Limits of Geometric Higher Normal Functions and Apéry Constant.

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(Joint work with V. Golyshev, M. Kerr).

§ Apéry constants in the A-model side.

- Apéry originally proved the irrationality of $\zeta(3)$ by aproximating it as $\lim_{n\to\infty} \frac{b_n}{a_n}$ with the solutions $a(t) := \sum a_n t^n$, $b(t) := \frac{1}{6}\sum b_n t^n$ of

$$\begin{cases} L\, a(t) = 0 \\ (\delta - 1) L\, b(t) = 0 \end{cases} \qquad \left(\delta := t\frac{\partial}{\partial t}\right)$$

$$L := \delta^3 - t(2\delta+1)(17\delta^2 + 17\delta + 5) + t^2(\delta^3+1)^3.$$

- L is the Picard-Fuchs operator for a family of K3, which is the Landau-Ginzberg model of the Fano 3fold $V_{12} = \mathcal{O}(5,10) \cap (7 \text{ planes})$ ([F. Beukers, C. Peters '84]).

- On the A-model side, V. Golyshev generalized the construction of the Apéry constant $\lim_{n\to\infty} \frac{a_n}{b_n}$ to some Fano 3folds with Picard number $= 1$:

Fano 3fold	V_{10}	V_{12}	V_{14}	V_{16}	V_{18}
$\lim_{n\to\infty} \frac{b_n}{a_n}$	$\frac{1}{10}\zeta(2)$	$\frac{1}{6}\zeta(3)$	$\frac{1}{7}\zeta(2)$	$\frac{7}{32}\zeta(3)$	$\frac{1}{3}L(\chi_3, 3)$

- Let X^0 be one of these V_i.
They are deduced from the $\mathcal{D}$-module structure on $H^*(X^0) \otimes \mathbb{C}[t^{\pm 1}]$ with the (small) quantum product
($\mathcal{D} := \mathbb{C}[(\lambda_{\underline{m}}), (\partial_{\underline{\lambda_m}})]$)

$\uparrow$ coeff. of Laurent polynomials for the LG-models.

$\hat{L}$: operator killing $1 \otimes 1 =: \sum \beta_{ij} t^i \delta_t^j$

$\mapsto$ Quantum Recursion on a sequence $(\hat{u}_k)$:

$$\hat{R} := \sum \beta_{ij} (k-i)^j \, \hat{u}_{k-i} = 0. \quad \text{for } {}^\forall k.$$

※. Conjecturally, $\hat{L}$ is the Fourier-Laplace transform of the PF-eq. L on the variable part $\mathcal{H}^{n-1}_v$ on the mirror LG-model.

- For a baisis $\{(\hat{u}^{(0)}), \dots, (\hat{u}^{(d)})\}$ of the solutions of $\hat{R}$ s.t. $\hat{u}^{(i)}_k = 0$ for $k < i$, $\hat{u}^{(i)}_i = \frac{1}{i!}$

with $u_k := k! \, \hat{u}_k$ (the inverse FL-transform),

Def

The Apéry constant of X° are

$$\alpha^{(i)}_{X_0} := \lim_{k \to \infty} \frac{u^{(i)}_k}{u^{(0)}_k}$$

- Golyshev's results are given by $(\mathcal{U}_k^{(0)}) = (a_k)$,

$$(\mathcal{U}_k^{(d-1)}) \; (= (\mathcal{U}_k^{(1)})) = (b_k)$$

- How can we construct $\alpha_{x_0}^{(i)}$ directly in the B-model side?

$\leadsto$

<u>Arithmetic Mirror Symmetry Conjecture.</u>

For each Fano n-fold X° with a toric degeneration, there exist a mirror 1-parameter family of CY $(n-1)$-folds over $\overline{\mathbb{Q}}$ s.t. $\alpha_{x_0}^{(i)}$ arises as a limit of a (higher) normal functions defined by a family of (higher) cycles pairing with a family of holomorphic form $\{\omega_t\}$.

<u>Thm</u> [V. Golyshev, M. Kerr, T. S '20]

This holds for V_{10}, V_{12}, V_{14}, V_{16}, V_{18}.

$\mapsto$ B-model side construction of Apéry constants.

We constructed the higher Chow cycles which defines the desired higher normal functions.

