Limits of Geometric Higher Normal Functions and Apéry Constants.

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(Joint work with V. Golyshev, M. Kerr).

§ Apéry constants in the A-model side.

- Apéry originally proved the irrationality of $\zeta(3)$ by aproximating it as $\lim_{n\to\infty} \frac{b_n}{a_n}$ with the solutions $a(t) := \sum a_n t^n$, $b(t) := \frac{1}{6} \sum b_n t^n$ of

$$\begin{cases} L\, a(t) = 0 & \leftarrow \text{homogeneous} \\ (\delta - 1) L\, b(t) = 0 & \leftarrow \text{inhomogeneous.} \end{cases} \qquad \left(\delta := t\frac{\partial}{\partial t}\right)$$

$$L := \delta^3 - t(2\delta+1)(17\delta^2 + 17\delta + 5) + t^2(\delta+1)^3.$$

- L is the Picard-Fuchs operator for a family of K3, which is the Landau-Ginzberg model of the Fano 3fold $V_{12} = \mathcal{O}(5,10) \cap (7 \text{ planes})$ ([F. Beukers, C. Peters '84]).

- On the A-model side, V. Golyshev generalized the construction of the Apéry constant $\lim_{n\to\infty} \frac{a_n}{b_n}$ to some Fano 3folds with Picard number $= 1$:

Fano 3fold	V_{10}	V_{12}	V_{14}	V_{16}	V_{18}
$\lim_{n\to\infty} \frac{b_n}{a_n}$	$\frac{1}{10}\zeta(2)$	$\frac{1}{6}\zeta(3)$	$\frac{1}{7}\zeta(2)$	$\frac{7}{32}\zeta(3)$	$\frac{1}{3}L(\chi_3, 3)$

- Let X^0 be one of these V_i.

They are deduced from the $\mathcal{D}$-module structure on $H^*(X^\circ) \otimes \mathbb{C}[t^{\pm 1}]$ with the (small) quantum product

$(\mathcal{D} := \mathbb{C}[(\lambda_{\underline{m}}), (\partial_{\underline{\lambda_m}})])$

$\uparrow$ coeff. of Laurent polynomials for the LG-models.

$\hat{L}$: operator killing $1 \otimes 1$ $=: \sum \beta_{ij} t^i \delta_t^j$

$\mapsto$ Quantum Recursion on a sequence $(\hat{u}_k)$:

$$\hat{R} := \sum \beta_{ij} (k-i)^j \, \hat{u}_{k-i} = 0. \quad \text{for } {}^\forall k.$$

※. Conjecturally, $\hat{L}$ is the Fourier-Laplace transform of the PF-eq. L on the variable part $\mathcal{H}^{n-1}_v$ on the mirror LG-model.

- For a baisis $\{(\hat{u}^{(0)}), \dots, (\hat{u}^{(d)})\}$ of the solutions of $\hat{R}$

s.t. $\hat{u}^{(i)}_k = 0$ for $k < i$, $\hat{u}^{(i)}_i = \frac{1}{i!}$

with $u_k := k!\, \hat{u}_k$ (the inverse FL-transform),

<u>Def</u>

The Apéry constant of X° are

$$\alpha^{(i)}_{X_0} := \lim_{k\to\infty} \frac{u^{(i)}_k}{u^{(0)}_k}$$

- Golyshev's results are given by $(u_k^{(0)}) = (a_k)$,

$$(u_k^{(d-1)}) \ (= (u_k^{(1)})) = (b_k)$$

- How can we construct $\alpha_{x_0}^{(i)}$ directly in the B-model side?

$\longmapsto$

Arithmetic Mirror Symmetry Conjecture.

For each Fano n-fold X° with a toric degeneration, there exist a mirror 1-parameter family of CY $(n-1)$-folds over $\overline{\mathbb{Q}}$ s.t. $\alpha_{x_0}^{(i)}$ arises as a limit of a (higher) normal functions defined by a family of (higher) cycles pairing with a family of holomorphic form $\{\omega_t\}$.

Thm [V. Golyshev, M. Kerr, T. S '20]

This holds for $V_{10}, V_{12}, V_{14}, V_{16}, V_{18}$.

$\rightsquigarrow$ B-model side construction of Apéry constants.

We constructed the higher Chow cycles which defines the desired higher normal functions.

