical Objects} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

This Talk:

$$\begin{tikzcd} Y \to \mathbb{D}^k & & K_{X_{\overline{\sigma}}} \\ & (B, \Delta, \mathscr{P}, \varphi) \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$K_{X_{\overline{\sigma}}}$: Canonical bundle of a smooth toric surface $X_{\overline{\sigma}}$

$$\begin{tikzcd} \beta_t & & i_* \mathcal{O}_C \\ & \beta_{trop} \arrow[ul] \arrow[ur] & \end{tikzcd}$$

$i_* \mathcal{O}_C$: C smooth ample curve in $X_{\overline{\sigma}}$

① G-S program for this case
(Gross - Siebert)

$$\begin{array}{c} i_* \mathcal{O}_C \\ ② \big\downarrow \\ \beta\text{-trop} \\ ③ \big\downarrow \quad \text{(Ruddat - Siebert)} \\ \beta_t \subseteq \mathcal{Y}_t \end{array}$$

① G-S model for $K_{X_{\bar{\sigma}}}$

$X_{\bar{\sigma}}$: Smooth toric surface

$\bar{\sigma}$: Smooth complete fan in $M_{\mathbb{R}} \cong \mathbb{R}^2$

$\{\bar{v}_1, \cdots, \bar{v}_p\} \longleftrightarrow \{\bar{D}_1, \cdots, \bar{D}_p\}$.

$\{\bar{D}_1, \cdots, \bar{D}_k\}$ basis of $\mathrm{Pic}(X_{\bar{\sigma}})$

$K_{X_{\bar{\sigma}}} = X_{\Sigma}$.

Σ : Fan in $M_{\mathbb{R}} \times \mathbb{R}$ over $\bar{\sigma}$

$\{v_0, \cdots, v_p\} \longmapsto \{D_0, \cdots, D_p\}$

$\{D_1, \cdots, D_k\}$ basis of $\mathrm{Pic}(K_{X_{\bar{\sigma}}})$

Δ: Polytope in $M_{\mathbb{R}} \cong \mathbb{R}^2$
with $\{\bar{v}_1, \cdots, \bar{v}_p\}$ vertices.

$\bar{\mathcal{P}}$: Polyhedral decomposition of Δ
by connecting the origin to $\bar{v}_i$

B: $\mathrm{Conv}\{\Delta \times (1,0),\ (0,0,0,1)\}$

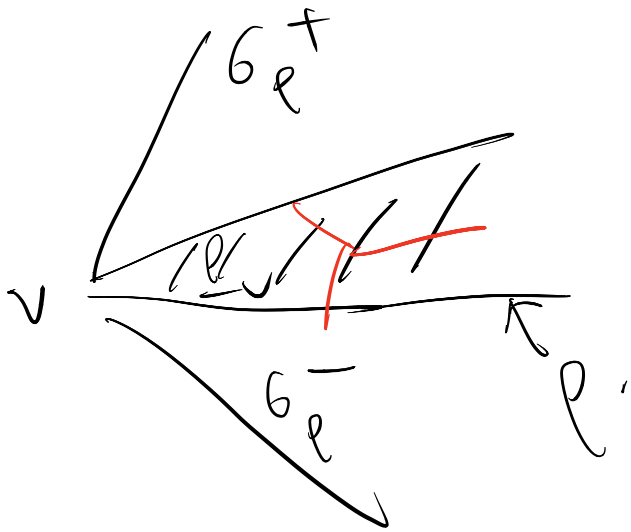
$\cup$

$\mathrm{Conv}\{\Delta \times (1,0),\ (0,0,1,-1)\}$

$\mathcal{P}$: Induced by $\bar{\mathcal{P}}$

$\boldsymbol{\Delta}$: 1st Barycentric decomp of
$\bar{\mathcal{P}}$ not containing vertices

Slabs: codim-1 cell ρ of $\mathscr{P}$ in Δ



$$\rho \setminus \Delta = \bigcup_{v \in \rho} \rho_v$$

Affine str. on $B_0 = B \setminus \Delta$

(U_σ, ψ_σ) , (U_v, ψ_v)

Int σ Star nbhd of v

$$\psi_\sigma : U_\sigma \to \mathbb{R}^3 \cong \Lambda_\sigma \otimes \mathbb{R}$$
$$(m, r_1, r_2) \mapsto (m, r_2)$$

$$\psi_v : U_v \to \mathbb{R}^4 / \mathbb{R}_{\geq 0} \cdot (\bar{v}, 1, 0) \cong \mathbb{R}^3 \cong \Lambda_v \otimes \mathbb{R}$$
$$(m, r_1, r_2) \mapsto (m - r_1 \bar{v}, r_2)$$

