X = smooth projective $/\mathbb{C}$, $D \subset X$ smooth divisor. $\mathcal{O}(D)$

$X_{D,r}$ = the stack of r^{th} roots of X along D.
(Cadman, Vistoli) For S = scheme, the S-points of $X_{D,r}$ are

$$\left\{ S \xrightarrow{f} X,\ \underset{\text{line bundle}}{M \to S},\ t \in H^0(M),\ f^*\mathcal{O}(D) \underset{\varphi}{\xrightarrow{\sim}} M^{\otimes r},\ f^*s_D = t^r \right.$$

$X_{D,r}$ has μ_r stack structure along D, and is isomorphic to X outside D.

- $\boxed{[GW_g(X_{D,r})]_{r^0} = GW_g^{rel}(X,D)}$ $(*)$

Natural question: generalize this equality to the case of D reducible

- If components of D are disjoint, then the argument for the smooth case works component by component, giving the same result.
- If the components are not disjoint, then it is natural to assume that $\boxed{D = D_1 + D_2 + \cdots + D_n}$ is simple normal crossing.
- What should be the RHS for $(*)$?
  In simple normal crossing (X,D), we get a log scheme. We can consider the log GW theory of Abramovich-Chen-Gross-Siebert.
- What should be the LHS?
  This talk (j. w/ Fenglong You)
- Does the equality hold?

  NO//

- Does the equality hold?

  NO//

$\boxed{\text{LHS}}$ $r_1, r_2, \cdots, r_n \in \mathbb{N}$. $X_{(D_1,r_1),(D_2,r_2),\cdots,(D_n,r_n)}$ ~ multiroot stack

- $GW_g(X_{(D_1,r_1),(D_2,r_2),\cdots,(D_n,r_n)})$ is polynomial in each r_i for $r_i >> 1$
- Then we take the constant term $(r_1^0 r_2^0 \cdots r_n^0)$ to get a collection of "invariants".
- $\mathcal{X}$ = smooth DM stack, $\mathcal{D} \subset \mathcal{X}$ smooth divisor
  $r \in \mathbb{N}$. $\mathcal{X}_{\mathcal{D},r}$ = root stack.

$\boxed{\text{Want}}$ $GW_g(\mathcal{X}_{\mathcal{D},r})$ is polynomial in r for $r >> 1$.

Proof ① degenerate to the normal cone of $\mathcal{D} \subset \mathcal{X}$:

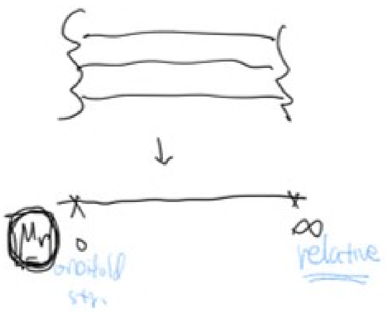
$$\mathcal{X}_{\mathcal{D},r} \rightsquigarrow \mathcal{X} \underset{\mathcal{D}=\mathcal{D}_\infty}{\cup} \mathbb{P}(N_{\mathcal{D}/\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}})_{\mathcal{D}_0,r}$$

degeneration formula for GW invs:

$$GW_g(\mathcal{X}_{\mathcal{D},r}) = \sum_{g_1+g_2=g} GW_{g_1}(\mathcal{X},\mathcal{D}) * GW_{g_2}(\mathbb{P}(N_{\mathcal{D}/\mathcal{X}} \oplus \mathcal{O}_{\mathcal{X}})_{\mathcal{D}_0,r}, \mathcal{D}_\infty)$$

$g_1+g_2=g$

② localization wrt fiberwise $\mathbb{C}^*$-action.



Contribution from maps to $\boxed{D_0}$

$N_{D/X} \to D$

$D_0 = \sqrt[r]{N_{D/X}/D}$ ~ μ_r-gerbe over D.

$\mathcal{L} \to D_0$ universal rth root of $N_{D/X}$

$$\begin{tikzcd} \mathcal{C} \arrow[r, "f"] \arrow[d, "\pi"] & D_0 \\ \overline{M}_{g,n}(D_0,\beta) & \end{tikzcd}$$

Contribution

$$\boxed{c_\circ(-R\pi_* f^*\mathcal{L}) \cap [\overline{M}_{g,n}(D_0,\beta)]^{vir}}$$

compute using Toen's Grothendieck Riemann-Roch for stacks (T.)

polynomial in r.

• Pixton's general polynomiality result (see [JPPZ I]) applies here.

$\left[GW_g(X_{(D_1,r_1),\dots,(D_n,r_n)})\right]_{r_1^\circ \dots r_n^\circ}$ form a "theory".

Nice properties

Nice properties

① It has string / dilaton / divisor eq'ns.

② It yields an associative ring called "relative quantum cohomology ring".

State space: $H := \bigoplus_{\vec{s} \in \mathbb{Z}^n} H_{\vec{s}}$, $H_{\vec{s}} = H^*(D_{I_{\vec{s}}})$

$I_{\vec{s}} := \{ i \mid s_i \neq 0 \} \subset \{1, \dots, n\}$.

$D_I = \bigcap_{i \in I} D_i$, $H^*(D_\phi) := H^*(X)$

$$(-,-) : H \times H \to \mathbb{C}, \quad ([\alpha]_{\vec{s}}, [\beta]_{\vec{s}'}) := \begin{cases} \int_{D_{I_{\vec{s}}}} \alpha \cup \beta & -\vec{s} = \vec{s}' \\ 0 & \text{else} \end{cases}$$

③ It has Givental's formalism: $\underset{\text{Lag. cone.}}{\mathcal{L}} \subset \underset{\text{symplectic vector space}}{\mathcal{H}}$

④ It has a "mirror theorem": $\underset{\text{explicit slice}}{I} \subset \mathcal{L}$ in good cases

allowing some computations of invariants in genus 0.

⑤ It is a partial CohFT, lacking the loop axiom.

⑥ It can be used, in some cases, for mirror constructions following Gross–Siebert's approach.

Outlook

① (Pandharipande) degeneration formula?

② relation w/ log GW?