A tropical analog of the Hodge conjecture for smooth algebraic varieties over trivially valued fields

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based on https://arxiv.org/abs/2009.04690

https://arxiv.org/abs/2009.04677

Cycle class map.

X: sm. proj. var. / $\mathbb{C}$.

the Hodge conj.: surjective

$$CH^p(X)\otimes\mathbb{Q} \to H^{p,p}(X) \cap H^{2p}(X, \mathbb{Q})$$

$$\searrow \; H^{p,p}_{trop}(X) \quad \circlearrowright \quad \nearrow ? \quad \text{future}$$

Liu's tropical cycle class map

Today: $\boxed{\text{isomorphism}}$

$H^{p,p}_{trop}(X)$: tropical cohomology.

- Gross-Siebert
  $H^q(X_0(B), \Omega^p) \simeq H^q(B, \wedge^p T^\vee)$
  $X_0(B)$: log Calabi-Yau; B: trop. affn. mfd; $T^\vee$: cotangent sp.
- Itenberg-Katzarkov-Mikhalkin-Zarkov
  a similar result

Main Theorem 1.

X: sm. algebraic variety over a field K (e.g. $\mathbb{C}$) w/ the trivial valuation

$K \to \{0, \infty\}$

$0 \mapsto \infty$

$a \neq 0 \mapsto 0$.

Then

$$CH^p(X)\otimes\mathbb{Q} \to H^{p,p}_{trop}(X)$$

is an isomorphism.

Rem.

- $\exists$ similar studies
  - '09. (announcement) Katzarkov-Kontsevich
  - '20 Zarkov's paper on Kontsevich's idea (to disprove the Hodge conj.)
- Many properties of trop. cohomology of smth. trop. var.
  - Lefschetz (1,1) — Jell-Rau-Shaw '18
  - Hodge-Riemann relation, hard Lefschetz — Adiprasito-Huh-Katz (matroid), Amini-Piquerez. (general)

Strategy

K_T^p : tropical K-groups. ($\mathbb{Q}$-coeff.)

— the Gersten resolution —

the residue map

$$0 \to \underset{\text{sheaf'n}}{K_T^p} \to \bigoplus_{x \in X^0} K_T^p(k(x)) \xrightarrow{d} \bigoplus_{x \in X^1} K_T^{p-1}(k(x)) \xrightarrow{d} \cdots$$

$$\cdots \to \bigoplus_{x \in X^i} K_T^{p-i}(k(x)) \to \cdots\cdots \xrightarrow{d} \bigoplus_{x \in X^{p-1}} K_T^1(k(x)) \xrightarrow{d} \bigoplus_{x \in X^p} K_T^0(k(x)) \to 0$$

$k(x)^\times \xrightarrow{\text{div.}} \mathbb{Q}$

X^i : codim i points.

$$\rightsquigarrow H^p(X_{zar}, K_T^p) \simeq CH^p(X)_{\mathbb{Q}}$$

Main Theorem 2. ($\Rightarrow$ Main Theorem)

$$H^{p,q}_{trop}(X) \simeq H^q(X_{Zar}, K^p_T)$$

Strategy.

Key. → $\cdot\ \mathbb{A}^1$-homotopy inv.
- étale excision
- corestriction map / fin. fields.
- ⋮

(algebro-geometric) a theorem on cohomology theories

$\Rightarrow$ $\mathcal{H}^{p,q}_{trop}$ (sheaf'n) has the Gersten resolution

$\to$ Main Thm 2.

Rem. (char 0)

By resolution of singularities (instead of the theorem on coh. theories),

"$H^{p,p}_{trop}(X) \to Hdg^{p,p}(X)$: surj

$\Rightarrow$ the Hodge conj. holds"

(∵ Reduce to good (schön) tropical varieties.)

K: a field w/ the trivial valuation

tropicalization

T_Σ : a toric var. $\longleftrightarrow \Sigma$; a fan

$\cup$

$\mathbb{G}_m^n = \operatorname{Spec} K[M]$: the dense orbit

$X \subset T_\Sigma$: a closed subvar.

