SYZ Mirror Symmetry of Del Pezzo Surfaces

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Mirror symmetry and related topics 2020, Kyoto University
Dec 9, 2020

Outline of the Talk

- Set-up of the Geometry
- SYZ Fibrations of Del Pezzo Surfaces and their Dual Fibrations
- Applications to Enumerative Geometry

The Easter Egg

Del Pezzo Surfaces

- A compact complex surface Y is **del Pezzo** if $-K_Y$ is ample.

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- $d = (-K_Y)^2$ is called the degree of the del Pezzo surface.
  Denote $d = 8'$ for the case of $\mathbb{P}^1 \times \mathbb{P}^1$.

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- $d = (-K_Y)^2$ is called the degree of the del Pezzo surface. Denote $d = 8'$ for the case of $\mathbb{P}^1 \times \mathbb{P}^1$.
- Every del Pezzo surface Y admits a smooth anti-canonical divisor $D \in |-K_Y|$.

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- $K_{\check{Y}} \cong \mathcal{O}_{\check{Y}}(-\check{D})$, where $\check{D}$ fibre. canonical bundle formula
- Possible singular fibres are classified by Kodaira, Perrson.
- An I_d fibre is an anti-canonical cycle consisting of a wheel of d (-2)-rational curves

Del Pezzo/RES as Log Calabi-Yau Pairs

- Y = del Pezzo surface, $D \in |-K_Y|$ smooth
  $\exists \Omega \in H^0(Y, K_Y(-D))$ non-vanishing mero. $(2,0)$-form.
- (Tian-Yau '90) $\exists$ **exact** Ricci-flat metric ω on $X = Y \setminus D$
  such that $2\omega^2 = \Omega \wedge \bar{\Omega}$.

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- Both X and $\check{X}$ are hyperKähler. $Sp(1) \cong SU(2)$

- (McMullen) The moduli space of (Y, D) is a fibration over j-line with fibres $\mathrm{Hom}(D^{\perp}, \mathbb{C}^*)/W$, of dimension $10 - d$.
- This is captured by the classical periods $\int \Omega$.

Deformation os Log CY Surfaces and Torelli Theorem

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- This is captured by the classical periods $\int \Omega$.
- (Gross-Hacking-Keel, Friedman) The moduli space of RES $\check{Y}$ with an I_d fibre $\check{D}$ is given by $\mathrm{Hom}(\check{D}^{\perp}, \mathbb{C}^*)/\check{W}$.
- $\exists$ distinguished pairs $(\check{Y}_e, \check{D}_e)$ with trivial periods in each deformation family for RES.
- This is captured by the classical periods $\int \check{\Omega}$.

Strominger-Yau-Zaslow Conjecture

Conjecture (Strominger-Yau-Zaslow '96)

- *Calabi-Yau manifolds admit* **special Lagrangian** *torus fibration near large complex structure limit.*
- *Mirror Calabi-Yau are constructed by dual torus fibration.*
- *Mirror complex structure receives quantum correction from holomorphic discs with special Lagrangian fibre. boundary conditions.*

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- (Harvey-Lawson '82) A submanifold L in X is special Lagrangian if $\omega|_L = 0$, $\mathrm{Im}\Omega|_L = 0$.
- Del Pezzo/RES cases are conjectured by Auroux '07.

New Special Lagrangian Fibrations I

Theorem (Collins-Jacob-L. '19)

Y=del Pezzo surface or RES, $D \in |-K_Y|$ smooth.
Then $X = Y \backslash D$ admits a special Lagrangian fibration with a special Lagrangian section with respect to the Tian-Yau metric.

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Then $X = Y \setminus D$ admits a special Lagrangian fibration with a special Lagrangian section with respect to the Tian-Yau metric.

- This solves conjectures of Yau and Auroux '07.
  1. $Y = \mathbb{P}^2$, with 3 nodal singular fibres.
  2. For generic (Y, D) with Y rational elliptic surface, there are 12 singular fibres.
- The base is $\mathbb{R}^2$ by uniformization theorem and theorem of Yau.

HK Rotation connecting dP/RES

Theorem (Collins-Jacob-L. '19)

Let $\check{X}$ be a suitable hypKähler rotation of X.

$$\check{\omega} = Re\Omega, \quad \check{\Omega} = Im\Omega + i\omega.$$

Then $\check{X}$ compactified to a RES $\check{Y}$ by adding an I_d fibre at infinity, where $d = (-K_Y)^2$.

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Then $\check{X}$ compactified to a RES $\check{Y}$ by adding an I_d fibre at infinity, where $d = (-K_Y)^2$.

- (Auroux-Kartzarkov-Orlov '05) showed that the above is the compactification of the Landau-Ginzburg mirror of Y.

$$D^b Coh(Y) \cong FS(W)$$

- The correspondence respects the deformation families.

