Gamma integral structure for an invertible polynomial of chain type

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joint work with Atsushi Takahashi.

Review : Gamma integral structure for $\mathbb{P}^n$

$\mathbb{P}^n$: n-dimensional projective space

$\downarrow$ the Gromov–Witten theory of $\mathbb{P}^n$

$^\exists$ Frobenius structure ($=$ the quantum cohomology of $\mathbb{P}^n$) on the complex manifold

$$M_{\mathbb{P}^n} := \bigoplus_{q \in \mathbb{Z}} H^{q,q}(\mathbb{P}^n).$$

The Gamma integral structure for an algebraic variety was introduced by Iritani and Katzarkov–Kontsevich–Pantev.

Define a morphism $\mathrm{ch}_\Gamma : K_0(\mathcal{D}^b(\mathbb{P}^n)) \longrightarrow H^*(\mathbb{P}^n)$ by

$$\mathrm{ch}_\Gamma([E]) := \widehat{\Gamma}_{\mathbb{P}^n}\mathrm{Ch}(E), \quad E \in \mathcal{D}^b(\mathbb{P}^n).$$

- $\widehat{\Gamma}_{\mathbb{P}^n} := \prod_{i=1}^{n} \Gamma(1+\delta_i)$: the Gamma class of $\mathbb{P}^n$,
  ($\delta_1, \dots, \delta_n$: the Chern roots of the tangent bundle of $\mathbb{P}^n$).
- $\mathrm{Ch}(E) := \sum_{i=1}^{\mathrm{rank}\,E} \mathbf{e}[\delta_i^E]$: (modefied) Chern roots of $E \in \mathcal{D}^b(\mathbb{P}^n)$,
  ($\delta_1^E, \dots, \delta_{\mathrm{rank}\,E}^E$: the Chern roots of E).

Here $\mathbf{e}[-] = \exp(2\pi\sqrt{-1}\cdot -)$.

Definition 1.1 (Iritani).

The Gamma integral structure of the total Hodge cohomology space $H^*(\mathbb{P}^n)$ is defined to be a K-framing given by

$$\frac{1}{(2\pi\sqrt{-1})^n}\mathrm{ch}_\Gamma(K_0(\mathcal{D}^b(\mathbb{P}^n))).$$

$(\mathcal{O}(0), \mathcal{O}(1), \ldots, \mathcal{O}(n))$: Beilinson's full exceptional collection on $\mathcal{D}^b(\mathbb{P}^n)$.
$\{\mathbf{b}_i\}_{i=0}^n$: homogeneous basis of $H^*(\mathbb{P}^n)$ such that $\mathbf{b}_i \in H^{i,i}(X)$.

- $\mathbf{S}$: matrix representation of the automorphism on $K_0(\mathcal{D}^b(X))$ induced by the Serre functor $\mathcal{S} := - \otimes \omega_{\mathbb{P}^n}[n]$ w.r.t. $\{[\mathcal{O}(i)]\}$.
- χ : the Euler matrix w.r.t. $\{\mathcal{O}(i)\}$.
- $\mathrm{ch}_\Gamma := (\mathrm{ch}_{\Gamma,1}, \ldots, \mathrm{ch}_{\Gamma,n+1})$ is the matrix such that i-th column $\mathrm{ch}_{\Gamma,i}$ is given by
$$\mathrm{ch}_{\Gamma,i} := \mathrm{ch}_\Gamma(\mathcal{O}(i)) \in H^*(X).$$
- $\widetilde{Q}$: the grading (diagonal) matrix on $H^*(X)$. That is,
$$\widetilde{Q}_{ii} := \left(i - \frac{n}{2}\right).$$
- η : the matrix representation of the Poincaré pairing w.r.t. $\{\mathbf{b}_i\}$.

Proposition 1.2 (Iritani).

The following equality holds:

$$\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma\right)^{-1}\mathbf{e}[\widetilde{Q}]\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma\right)=\mathbf{S},$$

$$\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma\right)^{T}\mathbf{e}\left[\frac{1}{2}\widetilde{Q}\right]\eta\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma\right)=\chi.$$

$\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma$ = the central connection matrix of the Frobenius manifold

χ = the Stokes matrix of the Frobenius manifold

The mirror object of $\mathbb{P}^n$ is the Landau–Ginzburg model with a primitive form

- $f_q : (\mathbb{C}^*)^n \longrightarrow \mathbb{C}, \quad f_q(x_1, \ldots, x_n) := x_1 + \cdots + x_n + \dfrac{q}{x_1 \ldots x_n}$
- $\zeta = \left[\dfrac{dx_1 \wedge \cdots \wedge dx_n}{x_1 \ldots x_n} \right]$

for $q \in \mathbb{C}^*$.

In the mirror side, there is a "natural" integral structure on $\Omega_{f_q} \cong H^n((\mathbb{C}^*)^n, \mathrm{Re}(f_q) \gg 0; \mathbb{C})$ induced by $H_n((\mathbb{C}^*)^n, \mathrm{Re}(f_q) \gg 0; \mathbb{Z})$.