§ Higher Chow cycles

- X : smooth quasi-projective variety / $\mathbb{C}$.
  The Chow groups $CH^p(X) := (\text{algebraic cycles}) / \sim_{rat}$ can be generalized to the higher Chow cycles. $CH^p(X, n)$.
- $CH^p(X, n)_{\mathbb{Q}} \cong H^{2p-n}_{\mathcal{M}}(X, \mathbb{Q}(p)) \cong Gr^p_\gamma K_n(X)_{\mathbb{Q}}$.
- A LG-model of the Fano threefold X° can be obtain from a general Laurant polynomial ϕ as a family of hypersurfaces

$$\mathcal{X}' := \{\overline{X_t : 1 - t\phi = 0}\} \subset \mathbb{P}_{\Delta_\phi}$$

in the toric variety $\mathbb{P}_{\Delta_\phi}$ after the MPCP desingularization

$$(\Delta_\phi := (\text{Convex hull of } \mathcal{M}_\phi := \{\underline{m} \in \mathbb{Z}^n \mid x^{\underline{m}} \text{ is in } \phi\}))$$

- More precisely, we need to resolve the non-torical singularites in $\mathcal{X}'$. We take a successive blow-up of $\mathbb{P}_\Delta$ along each component of the base locus Z.
- We construct a family of higher cycles on

$$\mathcal{X} := \{\tilde{X}_t\}_{t \in \mathbb{P}^1}$$
$$\mathcal{X}^* = \mathcal{X} \setminus (\text{Singular fibers})$$

<u>Case I.</u>

$V_{12}, V_{16}, V_{18} \quad \dots \quad \gamma \in CH^3(\mathcal{X}^*, 3).$

<u>Case II</u>

$V_{10}, V_{14} \quad \dots \quad \gamma \in CH^2(\mathcal{X}^*, 1).$

Fano 3fold	V_{10}	V_{12}	V_{14}	V_{16}	V_{18}
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- Each $\gamma \in CH^p(X, n)$ can be represented by an algebraic cycle of codim p in $X \times (\mathbb{P}^1 \setminus \{1\})^n$

- <u>Case I</u> ($\gamma \in CH^3(\mathcal{X}^*, 3)$.)

$\mathcal{X}^* \supset \mathbb{G}_m^3$ has the symbol $\{x, y, z\} \in CH^3(\mathbb{G}_m^3, 3)$.
(graphs of x, y, z).

<u>Def</u>

A Laurent polynomial ϕ is called tempered if $\{\underline{x}\} \in CH^n(\mathbb{G}_m^n, n)$ extends to a higher cycle in $CH^n(\mathcal{X}^*, n)$.

(※. $n=2$. case $\Leftrightarrow$ each edge polynomial is cyclotomic)

- For each V_i, a LG-model with the Minkowski polynomial ϕ is known and [G. da Silva Jr, '19] shows that ϕ is tempered.

$\mapsto$ We obtain $\gamma \in CH^3(\mathcal{X}^*, 3)$.

- <u>Case II.</u> ($\gamma \in CH^2(\mathcal{X}^*, 1)$).

Each cycle in $CH^p(X_t, 1)$ is represented by

$$\sum_j (f_j, Z_j) \quad \text{with} \quad \begin{array}{l} Z_j \subset X_t \ \text{codim. } p\text{-}1 \text{ cycle} \\ f_j : \text{rational function on } Z_j \end{array}$$

s.t.

$$\sum_j (\mathrm{div}\, f_j) = 0.$$

- As Z_j, we take an irreducible component C_j of the Base locus Z of $\mathcal{X}^*$.

If C_j is a rational curve, a choice of $C_j \xrightarrow{\cong} \mathbb{P}^1$ defines f_j. Put the boundaries "0" and "∞" on $C_i \cap C_j$ s.t. $\sum_j \mathrm{div}(f_j)$ cancels out.

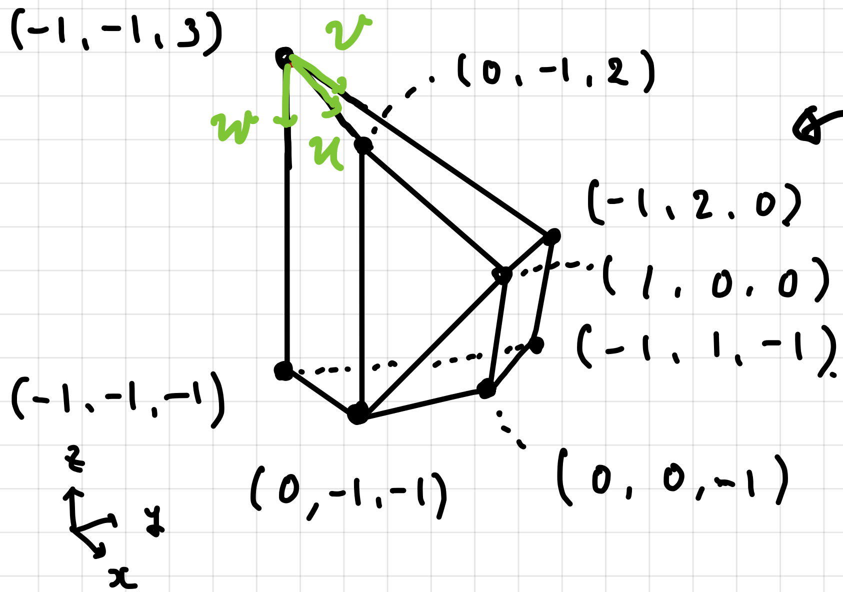
$\mapsto$ Since $Z_j \subset Z$, $\gamma_t := \sum_j (Z_j, f_j) \in CH^2(X_t, 1)$ can be extended to a (constant) family of higher cycles in $CH^2(\mathcal{X}^*, 1)$.