$X^{\mathrm{Ber}} := \{ v : \mathcal{O}_X \to \mathbb{R} \cup \{\infty\} : \text{valuations} \mid \text{trivial on } K \}$,
the Berkovich space

For $\sigma \in \Sigma$,

$$\mathrm{Trop} : \underset{\text{orbit}}{O(\sigma)^{\mathrm{Ber}}} \longrightarrow \operatorname{Hom}(M \cap \sigma^\perp, \mathbb{R}).$$

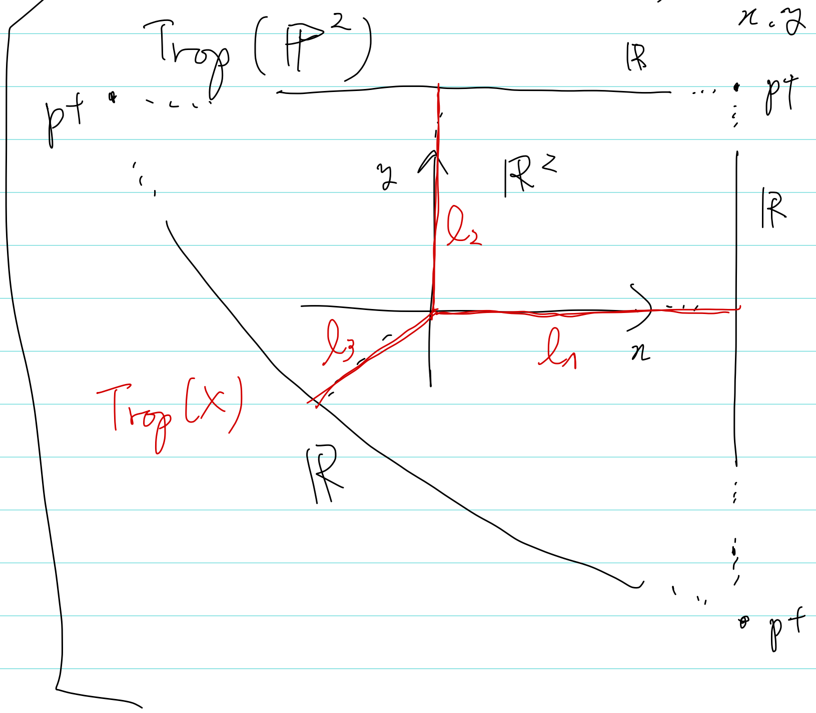
$$v \longmapsto v|_{M \cap \sigma^\perp}$$

$$\rightsquigarrow \mathrm{Trop}(X) := \bigsqcup_{\sigma \in \Sigma} \mathrm{Trop}((X \cap O(\sigma))^{\mathrm{Ber}})$$

: a finite union of cones of dim = dim X.

Example

$\mathbb{P}^1 = X = (x + y + 1 = 0) \subset \mathbb{P}^2_{x,y}$



$X^{\mathrm{Ber}} = \mathbb{P}^{1\,\mathrm{Ber}}$

x-adic val. $\longleftrightarrow l_1$

y- 〃 $\longleftrightarrow l_2$

∞- 〃 $\longleftrightarrow l_3$.

tropical cohomology

$P \subset \mathrm{Trop}(X)$: a cone

$\sigma_P \in \Sigma$,
$\mathrm{int}(P) \subset \mathrm{Trop}(\mathcal{O}(\sigma_P)^{\mathrm{Ber}})$

$$F^p(P) := F^p(P, \mathrm{Trop}(X)) := \Lambda^p(M \otimes \mathbb{Q}) \cap \sigma_P^{\perp} \Big/ \left(\begin{array}{l} f \sim 0 \\ \text{if } \; f|_{\mathrm{Span}(Q)} = 0 \\ \quad P \subset Q \subset \mathrm{Trop}(X) \\ \quad \sigma_P = \sigma_Q . \end{array} \right)$$