Theorem 1.3 (Galkin–Golyshev–Iritani, Iritani).

The Gamma integral structure on $H^*(X)$ is isomorphic to the natural one on Ω_{f_q}.

It is known by Coates–Corti–Iritani–Tseng and Iritani that for a weak Fano toric orbifold X the same statement of Theorem 1.3 is true.

Invertible polynomial

$f \in \mathbb{C}[z_1, \ldots, z_n]$: weighted homogeneous polynomial.

$\Leftrightarrow {}^{\exists} w_1, \ldots, w_n, d \in \mathbb{Z}_{\geq 1}$ such that

$$f(\lambda^{w_1} z_1, \ldots, \lambda^{w_n} z_n) = \lambda^d f(z_1, \ldots, z_n), \quad \lambda \in \mathbb{C}^*.$$

Definition 2.1.

A weighted homogeneous polynomial $f = f(\mathbf{z})$ is invertible if

- *f is non-degenerate. That is, f has at most an isolated critical point at the origin $\mathbf{z} = 0$.*
- *f is of the form*

  $$f(z_1, \ldots, z_n) = \sum_{i=1}^{n} \prod_{j=1}^{n} z_j^{\mathbb{E}_{ij}}$$

  for $\mathbb{E}_{ij} \in \mathbb{Z}_{\geq 0}$, $i, j = 1, \ldots, n$.
- *The matrix $\mathbb{E} := (\mathbb{E}_{ij})$ of size n is invertible over $\mathbb{Q}$.*

- *Jacobian algebra* : $\mathrm{Jac}(f) := \mathbb{C}[z_1, \ldots, z_n] \Big/ \left(\frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \right).$
- *Milnor number* : $\mu_f := \dim_{\mathbb{C}} \mathrm{Jac}(f).$
- $\mathbb{C}$-vector space Ω_f : $\Omega_f := \Omega^n(\mathbb{C}^n)/df \wedge \Omega^{n-1}(\mathbb{C}^n).$

  By choosing a nowhere vanishing n-form $d\mathbf{z} := dz_1 \wedge \cdots \wedge dz_n$ we have the following isomorphism

$$\mathrm{Jac}(f) \xrightarrow{\cong} \Omega_f, \quad [\phi(\mathbf{z})] \mapsto [\phi(\mathbf{z}) d\mathbf{z}].$$

  Ω_f is an analogue of the total Hodge cohomology for an algenraic variety.
- non-degenerate symmetric $\mathbb{C}$-bilinear form $J_f : \Omega_f \times \Omega_f \longrightarrow \mathbb{C}$:

$$J_f \left([\phi_1(\mathbf{z}) d\mathbf{z}], [\phi_2(\mathbf{z}) d\mathbf{z}] \right) := \mathrm{Res}_{\mathbb{C}^n} \begin{bmatrix} \phi_1(\mathbf{z})\phi_2(\mathbf{z}) dz_1 \wedge \cdots \wedge dz_n \\ \frac{\partial f}{\partial z_1}, \ldots, \frac{\partial f}{\partial z_n} \end{bmatrix}.$$

Let $f_1 \in \mathbb{C}[z_1, \dots, z_n]$ and $f_2 \in \mathbb{C}[z'_1, \dots, z'_m]$ be invertible polynomials.
The *Thom–Sebastiani sum* $f_1 \oplus f_2$ is defined by

$$f_1 \oplus f_2 := f_1 \otimes 1 + 1 \otimes f_2 \in \mathbb{C}[z_1, \dots, z_n] \otimes_{\mathbb{C}} \mathbb{C}[z'_1, \dots, z'_m].$$

Proposition 2.2 (Kreuzer–Skarke).

Any invertible polynomial f can be written as a Thom–Sebastiani sum $f = f_1 \oplus \cdots \oplus f_p$ of invertible ones f_ν, $\nu = 1, \dots, p$ of the following types:

- (chain type) $z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_{m-1}^{a_{m-1}} z_m + z_m^{a_m}, \quad m \geq 1$;
- (loop type) $z_1^{a_1} z_2 + z_2^{a_2} z_3 + \cdots + z_{m-1}^{a_{m-1}} z_m + z_m^{a_m} z_1, \quad m \geq 2$.

We have

- $\mathrm{Jac}(f_1 \oplus f_2) \cong \mathrm{Jac}(f_1) \otimes_{\mathbb{C}} \mathrm{Jac}(f_2)$,
- $\Omega_{f_1 \oplus f_2} \cong \Omega_{f_1} \otimes_{\mathbb{C}} \Omega_{f_2}$,
- $\mu_{f_1 \oplus f_2} = \mu_{f_1} \cdot \mu_{f_2}$, ...

Invertible polynomial with Group

Definition 2.3.