New Special Lagrangian Fibrations II

The mirror symmetry of log Calabi-Yau surfaces $(\check{Y}, \check{D})$ are studied by Gross-Hacking-Keel when $\check{D}$ is maximal degenerate.

Theorem (Collins-Jacob-L)

Let $\check{Y} = RES$ and $\check{D} = I_d$ singular fibre. Then $\check{X} = \check{Y} \backslash \check{D}$ admits a special Lagrangian fibration. Moreover, a suitable hyperKähler rotation $X' \to \mathbb{C}$ can be compactified to a rational elliptic surface Y' by adding an I_d singular fibre.

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- The complex affine structure on the base is asymptotically to that of Gross-Hacking-Keel.

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- The complex affine structure on the base is asymptotically to that of Gross-Hacking-Keel.
- It is natural to expect that it is the mirror SYZ fibraton.

Idea of the proof

- Construct a special Lagrangian torus for the model geometry (Calabi ansatz, semi-flat metric).
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  Caveat: the geometry is degenerate!
- The deformation of a special Lagrangian tori covered the non-compact Calabi-Yau surface via J-holo. curves theory.
  1. (Closedness) Sacks-Uhlenbeck-Gromov compactness theorem for the degenerate geometry.
  2. (Openness) Classification of possible singular fibres and analysis the their local deformations.

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$$\check{\mathcal{M}}_{Kah} := \{\text{CY metrics asympototic to } \omega_{sf,\epsilon}\}/\text{Aut}_0(\check{X}),$$

which is a cone with non-empty interior in $H^2(\check{X}, \mathbb{R}) \sim \mathbb{R}^{11-d}$.

SYZ Mirror Symmetry between Del Pezzo surfaces & RES

Theorem (Collins-Jacob-L.)

Under the mirror map near LCSL, $\mathcal{M}_{cpx} \longrightarrow \check{K}_{Kah}$

$$PD([\sigma_q]) + \Omega_q \mapsto \check{\mathbf{B}}_{\check{q}(q)} + i\frac{m[\check{\omega}_{\check{q}(q)}]}{\alpha_{\check{q}(q)}}$$

$$Im\tau_q = m\alpha_{\check{q}(q)},$$

the special Lagrangian fibration in dPs and RES

1. *exchange the complex and symplectic affine structures, and*
2. *the volume of the fibres are inverse to each other.*

$\alpha_{\check{q}(q)}$ is the additional variable in $\check{K}_{Kah}$.

Gravitational Instantons

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- (Chen-Chen '15) Classification of gravitational instantons with faster than quadratic curvature decay.
- (Hein '12) New gravitational instantons from RES of volume growth $r^2, r^{4/3}$ labeled as ALG*, ALH*, which curvature have no quadratic decay.

Application to "New" Gravitational Instantons

Theorem (Collins-Jacob-L.)

Given $(\check{Y}, \check{D})$, *there exists an extra* $\mathbb{R}$*-family of Ricci-flat metrics on* $\check{X}$ *with Hein's metrics are indexed by* $\mathbb{Z}$.

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- The special Lagrangian tori in $\check{X}$ hyperKähler rotate back to X to be special Lagrangian tori but with phase $\pi/2$.
- $D \cong \mathbb{Z}/(\mathbb{Z} \oplus a\mathbb{Z}i), a \in \mathbb{R}_+$ and leads to a contradiction for general choice of D.

Mirror Symmetry and Enumerative Geometry

Mirror Symmetry for Fano Manifolds

- Fano manifold $Y \leftrightarrow$ Landau-Ginzburg superpotential $W : \check{Y} \to \mathbb{C}$, where W is a holomorphic function.
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  1. $QH^*(Y) \cong Jac(W) \Leftarrow Fuk(X) \cong D^b Sing(W) \cong MF(W)$.
  2. $FS(W) \cong D^b Coh(Y)$.
  3. $SH^*(Y \setminus D) \cong PV^*(\check{Y}) \Leftarrow D^b \mathcal{W}(Y \backslash D) \cong D^b Coh(\check{Y})$.
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- How do we compute the LG superpotential?

Superpotential from Lagrangian Floer theory

- (Givental, Hori-Vafa) $Y=$ toric Fano, formula for W.
- (Cho-Oh) Y toric Fano, $L=$ moment torus fibre, then

$$W = W^{LF}(L).$$

  Write $b=\sum x_i e_i \in H^1(L,\Lambda_+)$ wrt basis e_i of $H^1(L,\mathbb{Z})$, then $m(e^b)=\sum_k m_k(b,\cdots,b)=W^{LF}(b)\mathbf{1}_L$.