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \;\; f \in$

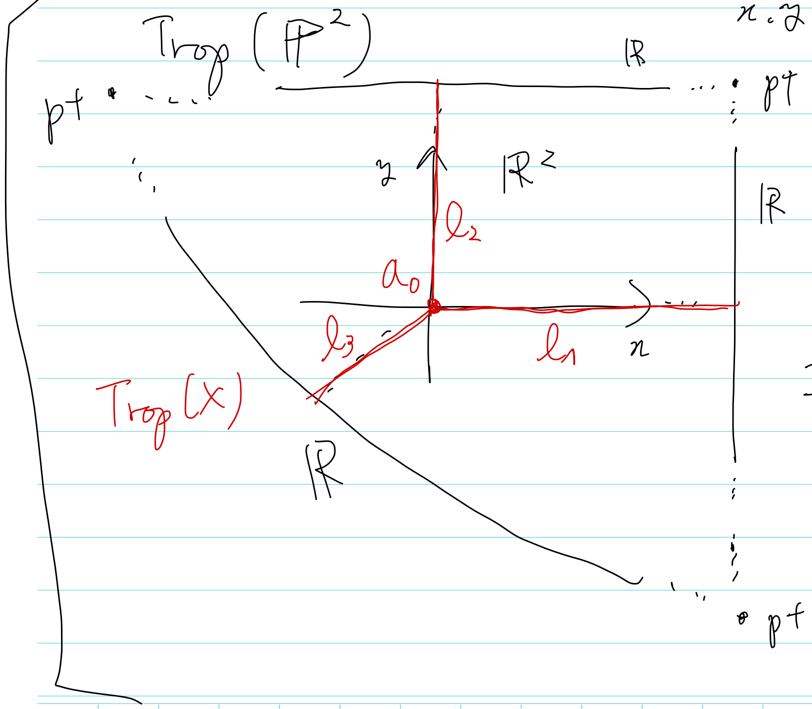
$$H^{p,q}_{\mathrm{trop}}(\mathrm{Trop}(X)) := \text{“} q\text{-th singular cohomology w/ } F^p\text{-coeff.”}$$

$$H^{p,q}_{\mathrm{trop}}(X) := \varinjlim_{\varphi} H^{p,q}_{\mathrm{trop}}(\mathrm{Trop}(\varphi(X)))$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & T_\Sigma \\ & T_{\Sigma'} \arrow[u, "\text{toric mor.}"'] \end{tikzcd}$$

Example

$\mathbb{P}^1 = X = (x+y+1=0) \subset \mathbb{P}^2_{x,y}$



$X^{\mathrm{Ber}} = \mathbb{P}^{1,\mathrm{Ber}}$

x-adic val. $\longleftrightarrow l_1$

y- 〃 $\longleftrightarrow l_2$

∞- 〃 $\longleftrightarrow l_3$.

$$H^{1,1}_{\mathrm{trop}}(\mathrm{Trop}(X)) = \bigoplus_i \underbrace{\mathrm{Span}(l_i)}_{\cong \, \mathbb{Q}}[l_i] \Big/ \mathrm{Im}\left(F^2(a_0) \to \bigoplus_i \mathrm{Span}(l_i)[l_i]\right)$$

$x \longmapsto [l_1] - [l_3] = \mathrm{div}\, x$

$y \longmapsto [l_2] - [l_3] = \mathrm{div}\, y$.

tropical K-groups, L/K : an ext.

$$K_M^p(L)_{\mathbb{Q}} := \wedge^p(L^\times)\otimes_{\mathbb{Z}}\mathbb{Q} \,/\, (a\wedge 1-a\wedge\cdots) \quad \text{Milnor K group.}$$

$$K_T^p(L/K) := \varinjlim_{\mathcal{C}/K} F^p(0, \mathrm{Trop}(\overline{\mathcal{C}(\mathrm{Spec}\,L)})).$$

$$\parallel$$

$$(K_T^p(L))$$

$$\begin{tikzcd} \mathrm{Spec}\,L \arrow[r] \arrow[dr] & \mathbb{G}_m^n \\ \mathcal{C} \ni & \mathbb{G}_m^{n'} \arrow[u] \end{tikzcd} \quad \text{torus morphisms}$$

$$\cancel{\simeq \wedge^p L^\times_{\mathbb{Q}}}$$

$$\bigcap_{v\in ZR(L/K)} \mathrm{Ker}(\wedge^p v_{\mathbb{Q}} : \wedge^p L^\times_{\mathbb{Q}} \to \wedge^p \mathbb{R}^{r_v})$$

$$ZR(L/K) := \left\{ v : L^\times \to (\mathbb{R}^{r_v}, \text{lexicographic order}) : \text{valuations} \;\middle|\; \text{trivial on } K \right\} / \sim$$