The group of maximal diagonal symmetries G_f *of* f *is defined as*

$$G_f := \{(\lambda_1, \dots, \lambda_n) \in (\mathbb{C}^*)^n \mid f(\lambda_1 z_1, \dots, \lambda_n z_n) = f(z_1, \dots, z_n)\} .$$

- Each element $g \in G_f$ has a unique expression of the form

$$g = (\mathbf{e}[\alpha_1], \dots, \mathbf{e}[\alpha_n]), \quad 0 \leq \alpha_i < 1.$$

- The *age* of $g \in G_f$ is defined to be the rational number

$$\mathrm{age}(g) := \sum_{i=1}^{n} \alpha_i.$$

For each $g \in G_f$, set

- $\mathrm{Fix}(g) := \{\mathbf{z} \in \mathbb{C}^n \,|\, g \cdot \mathbf{z} = \mathbf{z}\}$,
- $f^g := f|_{\mathrm{Fix}(g)} : \mathrm{Fix}(g) \longrightarrow \mathbb{C}$,
- $n_g := \dim_{\mathbb{C}} \mathrm{Fix}(g)$.

Definition 2.4.

Define a $\mathbb{Q}$-graded complex vector space Ω_{f,G_f} by

$$\Omega_{f,G_f} := \bigoplus_{g \in G_f} \Omega_{f,g}, \quad \Omega_{f,g} := (\Omega_{f^g})^{G_f}(-\mathrm{age}(g)).$$

For the pair (f, G_f) a non-degenerate symmetric $\mathbb{C}$-bilinear form $J_{f,G_f} : \Omega_{f,G_f} \times \Omega_{f,G_f} \longrightarrow \mathbb{C}$ is defined.

Mirror symmetry for invertible polynomials

$f \in \mathbb{C}[z_1, \dots, z_n]$: invertible polynomial.
$\widetilde{f} \in \mathbb{C}[x_1, \dots, x_n]$: the *Berglund–Hübsch transpose* of the polynomial f,

$$\widetilde{f}(x_1, \dots, x_n) := \sum_{i=1}^{n} \prod_{j=1}^{n} x_j^{\mathbb{E}_{ji}}.$$

It is expected by Berglund–Hübsch that the polynomial $\widetilde{f}$ is a mirror dual object corresponding to the pair (f, G_f).

Proposition 2.5 (Kreuzer).

There exists an isomorphism of $\mathbb{Q}$-graded $\mathbb{C}$-vector spaces

$$\mathbf{mir} : \Omega_{\widetilde{f}} \cong \Omega_{f,G_f}.$$

Invertible polynomial of chain type

From now on, we only consider invertible polynomials of chain type:

$$f_n := z_1^{a_1} z_2 + \cdots + z_{n-1}^{a_{n-1}} z_n + z_n^{a_n}, \quad n \geq 1$$

Then the Berglund–Hübsch transpose is given by

$$\widetilde{f}_n := x_1^{a_1} + x_1 x_2^{a_2} + \cdots + x_{n-1} x_n^{a_n}.$$

For simplicity, we assume that $a_i \geq 2$ for all $i = 1, \ldots, n$.

$$f_n(\mathbf{z}) = z_1^{a_1} z_2 + z_2^{a_2} z_3 + f_{n-2}(z_3, \ldots, z_n),$$
$$\widetilde{f}_n(\mathbf{x}) = \widetilde{f}_{n-2}(x_1, \ldots, x_{n-2}) + x_{n-2} x_{n-1}^{a_{n-1}} + x_{n-1} x_n^{a_n}.$$

Proposition 2.6 (Kreuzer).

Define sets B'_n, B_n of monomials in $\mathbb{C}[x_1,\dots,x_n]$ inductively as follows:

1. *($n=0$)* $B_0 := B'_0 = \{1\}$.
2. *($n=1$)* $B_1 := B'_1 = \left\{x_1^{k_1} \,|\, 0 \le k_1 \le a_1 - 2\right\}$.
3. *($n \ge 2$)* $B'_n := \left\{x_1^{k_1}x_2^{k_2}\cdots x_n^{k_n} \,\middle|\, \begin{array}{c} 0 \le k_i \le a_i - 1 \ (i=1,\dots,n-1) \\ 0 \le k_n \le a_n - 2 \end{array}\right\}$,
   and
$$B_n := B'_n \cup \left\{\phi^{(n-2)}(x_1,\dots,x_{n-2})x_n^{a_n-1} \,|\, \phi^{(n-2)}(x_1,\dots,x_{n-2}) \in B_{n-2}\right\}.$$

The set B_n defines a $\mathbb{C}$-basis of the Jacobian algebra $\mathrm{Jac}(\widetilde{f}_n)$. Namely, we have $\mathrm{Jac}(\widetilde{f}_n) = \langle [\phi^{(n)}(\mathbf{x})] \,|\, \phi^{(n)}(\mathbf{x}) \in B_n\rangle_{\mathbb{C}}$.