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- Write $z_i = e^{x_i}$ and for $\gamma \in H_2(Y, L)$ write $z^{\partial\gamma} = \prod_i z_i^{\langle \partial\gamma, e_i \rangle}$. Then
$$W(z_1, \cdots, z_n) = \sum_{\beta : MI(\beta)=2} n_\beta T^{\omega(\beta)} z^{\partial\beta},$$
where $n_\beta := \int_{[\mathcal{M}_1(X,L;\beta)]^{vir}} 1$.

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Lemma (L- '19)

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1. $\omega_i|_{\pi^{-1}(U_i)} = \omega_{TY}$
2. $[\omega_i] = k_i c_1(Y)$ *with* $k_i \nearrow \infty$.

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- L_u are Lagrangian wrt ω_i for $i \gg 0$.
- Together with the computation of $W(L_u)$ later, this is a realization of the renormalization procedure proposed by Hori-Vafa and Auroux.

Wall-Crossing of the Superpotentials (FOOO)

- A_∞ structure $\{m_k\}$ on $H^*(L_u)$ with Maurer-Cartan space

  $\mathcal{MC}_{weak}(L_u) := \{b \in H^1(L_u, \Lambda_+) | m(e^b) = c\mathbf{1}_L\}/\sim \cong H^1(L_u, \Lambda_+)$.

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- (Fukaya's trick) ϕ path from u_- to u_+, the pseudo-isotpy of A_∞ structures of $H^*(L_{u_\pm})$ induces $F_\phi : H^1(L_{u_-}) \cong H^1(L_{u_+})$ (without flux), where F_ϕ records holo. discs of Maslov index zero with $F_\phi \equiv id(\mathrm{mod}\Lambda_+)$.

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- If no negative Maslov index discs, $u_- = u_+$ and ϕ is contractible, then $F_\phi = id$.
- (wall-crossing formula) $W(b; u_-) = W(F_\phi(b); u_+)$.

Wall-Crossing w/ SLAG Fibration in CY Surfaces

- (Hitchin) $\exists$ integral affine structure on B_0.
- L_t bound MI=0 discs of $\gamma \in H_2(X, L_t)$, then L_t sit above an affine line l_γ. Advantage of SLag fibration!

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Theorem (L-'17)

In the surface case, if ϕ goes across l_γ, then

$$F_\phi : z^{\partial\gamma'} \mapsto z^{\partial\gamma'} f_\gamma^{\langle\gamma',\gamma\rangle}, \quad \log f_\gamma(u) = \sum_{d\geq 1} d\tilde{\Omega}(d\gamma; u) z^{d\gamma},$$

where $\tilde{\Omega}(\gamma; u)$ denotes the weighted count of MI=0 tropical discs.

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This is the form of the Kontsevich-Soibelman transformation.

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- $n_{\beta_0}(u) = 1 = n_{\beta_0}^{trop}$ for u near D and β_0 vanishing thimble.

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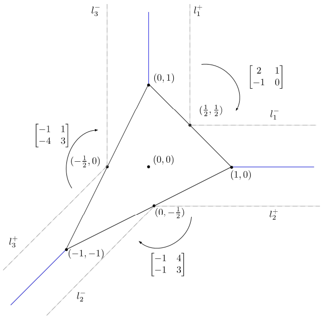
- Compare the wall-crossing formula.
- By induction on the number of bends of possible broken lines.

Towards the Equivalence of AG/SG Mirrors

Theorem (Bouseau '19, Lau-Lee-L., '20)

The complex affine structure of the SYZ fibration of $\mathbb{P}^2 \backslash E$ *coincides with the one of Carl-Pomperla-Siebert.*



More on $Y = \mathbb{P}^2$

- These lay out the foundation of the comparison of family Floer mirror with the mirror constructed in Gross-Siebert program.
- $W(L_u)$ are those appear in Vianna's work after Pascaleff-Tonkonog.
- Limit of certain open GW of MI=0 with limiting boundary condition to infinity coincides with closed GW with maximal tangency together with the work of Gräfnitz.
- Stability of $FS(W)$ and counting of stable objects mirror to geometric stability condition of $D^b(\mathbb{P}^2)$ from the work of Bousseau.

The Easter Egg

Fun application in algebraic geometry:

Theorem (? classical result)

Let $E \subseteq \mathbb{P}^2$ be a smooth cubic curve. Then there exists

$$\mathbb{Z}_3 \oplus \mathbb{Z}_3 \subseteq Aut(\mathbb{P}^2)$$

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1. The corresponding rational elliptic surface $\check{Y}$ is extremal $I_9 I_1^3$.
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1. The corresponding rational elliptic surface $\check{Y}$ is extremal $I_9 I_1^3$.
2. $\check{\Omega}$ is meromorphic.
3. $\exists \mathbb{Z}_3 \oplus \mathbb{Z}_3 \in \mathrm{Aut}(\check{Y})$ preserving $\check{\Omega}$ and $\breve{\omega}$. Uniqueness theorem!
4. The corresponding automorphisms of X extend over E.

THANK YOU!