§ Higher Chow cycles

- X : smooth quasi-projective variety $/ \mathbb{C}$.
  The Chow groups $CH^p(X) :=$ (algebraic cycles)$/\sim_{rat}$
  can be generalized to the higher Chow cycles $CH^p(X, n)$.
- $CH^p(X, n)_{\mathbb{Q}} \cong H_{\mathcal{M}}^{2p-n}(X, \mathbb{Q}(p)) \cong Gr_{\gamma}^{p} K_n(X)_{\mathbb{Q}}$.
- A LG-model of the Fano threefold X° can be obtain from a general Laurant polynomial ϕ as a family of hypersurfaces

$$\mathcal{X}' := \{ \overline{X_t : 1 - t\phi = 0} \} \subset \mathbb{P}_{\Delta_\phi}$$

in the toric variety $\mathbb{P}_{\Delta_\phi}$ after the MPCP desingularization

$$(\Delta_\phi := (\text{Convex hull of } \mathcal{M}_\phi := \{ \underline{m} \in \mathbb{Z}^n \mid x^{\underline{m}} \text{ is in } \phi \}))$$

- More precisely, we need to resolve the non-torical singularites in $\mathcal{X}'$. We take a successive blow-up of $\mathbb{P}_\Delta$ along each component of the base locus Z. $X_0 \cap X_t$.
- We construct a family of higher cycles on

$$\mathcal{X} := \{\tilde{X}_t\}_{t \in \mathbb{P}^1}$$
$$\mathcal{X}^* = \mathcal{X} \setminus (\text{singular fibers})$$

Case I.

$V_{12}, V_{16}, V_{18} \ \ldots \qquad \gamma \in CH^3(\mathcal{X}^*, 3).$

Case II

$V_{10}, V_{14} \ \cdots \qquad \gamma \in CH^2(\mathcal{X}^*, 1).$

Fano 3fold	V_{10}	V_{12}	V_{14}	V_{16}	V_{18}
$\lim\limits_{n\to\infty} \frac{b_n}{a_n}$	$\frac{1}{10}\zeta(2)$	$\frac{1}{6}\zeta(3)$	$\frac{1}{7}\zeta(2)$	$\frac{7}{32}\zeta(3)$	$\frac{1}{3}L(\chi_3, 3)$

(K-theory elevator.) $X_t \rightsquigarrow X_0 \leftarrow$ SNCD. $CH(X_0)$

higher cycles   higher cycles

Case I $CH^3(X_0, 3)$.

$$\begin{tikzcd} Z^3(X_0^{[0]}, 3) & & \\ Z^3(X_0^{[0]}, 4) \arrow[u] \arrow[r] & Z^3(X_0^{[1]}, 4) & \mathbb{Q}\cdot\mathfrak{Z}(3) \\ & Z^3(X_0^{[1]}, 5) \arrow[u] \arrow[r] & Z^3(X_0^{[2]}, 5) \arrow[u, "AJ."'] \end{tikzcd}$$

H.

point.

Case II. $CH^2(X_0, 1)$.

$$\begin{tikzcd} Z^2(X_0^{[0]}, 1) & & \\ Z^2(X_0^{[0]}, 2) \arrow[u] \arrow[r] & Z^2(X_0^{[1]}, 2) & \mathbb{Q}\,\mathfrak{Z}(2), \\ & Z^2(X_0^{[1]}, 3) \arrow[u] \arrow[r] & Z^2(X_0^{[2]}, 3) \arrow[u, "AJ."'] \end{tikzcd}$$

H

point.

- Each $\gamma \in CH^p(X, n)$ can be represented by an algebraic cycle of codim p in $X \times (\underbrace{\mathbb{P}^1 \setminus \{1\}}_{\text{"}\square^1\text{"}})^n$

- <u>Case I</u> ($\gamma \in CH^3(\mathcal{X}^*, 3)$.)

  $\mathcal{X}^* \supset \mathbb{G}_m^3$ has the symbol $\{x, y, z\} \in CH^3(\mathbb{G}_m^3, 3)$.
  (graphs of x, y, z).

<u>Def</u>

A Laurent polynomial ϕ is called tempered if $\{\underline{x}\} \in CH^n(\mathbb{G}_m^n, n)$ extends to a higher cycle in $CH^n(\mathcal{X}^*, n)$.

(※. $n=2$. case $\Leftrightarrow$ each edge polynomial is cyclotomic)

- For each V_i, a LG-model with the Minkowski polynomial ϕ is known and [G. da Silva Jr, '19] shows that ϕ is tempered.

$\mapsto$ We obtain $\gamma \in CH^3(\mathcal{X}^*, 3)$.

- <u>Case II.</u> ($\gamma \in CH^2(\mathcal{X}^*, 1)$).