For $i = 1,\dots,n$, set
$$d_i := a_1a_2\cdots a_i, \quad d_0 := 1.$$
$$\widetilde{\mu}_n := \sum_{i=0}^{n}(-1)^{n-i}d_i.$$

Corollary 2.7.

The Milnor number $\mu_{\widetilde{f}_n} = \dim_{\mathbb{C}} \mathrm{Jac}(\widetilde{f}_n)$ *is given by* $\widetilde{\mu}_n$.

Note that $\# B'_{\widetilde{f}_n} = d_n - d_{n-1}$.

$\{\zeta^{(n)}\}$: the basis of $\Omega_{\widetilde{f}_n}$ induced by the basis $\{\phi^{(n)}\}$.

$$\Omega_{\widetilde{f}_n} \cong \left(\bigoplus_{\mathbf{k}} \mathbb{C} \cdot \zeta^{(n)}_{\mathbf{k}} \right) \bigoplus \Omega_{\widetilde{f}_{n-2}}(-1),\ n \geq 2,$$

where $\mathbf{k} = (k_1, \dots, k_n)$ runs the set
$\{0 \leq k_i \leq a_i - 1\ (i = 1, \dots, n-1),\ 0 \leq k_n \leq a_n - 2\}$ and $\zeta^{(n)}_{\mathbf{k}}$ is the element corresponding to $[x_1^{k_1}, \dots, x_n^{k_n}]$.

Define two sets I_n' and I_n as follows:

1. $(n=0)\quad I_0 := I_0' := \{1\}$.
2. $(n=1)\quad I_1 := I_1' := \{1,2,\dots,a_1-1\}$.
3. $(n\geq 2)\quad I_n' := \{\kappa\in\{1,\dots,d_n\}\,|\,a_n\nmid\kappa\}$ and $I_n := I_n'\cup I_{n-2}$.

Remark: $\#I_n' = d_n - d_{n-1}$ and $\#I_n = \widetilde{\mu}_n$.

For every $\kappa\in I_n'$,

$$g_\kappa := \left(\mathbf{e}\left[\frac{1}{d_n}\kappa\right],\dots,\mathbf{e}\left[(-1)^{i-1}\frac{d_{i-1}}{d_n}\kappa\right],\dots,\mathbf{e}\left[(-1)^{n-1}\frac{d_{n-1}}{d_n}\kappa\right]\right)\in G_{f_n}$$

has order d_n and $\mathrm{Fix}(g_\kappa)=\{0\}$.

Define rational numbers $\omega_{\kappa,i}^{(n)},\ i=1,\dots,n$ by

$$\omega_{\kappa,i}^{(n)} := (-1)^{i-1}\frac{d_{i-1}}{d_n}\cdot\kappa - \left\lfloor(-1)^{i-1}\frac{d_{i-1}}{d_n}\cdot\kappa\right\rfloor$$

so that

$$g_\kappa = \left(\mathbf{e}\left[\omega_{\kappa,1}^{(n)}\right],\dots,\mathbf{e}\left[\omega_{\kappa,n}^{(n)}\right]\right).$$

$^{\exists}$ natural inclusion map

$$G_{f_{n-2}} \hookrightarrow G_{f_n}, \quad (\mathbf{e}[\alpha_1], \ldots, \mathbf{e}[\alpha_{n-2}]) \mapsto (\mathbf{e}[\alpha_1], \ldots, \mathbf{e}[\alpha_{n-2}], 1, 1).$$

We construct a basis $\{\xi_\kappa^{(n)}\}_{\kappa \in I_n}$ of the $\mathbb{C}$-vector space $\Omega_{f_n, G_{f_n}}$ satisfying

$$\Omega_{f_n, G_{f_n}} \cong \left(\bigoplus_{\kappa \in I'_n} \mathbb{C} \cdot \xi_\kappa^{(n)} \right) \bigoplus \Omega_{f_{n-2}, G_{f_{n-2}}}(-1), \quad n \geq 2.$$

as follows:

1. $(n = 0)$ Set $\xi_1^{(0)} := 1 \quad (1 \in I_0 = \{1\})$.
2. $(n = 1)$ Set $\xi_\kappa^{(1)} := \mathbf{1}_{g_\kappa}, \quad \kappa \in I_1$.
3. $(n \geq 2)$ Set

$$\xi_\kappa^{(n)} := \begin{cases} \mathbf{1}_{g_\kappa} & \kappa \in I'_n, \\ [\overline{\xi}_\kappa^{(n-2)} \wedge d(z_{n-1}^{a_{n-1}}) \wedge dz_n], & \kappa \in I_{n-2}, \end{cases}$$

where $\overline{\xi}_\kappa^{(n-2)}$ does a differential form representing $\xi_\kappa^{(n-2)}$.

For every $\mathbf{k}$ satisfying $\mathbf{x}^{\mathbf{k}} := x_1^{k_1} \cdots x_n^{k_n} \in B'_{\widetilde{f}_n}$, one can define a rational number $\omega^{(n)}_{\mathbf{k},i}$ for $i = 1, \ldots, n$.