Example of Case II. (V_{14}).

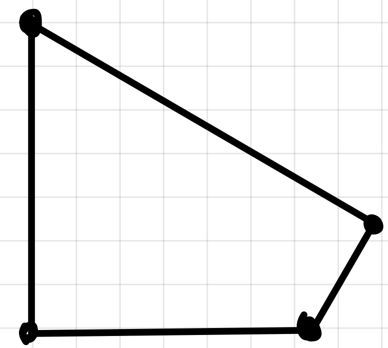
A mirror LG-model of V_{14} is given by

$$\phi = \frac{(1+x+y+z)^2}{x} + \frac{(1+x+y+z)(1+y+z)(1+z)^2}{xyz}.$$

$\Delta = \Delta_\phi$:



$\underline{u} = (u, v, w)$.



(The back facet of Δ)

$$u = 0 \Rightarrow \phi = (1 - w - v)^2 (v - (1-w)^2) = 0$$

§ Higher Normal Functions.

- Since $CH^p_{hom}(X_t, n) \overset{\simeq}{\underset{n>0}{}} CH^p(X_t, n)$ when X_t is smooth and projective, we can generalize the ordinal Abel-Jacobi map to the higher Abel-Jacobi map

$$AJ^{p,n}_{X_t} : CH^p(X_t, n) \longrightarrow J^{p,n}(X_t)$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \parallel$$
$$\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{Ext}^1_{MHS}(\mathbb{Q}, H^{2p-n-1}(X_t)(p)).$$

- A given family of higher cycles $\mathcal{Z}$ defines a holomorphic horizontal section in $\Gamma(S, \mathcal{J}^{p,n}(\mathcal{X}^*))$ (with $\mathcal{X}^* \longrightarrow S \subset \mathbb{P}^1$), which is called a geometric higher normal function $\nu_{\mathcal{Z}}(s)$.

※. We may consider $\mathcal{V}_Z$ as an extension of a variation of mixed Hodge structure (VMHS)

$$0 \to \mathcal{H} \to \mathcal{V} \to \mathbb{Q}(0) \to 0,$$

which is a general higher normal function.

When $\mathcal{V}$ has the limiting mixed Hodge structure (LMHS) on each singular fiber,

we say $\mathcal{V}$ is an admissible normal function (ANF).

Geometric HNF is an example of ANF.

([J.-L. Brylinski, S. Zucker])

- For the variable part $\mathcal{H}_v^{2p-r}$ of the cohomology of general fibers, with the projection $ANF(\mathcal{H}) \twoheadrightarrow ANF(\mathcal{H}_v^{2p-r}(p))$, we obtain

$$AJ^v : CH^p(\mathfrak{X}^*, r) \longrightarrow ANF(\mathcal{H}_v^{2p-r}(p)).$$

(<u>Beilinson-Hodge Conjecture.</u>

AJ^v is surjective.).

- For the <u>Case I</u> ($\gamma \in CH^3(\mathfrak{X}^*, 3)$), the induced geometric HNF has another interpretaition relating to the A-model side:

With $\overline{H}^k(X_t^*) := \mathrm{Coker}(H^k(\mathbb{G}_m^n) \to H^k(X_t^*))$

$\quad\quad X_t^* := X_t \cap \mathbb{G}_m^n$

$\underline{H}_k(X_t^*) := \mathrm{Ker}(H_k(X_t^*) \to H_k(\mathbb{G}_m^n))$,

we may consider an extension of VMHS

$$(1) \quad 0 \to \overline{H}^{n-1}(X_t^*) \to \mathcal{V}_{\phi,t} \to H^n(\mathbb{G}_m^n) \to 0$$

or dualy

$$(2) \quad 0 \to H^n(\mathbb{P}_\Delta, \mathbb{D}_\Delta) \to \mathcal{V}^\vee_{\phi,t}(-n) \to \underline{H}_{n-1}(X_t^*)(-n) \to 0$$