$\mathrm{Spec}\,K[M]$
$\parallel$
$\mathbb{G}_m^n$

$$\boxed{\begin{array}{l} ZR(L/K) \twoheadrightarrow \Lambda : \text{a fan str. of } \mathrm{Trop}(\overline{\mathcal{C}(\mathrm{Spec}\,L)}) \\ v=(v_1,\dots,v_r) \mapsto P \\ : L^\times \to \mathbb{R}^r \quad \text{s.t. } \sum_{i=1} \prod_{j=1}^{i} \varepsilon_j v_i \in \mathrm{int}(P) \\ v_i|_M \in \mathrm{Hom}(M,\mathbb{R}) \quad \text{for } 0<\varepsilon_j \ll 1 \end{array}}$$

$\rightsquigarrow$ K_T^p : satisfies good properties (Rost's cycle module)

the Gersten resolution

the residue map

$$0 \to \underset{\text{sheaf'n}}{\mathcal{K}_T^p} \to \bigoplus_{x \in X^0} K_T^p(k(x)) \xrightarrow{d} \bigoplus_{x \in X^1} K_T^{p-1}(k(x)) \xrightarrow{d} \cdots$$

$$k(x)^\times \xrightarrow{\mathrm{div}} \mathbb{Q}$$

$$\cdots \to \bigoplus_{x \in X^i} K_T^{p-i}(k(x)) \to \cdots\cdots \xrightarrow{d} \bigoplus_{x \in X^{p-1}} K_T^1(k(x)) \xrightarrow{d} \bigoplus_{x \in X^p} K_T^0(k(x)) \to 0$$

X^i : codim i points.

$$\rightsquigarrow \quad H^p(X_{zar}, \mathcal{K}_T^p) \simeq CH^p(X)_{\mathbb{Q}}$$

Main Theorem 2

X : sm. alg. var / K.

$$\Rightarrow \quad H^{p,q}_{trop}(X) \simeq H^q(X_{zar}, \mathcal{K}_T^p)$$

Proof of Main Theorem ②).

fix r

$$E_1^{p,q} := \bigoplus_{\substack{x \in X^p \\ \text{codim } p}} \varinjlim_{\substack{x \in U \subset X \\ \text{: op. nbd}}} H^{r,p+q}_{trop-\{x\}}(U) \Rightarrow H^{r,p+q}_{trop}(X)$$

($\mathbb{A}^1$-homotopy invariance
étale excision
$\exists$ corestriction map / fin. fields.)

By a theorem on cohomology theories,
(Bloch, Quillen, Bloch–Ogus, Gabber, Rost (cycle module)
$\underbrace{\text{Colliot-Thélène–Hoobler–Kahn}}$, ...
we use ↑ formulation.)

We have $E_2^{p,q} = H^p(X_{zar}, \mathcal{H}^{r,q}_{trop})$ ← sheaf'n

Easy Lemma,

(1) $E_2^{p,q} = 0$ $(q \neq 0)$ $\rightsquigarrow E_2^{p,0} \cong H^{r,p}_{trop}(X)$

(2) $\mathcal{H}^{r,0}_{trop} \simeq K^r_T$ $\rightsquigarrow E_2^{p,0} \simeq H^p(X_{zar}, K^r_T)$

$\rightsquigarrow$ Main Theorem 2

$\square$

Sketch of proof of $\mathbb{A}^1$-homotopy invariance).

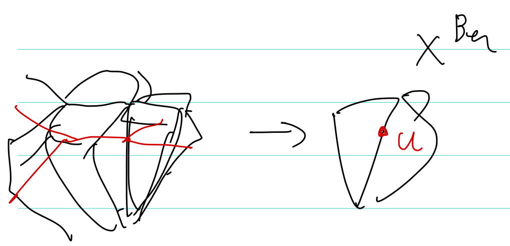
$\pi : X\times\mathbb{A}^{1,\mathrm{Ber}} \to X^{\mathrm{Ber}}$: the projection

$H^{p,q}_{\mathrm{trop}} = H^q(\mathrm{Ber}_{\mathrm{sp.}}, \mathcal{F}^p)$.