Proposition 2.8.

There exists a bijection of sets of $d_n - d_{n-1}$ elements

$$\psi : \left\{ \mathbf{k} = (k_1, \ldots, k_n) \,\middle|\, \begin{matrix} 0 \leq k_i \leq a_i - 1 \ (i = 1, \ldots, n-1) \\ 0 \leq k_n \leq a_n - 2 \end{matrix} \right\} \xrightarrow{\cong} I'_n,$$

such that $\omega^{(n)}_{\mathbf{k},i} = \omega^{(n)}_{\psi(\mathbf{k}),i}$ for each $i = 1, \ldots, n$.

Proposition 2.9.

There exists an isomorphism

$$\mathbf{mir} : (\Omega_{\widetilde{f}_n}, J_{\widetilde{f}_n}) \cong (\Omega_{f_n, G_{f_n}}, J_{f_n, G_{f_n}}), \quad \zeta^{(n)}_{\mathbf{k}} \mapsto \xi^{(n)}_{\kappa}.$$

Moreover, the matrix representation $\eta^{(n)}$ of $J_{f_n, G_{f_n}}$ with respect to the basis $\{\xi^{(n)}_{\kappa}\}_{\kappa \in I_n}$ is given by

1. *($n = 0$)* $\eta^{(0)} = (1)$,
2. *($n = 1$)*
$$\eta^{(1)} = \left(\frac{1}{a_1} \delta_{\kappa+\lambda, a_1} \right),$$
3. *($n \geq 2$)*
$$\eta^{(n)} = \begin{pmatrix} \frac{1}{d_n} \delta_{\kappa+\lambda, d_n} & 0 \\ 0 & -\frac{1}{a_n} \eta^{(n-2)} \end{pmatrix},$$

where κ and λ run the set I'_n.

Define a diagonal matrix $\widetilde{Q}^{(n)}$ of size $\widetilde{\mu}_n$ inductively as follows:

1. $(n = 0)$ Set $\widetilde{Q}^{(0)} := (0)$,
2. $(n = 1)$ Set $\widetilde{Q}^{(1)} := \left(\left(\omega^{(1)}_{\kappa,1} - \frac{1}{2}\right) \delta_{\kappa\lambda}\right)$,
3. $(n \geq 2)$ Set
$$\widetilde{Q}^{(n)} := \begin{pmatrix} \widetilde{P}^{(n)} & 0 \\ 0 & \widetilde{Q}^{(n-2)} \end{pmatrix},$$
where $\widetilde{P}^{(n)} = (\widetilde{P}^{(n)}_{\kappa\lambda})$ is a matrix of size $(d_n - d_{n-1})$ given by
$$\widetilde{P}^{(n)}_{\kappa\lambda} := \left(\sum_{l=1}^{n} \left(\omega^{(n)}_{\kappa,l} - \frac{1}{2}\right)\right) \delta_{\kappa\lambda}, \quad \kappa, \lambda \in I'_n.$$

Remark: The matrix $\widetilde{Q}^{(n)}$ corresponds to the exponents of $\widetilde{f}_n$ shifted by $-\dfrac{n}{2}$.

Proposition 2.10.

For each $\mathbf{k} = (k_1, \ldots, k_n)$ *such that* $\mathbf{x}^{\mathbf{k}} \in B'_n$, *we have*

$$\int_{(\mathbb{R}_{\geq 0})^n} e^{-\widetilde{f}_n(\mathbf{x})} \mathbf{x}^{\mathbf{k}} d\mathbf{x} = \frac{1}{d_n} \prod_{l=1}^{n} \Gamma(1 - \omega^{(n)}_{d_n - \psi(\mathbf{k}), l}),$$

where $\Gamma(s)$ *denotes the Gamma function.*

Remark : We can consider the number such as $\omega_{\kappa,i}$ and $\omega_{\mathbf{k},i}$ for any invertible polynomial. Hence, this formula can be generalize to any invertible polynomial.

Maximally graded matrix factorizations

$$S_n := \mathbb{C}[z_1, \dots, z_n],$$
$$L_{f_n} := \left(\bigoplus_{i=1}^{n} \mathbb{Z}\vec{z}_i \oplus \mathbb{Z}\vec{f}_n \right) \Big/ \left(\vec{f}_n - \sum_{j=1}^{n} \mathbb{E}_{ij}\vec{z}_j;\ i = 1, \dots, n \right).$$

$\mathrm{HMF}_{S_n}^{L_{f_n}}(f_n)$: the homotopy category of L_{f_n}-graded matrix factorizations.

Proposition 3.1 (Aramaki–Takahashi).