Each cycle in $CH^p(X_t, 1)$ is represented by

$$\sum_j (f_j, Z_j) \quad \text{with} \quad Z_j \subset X_t \text{ codim. } p-1 \text{ cycle}$$

$$f_j : \text{rational function on } Z_j$$

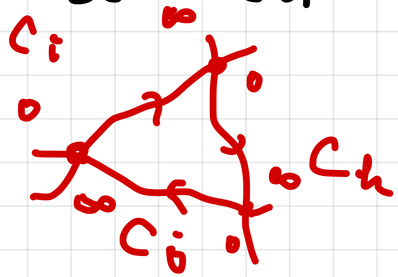
s.t.

$$\sum_j (\operatorname{div} f_j) = 0.$$

0 & ∞ cancels out.
$\in \mathbb{P}^1$

- As Z_j, we take an irreducible component C_j of the Base locus Z of $\mathcal{X}^*$.

If C_j is a rational curve, a choice of $C_j \xrightarrow{\cong} \mathbb{P}^1$ defines f_j. Put the boundaries "0" and "∞" on $C_i \cap C_j$ s.t. $\sum_j \operatorname{div}(f_j)$ cancels out.



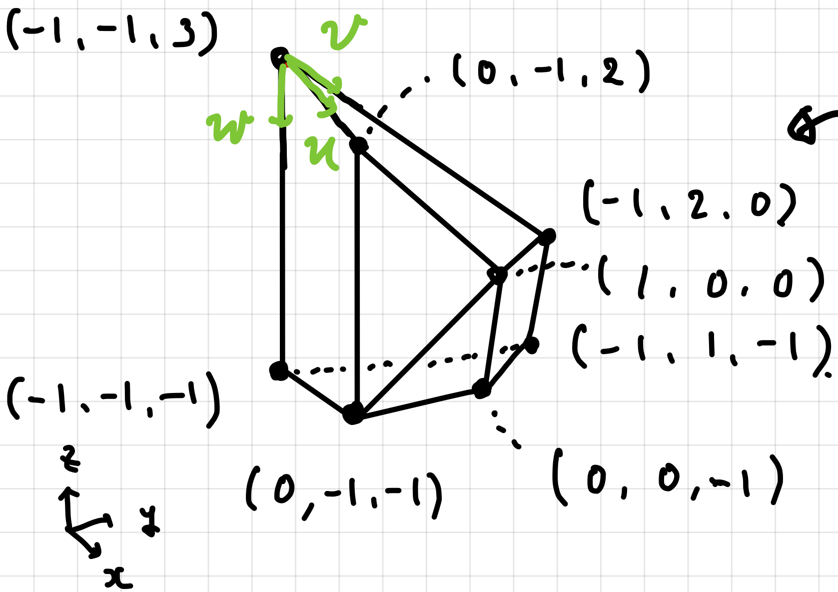
$\mapsto$ Since $Z_j \subset Z$, $\gamma_t := \sum_j (Z_j, f_j) \in CH^2(X_t, 1)$ can be extended to a (constant) family of higher cycles in $CH^2(\mathcal{X}^*, 1)$.

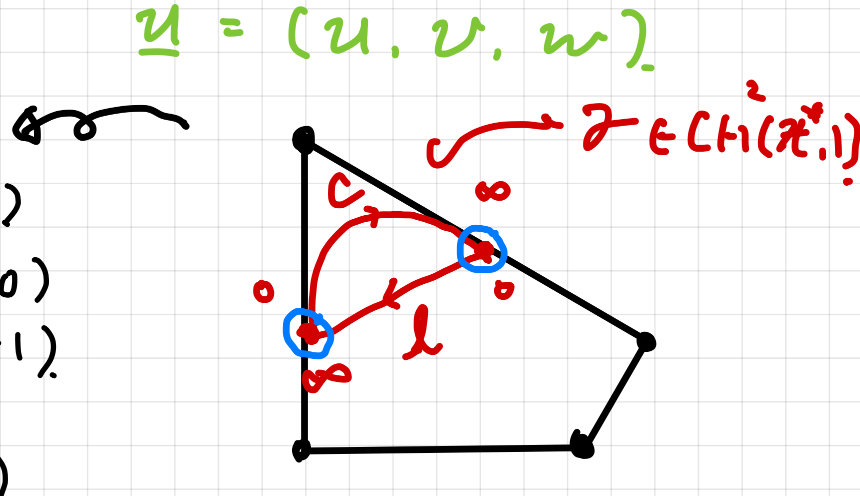
Example of Case II. (V_{14}).