It suffices to show

$$\boxed{R^i\pi_* \mathcal{F}^p_{(X\times\mathbb{A}^1)^{\mathrm{Ber}}} = \begin{cases} \mathcal{F}^p_{X^{\mathrm{Ber}}} & i=0 \\ 0 & i\geq 1.\end{cases}}$$

$u\in X^{\mathrm{Ber}}$. $(*)_u \cong \bigoplus_i \bigwedge^{p-i} \mathrm{Span}(u)^\vee \otimes H^q(\pi^{-1}(u), \mathcal{F}^p_{X\times\mathbb{A}^1}/\mathrm{Span}(u)^\vee)$.

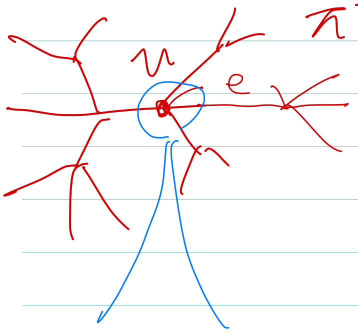


$v\in\pi^{-1}(u)$

$$\text{cones} \Leftarrow \overline{\{v\}} \subset ZR(k(\mathrm{supp}(v))/K)$$

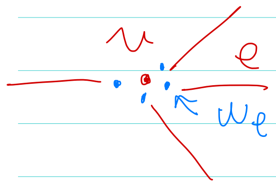
$$v \mapsto \mathrm{Span}(v) \qquad \wr\wr\ \text{big} \qquad w \mapsto w|_{v^{-1}(0)}$$

$$\text{"="}\ ZR(k(v)/K)$$



$\pi^{-1}(u) = \mathbb{A}^{1,\mathrm{Ber}}_{k(\mathrm{supp}(u))_u} \supset \mathbb{A}^{1,\mathrm{ad}}_{k(\mathrm{supp}(u))_u} = \{\text{valuations to } \mathbb{R}^r\ (\forall r)\}$

$$\rightsquigarrow \mathcal{F}^p_{X\times\mathbb{A}^1}/\mathrm{Span}(u)^\vee(v) \simeq K^p_T(k(v)),$$



$\{w_e\}_e = \overline{\{v\}}\setminus\{v\}$.

$\mathcal{F}^p_{X\times\mathbb{A}^1}/\mathrm{Span}(u)^\vee(e)$.

$$= K^p_T(k(w_e)) \oplus \underbrace{\mathbb{Q}\langle a-b\rangle^\vee}_{\substack{a,b\in e \\ \cong\ \mathbb{Q}}} \otimes K^{p-1}_T(k(w_e))$$

$\overline{w_e}$: a discrete valuation on $k(v)$,

$$\left(\partial_{\overline{w_e}} \circ (\pi_{\overline{w_e}} \wedge -), \partial_{\overline{w_e}}\right) : K_T^p(k(v)) \longrightarrow K_T^p(k(w_e)) \oplus K_T^{p-1}(k(w_e))$$

$$\begin{array}{c} \pi_{\overline{w_e}} \wedge a_1 \wedge \cdots \wedge a_{p-1} \\ + \\ b_1 \wedge \cdots \wedge b_p \end{array} \longmapsto (\overline{b_1} \wedge \cdots \wedge \overline{b_p}, \overline{a_1} \wedge \cdots \wedge \overline{a_{p-1}})$$

$(\overline{w_e}(a_i) = \overline{w_e}(b_j) = 0)$

Fact

- $k(v) = k'(\underset{\substack{\uparrow \\ \text{indeterminant}}}{x})$ $\quad k'/k(u)$: fin. ext.
- $\overline{\{v\}} \subset \mathbb{A}^{1,\mathrm{ad}}_{k(\mathrm{supp}(u))/u} \overset{1:1}{\longleftrightarrow} ZR(k(v)/k(u)) = \mathbb{P}^1_{k'}$
- $0 \to K_T^p(k') \to K_T^p(k'(x)) \xrightarrow{(\partial_x)} \bigoplus_{x \in \mathbb{A}^1_{k'},\ \text{closed}} K_T^{p-1}(k(x)) \to 0$ (exact)

$\rightsquigarrow$ We can compute $\bigoplus_e (\partial_{\overline{w_e}} \circ (\pi_{w_e} \wedge -), \partial_{\overline{w_e}})$ explicitly!

$\rightsquigarrow$ $\mathbb{A}^1$-homotopy invariance !! $\quad \square$