There exists a full exceptional collection $(E_1, \dots, E_{\widetilde{\mu}_n})$ *of* $\mathrm{HMF}_{S_n}^{L_{f_n}}(f_n)$ *such that* $\chi(E_i, E_j) = \chi_{ij}^{(n)}$ *where* $\chi^{(n)}$ *is a matrix defined by* $\chi^{(n)} = 1/\varphi_n(N)$, $N = (\delta_{i+1,j})$, *and*

$$\varphi_n(t) := \prod_{i=0}^{n} \left(1 - t^{d_i}\right)^{(-1)^{n-i}} \quad (n \geq 1), \quad \varphi_0(t) := 1 - t.$$

$\mathbf{S}^{(n)}$: the matrix representation of the automorphism on $K_0(\mathrm{HMF}_{S_n}^{L_{f_n}}(f_n))$ induced by the Serre functor with respect to the basis $\{[E_i]\}_{i=1}^{\widetilde{\mu}_n}$.

Definition 3.2.

Define a matrix $\mathrm{ch}_\Gamma^{(n)} := \left(\mathrm{ch}_{\Gamma,1}^{(n)}, \cdots, \mathrm{ch}_{\Gamma,\widetilde{\mu}_n}^{(n)}\right)$ *of size* $\widetilde{\mu}_n$ *as follows:*

1. $(n = 0)$ *Set* $\mathrm{ch}_\Gamma^{(0)} := (1)$.
2. $(n = 1)$ *Set*

$$\left(\mathrm{ch}_{\Gamma,j}^{(1)}\right)_\kappa := \Gamma\left(1 - \omega_{\kappa,1}^{(1)}\right)\left(1 - \mathbf{e}\left[\omega_{\kappa,1}^{(1)}\right]\right)\mathbf{e}\left[\omega_{\kappa,1}^{(1)}(j-1)\right], \ \kappa \in I_1 = I_1'.$$

3. $(n \geq 2)$ *Set*

$$\left(\mathrm{ch}_{\Gamma,j}^{(n)}\right)_\kappa := \begin{cases} c_\kappa^{(n)}\mathbf{e}\left[(-1)^{n-1}\omega_{\kappa,1}^{(n)}(j-1)\right], & \text{if } \kappa \in I_n', \\ 2\pi\sqrt{-1}\ \left(\mathrm{ch}_{\Gamma,j}^{(n-2)}\right)_\kappa, & \text{if } \kappa \in I_{n-2} = I_n \setminus I_n', \end{cases}$$

where

$$c_\kappa^{(n)} := \begin{cases} \displaystyle\prod_{l=1}^{n}\Gamma\left(1 - \omega_{\kappa,l}^{(n)}\right) \cdot \prod_{i=1}^{m}\left(1 - \mathbf{e}\left[\omega_{\kappa,2i-1}^{(2m-1)}\right]\right), & \text{if } \ n = 2m-1, \\ \displaystyle\prod_{l=1}^{n}\Gamma\left(1 - \omega_{\kappa,l}^{(n)}\right) \cdot \prod_{i=1}^{m}\left(1 - \mathbf{e}\left[\omega_{\kappa,2i}^{(2m)}\right]\right), & \text{if } \ n = 2m. \end{cases}$$

- The first part $\prod_{l=1}^{n} \Gamma(1-\omega_{\kappa,l}^{(n)})$ of $c_\kappa^{(n)}$ can be considered as $\widehat{\Gamma}_{f_n, G_{f_n}}$ on the κ-sector Ω_{f_n, g_κ} (cf. Chiodo–Iritani–Ruan).
- The last part of $c_\kappa^{(n)}$ can be considered as $\mathrm{Ch}(E_1)$ on the κ-sector (cf. Polishchuk–Vaintrob).
- The part $\mathbf{e}\left[(-1)^{n-1}\omega_{\kappa,1}^{(n)}(j-1)\right]$ comes from the auto-equivalence $((-1)^n j\vec{z}_1)$ whose matrix representation is given by

$$\begin{pmatrix} \left(\mathbf{e}\left[(-1)^{n-1}\omega_{\kappa,1}^{(n)}(j-1)\right]\delta_{\kappa\lambda}\right) & 0 \\ 0 & (1) \end{pmatrix},$$

  which acts on the vector $\mathrm{ch}_{\Gamma,1}^{(n)}$ to get $\mathrm{ch}_{\Gamma,j}^{(n)}$.

Therefore, $\mathrm{ch}_{\Gamma,j}^{(n)}$ can be considered as the matrix representation of "$\widehat{\Gamma}_{f_n, G_{f_n}} \mathrm{Ch}(E_j)$" with respect to the basis $\{\xi_\kappa^{(n)}\}_{\kappa \in I_n}$.

Main Theorem 1

Theorem 3.3 (O-Takahashi).

We have the following equality:

$$\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma^{(n)}\right)^{-1}\mathbf{e}\left[\widetilde{Q}^{(n)}\right]\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma^{(n)}\right)=\mathbf{S}^{(n)}, \tag{1}$$

$$\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma^{(n)}\right)^{T}\mathbf{e}\left[\frac{1}{2}\widetilde{Q}^{(n)}\right]\eta^{(n)}\left(\frac{1}{(2\pi)^{\frac{n}{2}}}\mathrm{ch}_\Gamma^{(n)}\right)=\chi^{(n)}. \tag{2}$$

This theorem is an analogue of Theorem 1.2

(Sketch of proof)

Induction on n. (1) follows from (2).