A mirror LG-model of V_{14} is given by

$$\phi = \frac{(1+x+y+z)^2}{x} + \frac{(1+x+y+z)(1+y+z)(1+z)^2}{xyz}.$$

$\Delta = \Delta_\phi$:





(The back facet of Δ)

$$u = 0 \Rightarrow \phi = (\underbrace{1-w-v}_{\text{line } \ell})^2(\underbrace{v-(1-w)^2}_{\text{conic } C.}) = 0$$

§ Higher Normal Functions.

- Since $CH^p_{hom}(X_t, \underset{\substack{\vee \\ 0}}{n}) \cong CH^p(X_t, n)$ when X_t is smooth and projective, we can generalize the ordinal Abel-Jacobi map to the **higher Abel-Jacobi map**

$$AJ^{p,n}_{X_t} : CH^p(X_t, n) \longrightarrow \underset{\substack{\parallel \\ \mathrm{Ext}^1_{MHS}(\mathbb{Q},\, H^{2p-n-1}(X_t)(p)).}}{J^{p,n}(X_t)}$$

- A given family of higher cycles $\mathfrak{Z}$ defines a <u>holomorphic horizontal section</u> in $\Gamma(S, \mathcal{J}^{p,n}(\mathcal{X}^*))$ (with $\mathcal{X}^* \longrightarrow S \subset \mathbb{P}^1$), which is called

  a **geometric higher normal function** $\nu_{\mathfrak{Z}}(\lambda)$.

※. We may consider $\mathcal{V}_{\mathfrak{z}}$ as an extension of a variation of mixed Hodge structur (VMHS)

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Q}(0) \to 0,$$

which is a general higher normal function.

When $\mathcal{V}$ has the limiting mixed Hodge structure (LMHS) on each singular fiber,

we say $\mathcal{V}$ is an admissible normal function. (ANF)

Geometric HNF is an example of ANF.

([J.-L. Brylinski, S. Zucker])

- For the variable part $\mathcal{H}_v^{2p-r}$ of the cohomology of general fibers, with the projection $ANF(\mathcal{H}) \twoheadrightarrow ANF(\mathcal{H}_v^{2p-r}(p))$, we obtain

$$AJ^v : CH^p(\mathcal{X}^*, r) \longrightarrow ANF(\mathcal{H}_v^{2p-r}(p)).$$

(<u>Beilinson - Hodge Conjecture.</u>
AJ^v is surjective.).

- For the <u>Case I</u> ($\gamma \in CH^3(\mathcal{X}^*, 3)$), the induced geometric HNF has another interpretaition relating to the A-model side:

With $\overline{H}^k(X_t^*) := \mathrm{Coker}\left(H^k(\mathbb{G}_m^n) \to H^k(X_t^*)\right)$

$X_t^* := X_t \cap \mathbb{G}_m^n$

$\underline{H}_k(X_t^*) := \mathrm{Ker}\left(H_k(X_t^*) \to H_k(\mathbb{G}_m^n)\right)$,

we may consider an extension of VMHS

$$(1) \qquad 0 \to \overline{H}^{n-1}(X_t^*) \to \mathcal{V}_{\phi,t} \to H^n(\mathbb{G}_m^n) \to 0$$

or dualy

$\mathbb{Q}(0) \cong$ $\qquad H^n(\mathbb{P}_\Delta \setminus X_t, D_\Delta | Z) \cong \mathbb{Q}(-n)$ $\qquad$ torus complement

$$(2) \qquad 0 \to H^n(\mathbb{P}_\Delta, \mathbb{D}_\Delta) \to \mathcal{V}_{\phi,t}^\vee(-n) \to \underline{H}_{n-1}(X_t^*)(-n) \to 0$$

$\mathbb{D}_\Delta := \mathbb{P}_\Delta \setminus \mathbb{G}_m^n$

and the extension classes are given by

$$(1) \quad \cdots \quad \langle \mathcal{V}_{\phi,t}, \gamma \rangle \equiv_{\mathbb{Q}(n)} \langle AJ(\{\underline{x}\}|_{X_t^*}), \gamma \rangle$$

$$(2) \quad \cdots \quad (\text{The period}) = \int_{(-1)^{n-1} \cap_i T_{x_i} (= \mathbb{R}_{<0})} \frac{d \log \underline{x}}{1 - t\phi(\underline{x})}$$

$$\equiv_{(2\pi i)^n \int_\eta \omega_t} \langle \widetilde{\mathcal{V}}_{\phi,t}, [\omega_t] \rangle .$$

Thm (A. Huang, B. Lian, S-T. Yau, X. Zhu '16)

$$\hat{\tau}^{\Delta}_{GKZ} := \tau^{a=(1,\underline{0}),\,\{1\}\times\Delta}_{GKZ} \cong H^n(\mathbb{P}_\Delta \setminus X_t,\ \mathbb{D}_\Delta \setminus Z).$$

(with ∇^{GM} for the $\mathcal{D}$-module str.)

and $\langle \tilde{\nu}_{\phi,t}, [\omega_t] \rangle$ is a GKZ-integral to obtain a local solution of the GKZ-system.