Monodromy:

$$T_\gamma : \Lambda_v \to \Lambda_v$$

$$(m, r) \mapsto (m + r(v - v'), r)$$

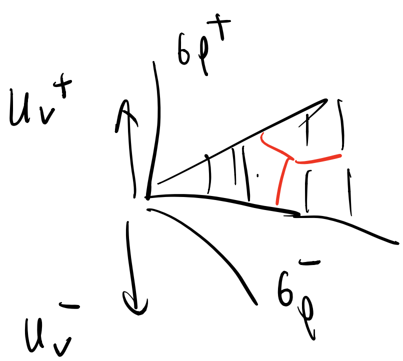
for $\gamma : U_v \to U_{\sigma^-} \to U_{v'} \to U_{\sigma^+} \to U_v$

Central fiber: $\mathcal{Y}_0 = \varinjlim_{\tau \in \mathscr{P}} \mathbb{P}_\tau$

$$\downarrow \text{ smoothen}$$

$$\mathcal{Y}_t$$

Local model near U_v:



$$\mathcal{Y}_0 : \mathrm{Spec}\left(\mathbb{C}[\Lambda_\rho][u_v^+, u_v^-]/(u_v^+ u_v^-)\right)$$

$$\downarrow$$

$$\mathcal{Y}_t : \mathrm{Spec}\left(\mathbb{C}[\Lambda_\rho][u_v^+, u_v^-][[t_1, \dots, t_n]]/(u_v^+ u_v^- - t^{k_v} f_v)\right)$$

$$\Vert$$

$$\mathrm{Spec}(R_v)$$

MPL: φ

$(U_\sigma, \varphi_\sigma), (U_v, \varphi_v)$ induced by

$$\tilde{\phi}(v^+) = q \neq 0 \in \mathbb{R}^k$$

$$\tilde{\phi}(v^-) = (0,\dots,0) \in \mathbb{R}^k$$

$$\tilde{\phi}(v_i) = \begin{cases} (0,\dots,0,\underset{\uparrow\, i\text{th}}{-1},0,\dots,0) & i \in \{1,\dots,k\} \\ (0,\dots,0) & \text{otherwise} \end{cases}$$

Kink κ_v: difference of slopes of φ_v along the third coordinate

$$\kappa_v = \varphi_v(0,0,1) + \varphi_v(0,0,-1)$$
$$= q - \tilde{\phi}(v)$$

f_v: Slab functions $\in \mathbb{C}[\Lambda_P][[t_1,\dots,t_k]]$

- const term of $f_v = 1$
- $f_{v'} = z^{v-v'} \cdot t^{\kappa_v - \kappa_{v'}} f_v$

- (Normalization Condition)

$$\log(f_v) = \sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} (f_v - 1)^i$$

has no terms of t^2

$$\begin{array}{ccc} u_v^+ \leftrightarrow z^{-v} z^{(0,0,1)} & \operatorname{Spec} R_v & u_v^+ \leftrightarrow z^{(0,0,1)} \\ u_v^- \leftrightarrow z^{(0,0,-1)} \swarrow & & \searrow u_v^- \leftrightarrow z^{(0,0,-1)} \\ \operatorname{Spec} \mathbb{C}[[t_1, \dots, t_k]][\Lambda_{\sigma^+}] & & \operatorname{Spec} \mathbb{C}[[t_1, \dots, t_k]][\Lambda_{\sigma^-}] \end{array}$$

$$\mathcal{Y} := \varinjlim \{ \operatorname{Spec} R_v, \operatorname{Spec} \mathbb{C}[[t_1, \dots, t_k]][\Lambda_{\sigma^+}], \operatorname{Spec} \mathbb{C}[[t_1, \dots, t_k]][\Lambda_{\sigma^-}] \}$$

$$\begin{array}{c} \mathcal{Y} \\ \downarrow \\ \mathbb{D}^k \end{array}$$

② : Central charge of $i_* \mathcal{O}_C$

$\downarrow$

β trop

$C \subseteq X_{\bar{\sigma}}$ smooth curve

$C \sim \overline{D} = \sum a_i \overline{D}_i$ ample divisor

$$\int_{K_{X_{\bar{\sigma}}}} t^{\vec{D}} \cdot \hat{\Gamma}_{K_{X_{\bar{\sigma}}}} (2\pi i)^{\deg/2} \mathrm{ch}(i_* \mathcal{O}_C)$$

$$= (2\pi i)^2 \sum_{j=1}^{k} \log t_j \, \ell(F_j)$$

$$+$$

$$(2\pi i)^3 (1-g)$$

$\downarrow$

Combinatorial information of $\nabla_c = \{ n \mid \langle n, \bar{v}_i \rangle \geq -a_i \}$