The following lemma implies (2).

Lemma 3.4.

Let

$$p_n(t) := \frac{1}{\varphi_n(t)} \cdot (1 - t^{d_n}) = \prod_{i=1}^{n} \left(1 - t^{d_{i-1}}\right)^{(-1)^{n-i}} \quad (n \geq 1), \quad p_0(t) := 1.$$

We have

$$\chi_{i,j}^{(n)} = \frac{1}{d_n} \sum_{a=1}^{d_n} p_n\left(\mathbf{e}\left[\frac{a}{d_n}\right]\right) \mathbf{e}\left[\frac{a}{d_n}(i-j)\right].$$

$p_n(t)$ is the Poincaré polynomial of the L_{f_n}-graded ring

$$\bigoplus_{\vec{l} \in L_{f_n}} \mathrm{HMF}_{S_n}^{L_{f_n}}(f_n)(E_1, E_1(\vec{l})).$$

Integral structures

$^\exists$ two isomorphisms

$$H_n(\mathbb{C}^n, \mathrm{Re}(\widetilde{f}_n) \gg 0; \mathbb{Z}) \cong H_n(\mathbb{C}^n, \widetilde{f}_n^{-1}(1); \mathbb{Z}) \cong H_{n-1}(\widetilde{f}_n^{-1}(1); \mathbb{Z}).$$

Define a "Poincaré Duality" map $\mathbb{D} : H_{n-1}(\widetilde{f}_n^{-1}(1); \mathbb{Z}) \to \Omega_{\widetilde{f}_n}$ by

$$\mathbb{D}(L) := \frac{1}{(2\pi\sqrt{-1})^n} \sum_{\kappa \in I_n} \left(\sum_{\lambda \in I_n} \eta^{\lambda\kappa} \int_\Gamma e^{-\widetilde{f}_n} \zeta_\lambda^{(n)} \right) \zeta_\kappa^{(n)}, \quad L \in H_{n-1}(\widetilde{f}_n^{-1}(1); \mathbb{Z})$$

where $\Gamma \in H_n(\mathbb{C}^n, \mathrm{Re}(\widetilde{f}_n) \gg 0; \mathbb{Z})$ is corresponding to $L \in H_{n-1}(\widetilde{f}_n^{-1}(1); \mathbb{Z})$.

Define the integral structure $\Omega_{\widetilde{f}_n;\mathbb{Z}}$ of $\Omega_{\widetilde{f}_n}$ by

$$\Omega_{\widetilde{f}_n;\mathbb{Z}} := \mathbb{D}\left(H_{n-1}(\widetilde{f}_n^{-1}(1); \mathbb{Z}) \right).$$

Note that $H_{n-1}(\widetilde{f}_n^{-1}(1); \mathbb{Z}) \cong K_0(\mathcal{D}^b\mathrm{Fuk}^{\to}(\widetilde{f}_n))$.

Homological mirror symmetry conjectures

$$\mathcal{D}^b\mathrm{Fuk}^{\to}(\widetilde{f}_n) \cong \mathrm{HMF}^{L_f}_{S}(f_n).$$

Definition 3.5.

We define a K-group framing $\mathrm{ch}^{(n)}_{\Gamma} : K_0(\mathrm{HMF}^{L_{f_n}}_{S_n}(f_n)) \longrightarrow \Omega_{f_n,G_{f_n}}$ *by*

$$\mathrm{ch}^{(n)}_{\Gamma}([E_j]) := \sum_{\kappa \in I_n} \mathrm{ch}^{(n)}_{\Gamma,\kappa j}\, \xi^{(n)}_{\kappa}.$$

We call the image $\Omega_{f_n,G_{f_n};\mathbb{Z}} := \dfrac{1}{(2\pi\sqrt{-1})^n}\mathrm{ch}^{(n)}_{\Gamma}\left(K_0(\mathrm{HMF}^{L_{f_n}}_{S_n}(f_n))\right)$ *the* *Gamma integral structure of* $\Omega_{f_n,G_{f_n}}$.

$(2\pi\sqrt{-1})^{-n}$ is due to the fact that $\Omega_{f_n,G_{f_n}}$ has the natural weight n from the view point of the Hodge theory.

Main Theorem 2

$\exists$ $\mathbb{Z}/d_n\mathbb{Z}$-action on $\Omega_{\widetilde{f}_n;\mathbb{Z}}$ and $\Omega_{f_n,G_{f_n};\mathbb{Z}}$.