$$\mathbb{D}_\Delta := \mathbb{P}_\Delta \setminus \mathbb{G}_m^n$$

and the extension classes are given by

$$(1) \quad \cdots \quad \langle \mathcal{V}_{\phi,t}, \gamma \rangle \equiv_{\mathbb{Q}(n)} \langle AJ(\{\underline{x}\}|_{X_t^*}), \gamma \rangle$$

$$(2) \quad \cdots \quad (\text{The period}) = \int_{(-1)^{n-1}\bigcap_i T_{x_i} (= \mathbb{R}_{<0})} \frac{d\log \underline{x}}{1 - t\phi(\underline{x})}$$

$$\equiv_{(2\pi i)^n \int_\eta \omega_t} \langle \widetilde{\mathcal{V}}_{\phi,t}, [\omega_t] \rangle .$$

Thm (A. Huang, B. Lian, S-T. Yau, X. Zhu '16)

$$\hat{\tau}^{\Delta}_{GKZ} := \tau^{a=(1,\underline{0}),\{1\}\times\Delta}_{GKZ} \cong H^n(\mathbb{P}_\Delta \setminus X_t, \mathbb{D}_\Delta \setminus Z).$$

(with ∇^{GM} for the $\mathcal{D}$-module str.).

and $\langle \tilde{\nu}_{\phi,t}, [\omega_t] \rangle$ is a GKZ-integral to obtain a local solution of the GKZ-system.

$\downarrow$ generalize from $\nu_{\phi,t}$ to ν_γ.

Def

For $\nu \in ANF(\mathcal{H}^{n-1}_{\nu}(p))$, the truncated HNF associated to ν is defined by

$$V(t) := \langle \tilde{\nu}(t), [\omega_t] \rangle \qquad (\tilde{\nu} := \nu_{\mathbb{Q}} - \nu_F)$$

This $V(t)$ defines an inhomogeneous PF-equation

$$LV(t) := g(t) \in \mathbb{C}(t)$$

Lem

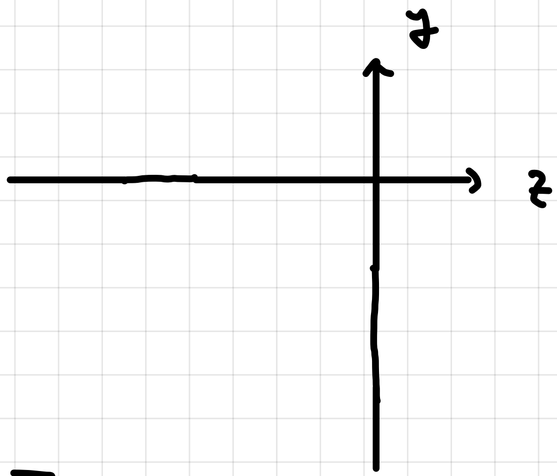
In our five cases of Fano threefolds, $\deg(g) = 1$.

- We normalize V to $\tilde{V}$ s.t. $g(t) = -t$.

  From the construction of $\mathcal{X}$, in both Case I and II, ν_g has no singularities (= "obstruction to extend $\mathcal{X}_t$ to a singular fiber") at $t = 0$.

Then the limit of $\tilde{V}(t)$ at $t=0$ is the desired Apéry constant.

Example (V_{14}. Apéry = $\frac{1}{7}\zeta(2)$)



$$V(t) = (2\pi i)^2 \int_{\Gamma_t} \omega_t$$

$$\overset{(t\ll 0)}{=} \frac{1}{(2\pi i)^2} \int_{A\times S^1} \frac{d\log u}{1 - t\phi}$$

$$\overset{\text{expand}}{=} \sum_{k\geq 0} t^k \int_A \left(\text{constant in } \phi \text{ w.r.t. } u \right)^k \frac{dv}{v} \wedge \frac{dw}{w}$$

$\Gamma_t = \{ (u, v, w) \in X_t \}$.

$\uparrow$

$A = \{ (v, w) \mid -1 \leq w \leq 0$

$-(w+1) \leq v \leq -(w+1)^2 \}$

coefficient for $k=0$: $\zeta(2)$.

for $k=1$: $-7+4\zeta(2)$.

$$\mapsto \quad V(t) = \zeta(2) + (-7 + 4\zeta(2))t + \cdots$$

$$(\mapsto \quad L(V(t)) = -7t)$$

$$\mapsto \quad \tilde{V}(t) = \frac{1}{7}\zeta(2) + \left(-1 + \frac{4}{7}\zeta(2)\right)t + \cdots$$

$$\xrightarrow[t\to 0]{\lim} \quad \frac{1}{7}\zeta(2).$$