$\downarrow$

Indentify ∇_c with the central component of $\Delta \setminus \blacktriangle$

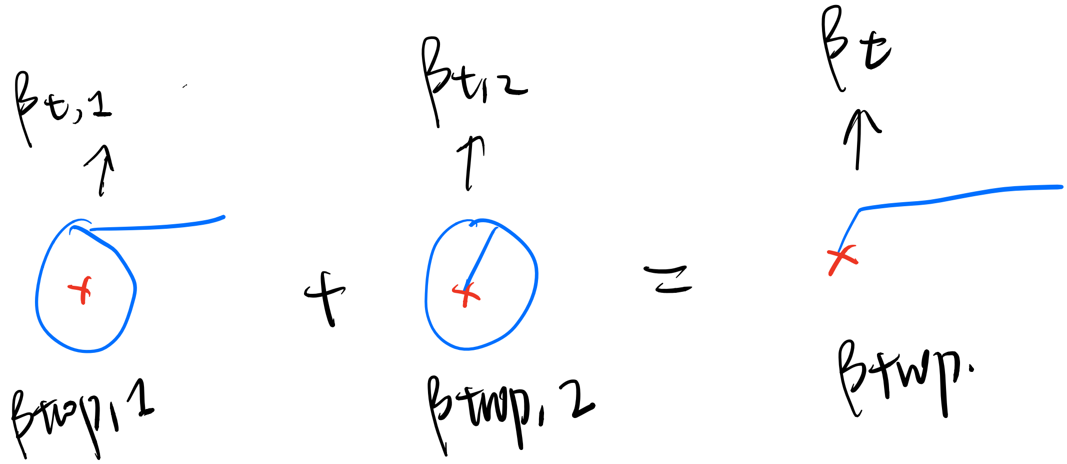
$\downarrow$

Take a triangulation of ∇_c and take the dual graph

$\downarrow$

Give a tropical curve in B

③: $\beta_{trop} \to \beta_t$



Def (Tropical 1-cycle):

$(h: \Gamma \to B \setminus \Delta,\ \xi_e \in \Gamma(h(e), \Lambda|_{h(e)}))$

closed graph

- $v \in \Gamma$ trivalent
- $h(v) \in \mathrm{Int}(\sigma)$
- $h(e) \cap$ codim 2 cell $= \emptyset$

$h(e) \cap$ codim 1 cell = isolated pts

• (Balancing condition)

$v \in \rho$

$$\sum_{v \in e} \varepsilon_{e,v} \xi_e|_v = 0$$

$\varepsilon_{e,v} \in \{-1, 1\}$ orientation of

e at v.

$$\Omega_t := d\log z_1 \wedge d\log z_2 \wedge d\log z_3$$

Thm (R.-S):

$$\int_{\beta_{t,1}} \Omega_t = (2\pi i)^2 \sum_{p \in \beta_e \cap \underline{\rho}_v} \operatorname{sgn}(p) (\xi_e)_3 \log(t^{\kappa_v}) - \frac{V}{2} (2\pi i)^3$$

where $\operatorname{sgn}(p) = 1$ if from σ^- to σ^+

$V = \#\{\text{vertices of } \beta_{trop}\}$.

$$\boxed{\int_{\beta_{t,\mathrm{loop}}} \Omega_t = (2\pi i)^3}$$

$\beta_t \to$ Lagrangian (Matessi, Ruddat – Mak)

$$\int_{\beta_t} \Omega_t = (2\pi i)^2 \sum_{j=1}^{k} \log t_j \, \ell(F_j) + (2\pi i)^3 \left(\frac{1}{2} \# \{ \beta_{t\mathrm{trop}} \cap \Delta \} - \frac{1}{2} \# \{ \text{vertices of } \beta_{t\mathrm{trop}} \} \right)$$

$$= 1 - \#\{\text{interior lattice pts of } \nabla_C\}$$

$$= 1 - g$$