- $\mathbb{Z}/d_n\mathbb{Z}$-action on $\mathbb{C}[x_1,\dots,x_n]$ given by

$$(x_1,\dots,x_i,\dots,x_n)\mapsto\left(\mathbf{e}\left[\frac{1}{d_1}\right]x_1,\dots,\mathbf{e}\left[\frac{(-1)^{i-1}}{d_i}\right]x_i,\dots,\mathbf{e}\left[\frac{(-1)^{n-1}}{d_n}\right]x_n\right)$$

  induces the one on $\Omega_{\widetilde{f}_n;\mathbb{Z}}$.
- The grading shift functor $(\vec{z}_1)$ on $\mathrm{HMF}^{L_{f_n}}_{S_n}(f_n)$ induces the action on $\Omega_{\widetilde{f}_n;\mathbb{Z}}$.

Theorem 3.6 (O-Takahashi).

The mirror isomorphism $\mathbf{mir}:\Omega_{\widetilde{f}_n}\cong\Omega_{f_n,G_{f_n}}$ *induces an isomorphism of integral structures* $\Omega_{\widetilde{f}_n;\mathbb{Z}}\cong\Omega_{f_n,G_{f_n};\mathbb{Z}}$ *and the isomorphism is* $\mathbb{Z}/d_n\mathbb{Z}$*-equivariant.*

This theorem was proven for the ADE case by Milanov–Zha.

(Strategy of proof)

We show

$$\mathrm{ch}^{(n)}_{\Gamma,\kappa 1} = \sum_{\lambda \in I_n} \eta^{\lambda\kappa} \int_{\Gamma_1} e^{-\widetilde{f}_n} \zeta^{(n)}_{\lambda}$$

and $\mathbb{Z}/d_n\mathbb{Z}$-equivariance.

- $(n = 1)$: By direct calculation.
- $(n = 2)$: By direct calculation based on Milanov–Zha.
- $(n \geq 3)$: By induction on n.
  - (Step 1) : If the case $n = 2m$ is true, then so is $n = 2m + 1$.
  - (Step 2) : If the case $n = 2m - 1$ is true, then so is $n = 2m$.

The way of proof (Step 1) and (Step 2) are different.

Bridgeland stability condition

In the A-model side, it is expected that the oscillatory integral $\int e^{-\widetilde{f}_n(\mathbf{x})} d\mathbf{x}$ induces a stability condition on $\mathcal{D}^b\mathrm{Fuk}^{\to}(\widetilde{f}_n)$. By Theorem 3.6, the mirror object dual to $\int_{\Gamma_j} e^{-\widetilde{f}_n(\mathbf{x})} d\mathbf{x}$ is given by $\sum_{\lambda \in I_n} \eta_{\psi(\mathbf{0})\lambda} \mathrm{ch}^{(n)}_{\Gamma,\lambda j}$.

Remark : $[d\mathbf{x}]$ is induced by the "canonical" primitive form for $\widetilde{f}_n$. Here, "canonical" means that this primitive form is determined by exponents of $\widetilde{f}_n$.

We expect the following conjecture:

Conjecture 3.7.

There exists a Bridgeland stability condition σ *on* $\mathrm{HMF}^{L_{f_n}}_{S_n}(f_n)$ *such that its stability function* $Z_\sigma : K_0(\mathrm{HMF}^{L_{f_n}}_{S_n}(f_n)) \longrightarrow \mathbb{C}$ *is given as follows:*
if $n = 2m-1$*, then*

$$Z_\sigma([E_j]) := \frac{1}{(2\pi\sqrt{-1})^n}\mathbf{e}\left[-\frac{j-1}{d_n}\right]\prod_{i=1}^{m}\left(1-\mathbf{e}\left[-\omega^{(n)}_{2i-1}\right]\right)\cdot\int_{(\mathbb{R}_{\geq 0})^n} e^{-\widetilde{f}_n(\mathbf{x})}d\mathbf{x},$$

and if $n = 2m$*, then*

$$Z_\sigma([E_j]) := \frac{1}{(2\pi\sqrt{-1})^n}\mathbf{e}\left[\frac{j-1}{d_n}\right]\prod_{i=1}^{m}\left(1-\mathbf{e}\left[-\omega^{(n)}_{2i}\right]\right)\cdot\int_{(\mathbb{R}_{\geq 0})^n} e^{-\widetilde{f}_n(\mathbf{x})}d\mathbf{x},$$

where $\omega^{(n)}_i$ *is the* i*-th rational weight of* f_n*.*
Moreover, this stability condition σ *is of Gepner type with respect to the auto-equivalence* $(\vec{z}_1)$ *and* $\mathbf{e}[1/d_n] \in \mathbb{C}$*:* $(\vec{z}_1).\sigma = \sigma.\mathbf{e}\left[\frac{1}{d_n}\right]$*.*

This conjecture is true for some cases.

Based on the results by Takahashi and Kajiura–Saito–Takahashi, we obtain the following

Proposition 3.8.

Conjecture 3.7 holds for $n = 1$ *and for invertible polynomials of ADE type in two and three variables which is the Thom–Sebastiani sum of invertible polynomials of chain type.*

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Thank you for your attention !