$\rightsquigarrow$ generalize from $\nu_{\phi,t}$ to ν_g.

Def

For $\nu \in ANF(\mathcal{H}^{n-1}_v(p))$, the truncated HNF associated to ν is defined by

$$V(t) := \langle \tilde{\nu}(t), [\omega_t] \rangle \qquad (\tilde{\nu} := \nu_Q - \nu_F)$$

This $V(t)$ defines an inhomogeneous PF-equation

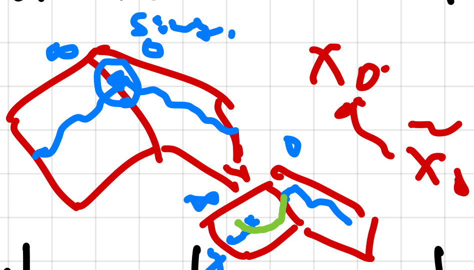
$$L\,V(t) := g(t) \in \mathbb{C}(t)$$

Lem

In our five cases of Fano threefolds, $\deg(g) = 1$.

- We normalize V to $\tilde{V}$ s.t. $g(t) = -t$.

From the construction of $\mathcal{X}$, in both Case I and II, ν_g has no singularities (= "obstruction to extend $\mathcal{X}_t$ to a singular fiber") at $t = 0$.

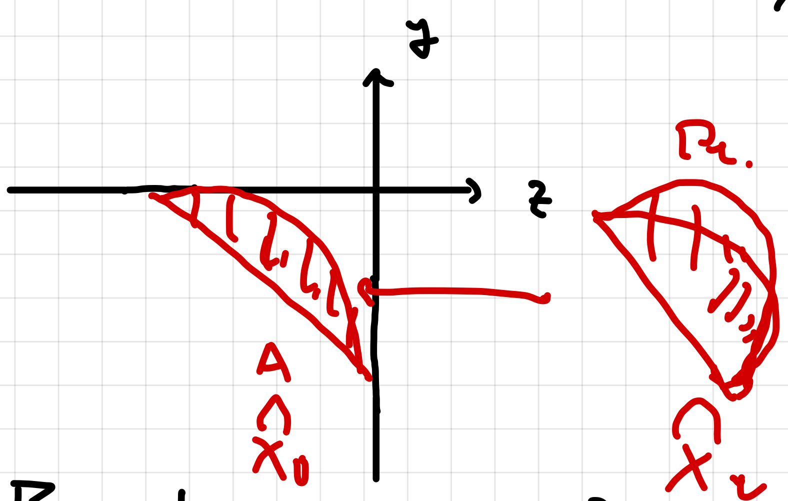


Then the "limit of $\tilde{V}(t)$" at $t=0$ is the desired Apéry constant.

Ker(sing). "singularity"

Example (V_{14}. Apéry $= \frac{1}{7}\zeta(2)$)

$$V(t) = (2\pi i)^2 \int_{\Gamma_t} \omega_t$$



$$\Gamma_t = \{ (u,v,w) \in X_t \}.$$
$$\uparrow$$
$$A = \{ (v,w) \mid -1 \le w \le 0,\ -(w+1) \le v \le -(w+1)^2 \}$$

$$\overset{(t \ll 0)}{=} \frac{1}{(2\pi i)^2} \int_{A \times S^1} \frac{d \log u}{1 - t\phi}$$

$$\overset{\text{expand}}{=} \sum_{k \ge 0} t^k \int_A \left((\text{constant in } \phi \text{ w.r.t. } u \right)^k \frac{dv}{v} \wedge \frac{dw}{w}$$

coefficient for $k=0$: $\zeta(2)$.

for $k=1$: $-7+4\zeta(2)$.

$$\mapsto \quad V(t) = \zeta(2) + (-7+4\zeta(2))t + \cdots$$

$$\left(\mapsto \quad L(V(t)) = -7t \right).$$

$$\mapsto \quad \tilde{V}(t) = \frac{1}{7}\zeta(2) + \left(-1 + \frac{4}{7}\zeta(2)\right)t + \cdots$$

$$\xrightarrow[t \to 0]{\lim} \frac{1}{7}\zeta(2).$$