Lagrangian Torus Fibration Models of Fano Threefolds

Thomas Prince

Mirror Symmetry and Related Topics

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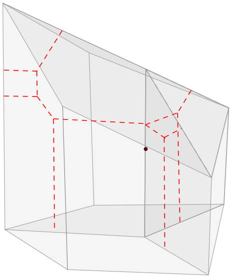

Topological Mirror Symmetry

In this talk we take the point of view suggested by the Strominger–Yau–Zaslow (SYZ) conjecture: that Mirror Symmetry can be realised by dualising (special) Lagrangian torus fibrations. Our particular inspiration is the following result of Gross from the paper 'Topological Mirror Symmetry' (2001).

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Topological Mirror Symmetry

The quintic threefold X has a 'well-behaved' torus fibration $\pi\colon X \to B$. Moreover there is a duality for well-behaved fibrations such that if $\breve{\pi}\colon \check{X} \to B$, is the dual fibration in this case, $\check{X}$ is diffeomorphic to a non-singular minimal model of the mirror quintic.

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Note that as a consequence of the dual fibration structures we have that $H^{ev}(X, \mathbf{R}) \cong H^3(\check{X}, \mathbf{R})$ and $H^3(X, \mathbf{R}) \cong H^{ev}(\check{X}, \mathbf{R})$ as expected.

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Integral Affine structure

The base manifold $B \cong S^3$, is equipped with an *affine structure* which contains certain singularities. In 'simple' situations (which will always hold in our constructions) there is a construction of $\pi\colon X \to B$ from B together with its affine structure.

In general, we denote the total space of the torus fibration constructed from B as $\check{X}(B)$.

Torus fibrations on Fano manifolds

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Main Statement

Given a Fano threefold X there is an integral affine manifold (with corners and singularities) B such that the numerical invariants of the pair $(\check{X}(B), \pi^{-1}(\partial B))$ coincide with those of (X, D) for a divisor $D \in |-K_X|$. In the case that X has Picard rank one we show that $\check{X}(B)$ is homeomorphic to X.

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The numerical invariants of X we refer to are $-K_X^3$, $h^{2,1}$, ρ_X and the index of $-K_X$. Note that our choice of codimension 2 submanifold $D := \pi^{-1}(\partial B)$ is completely analogous to the toric situation.

Fano classification

We briefly recall the classification of the 105 classes of Fano threefolds completed by Mori–Mukai, building on work of Fano and Iskovskikh. In particular, we summarise the classification below according to properties relevant later.

Picard rank	Total	Toric	$-K_X$ not very ample
1	17	1	2
2	36	4	3
3	31	7	0
4	13	4	0
5	3	2	0
≥ 6	5	0	2

Check: 105 total, 18 smooth toric, and 98 with very ample $-K_X$.

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2. For each such affine manifold B, compute invariants of $\check{X}(B)$.
3. In the rank one case use topological classification results to prove $\check{X}(B)$ is homeomorphic to a Fano manifold.

$$\begin{array}{ccc} X_P & \rightsquigarrow & \check{X}(B) \\ \downarrow & & \downarrow \\ P^\circ & \rightsquigarrow & B \end{array}$$

Idea: start on the top left, and finish on the top right going via the deformation of affine structures.

Fano classification and Mirror Symmetry

Our approach dovetails with the computation of quantum periods of Fano threefolds completed by Coates–Corti–Galkin–Kasprzyk. Starting from any Fano 3-fold X the authors give Laurent polynomials f such that

$$\pi_f(t) := \int_{T^3} \frac{1}{1 - tf} \bigwedge_{i=1}^{3} \frac{dx_i}{x_i}$$

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Compatibility with Mirror Symmetry

Every B we consider such that $\check{X}(B)$ models a Fano threefold X is constructed by deforming the affine structure on a polytope P°, such that P supports a Laurent polynomial mirror to X.

Affine manifolds with corners...

We first need to describe the objects B which appear in our constructions.

Integral Affine manifold

A manifold B with a maximal atlas $\mathcal{A}$ with charts in the group of integral affine transformations $\mathrm{GL}_n(\mathbf{Z}) \rtimes \mathbf{Z}^n$.

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Note that for such a manifold $\check{X}(B)$ is easy to define:

$$\check{X}(B) := T^\star B / \check{\Lambda}.$$

...and singularities.

Affine manifold with corners and singularities

A triple $(B, \mathcal{A}, \Delta)$, where B is a topological manifold with boundary, $\mathcal{A}$ is an atlas making $B_0 := B \setminus \Delta$ an affine manifold with corners and Δ is a union of submanifolds of codimension at least two.

In fact we only consider a very special class of singularities (simple) which we can describe explicitly. All of these are derived from the focus-focus singularity in dimension 2.

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focus-focus singularity

This is an affine structure with singularities on $\mathbf{R}^2$, with $\Delta = \{0\}$, and monodromy of Λ equal to the monodromy defined on integral homology by a Dehn twist.

Indeed, $\check{X}(B_0)$ is (topologically) exactly the usual local model in Picard–Lefschetz theory.

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An affine manifold with *simple* singularities is a triple $(B, \mathcal{A}, \Delta)$ such that B_0 is an affine manifold with corners and for any point $b \in \Delta$ there is neighbourhood U of $b \in B$ such that U is affine isomorphic to one of the following possibilities.

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1. The product of a focus-focus singularity times an open interval.
2. A *positive node*. This is a trivalent point of Δ with local monodromies

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

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3. A *negative node*. This is a trivalent point of Δ with local monodromies given by the transpose of the matrices given for the positive node.

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3. A *negative node*. This is a trivalent point of Δ with local monodromies given by the transpose of the matrices given for the positive node.
4. The product of a focus-focus singularity times a half-open interval.

Compactifications

We have a very straightforward definition of $\check{X}(B_0) \to B_0$, but how should we extend this to $\pi\colon \check{X}(B) \to B$? The problem is a local one, and solution to this is given in detail in *D-Branes and Mirror Symmetry*, Chapter 6. For each type of point in Δ we describe the topology of the fibre we insert.

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2. Over a positive node insert a T^3 with a collapsed T^2.
3. Over a negative node insert a T^3 with a pair of solid tori attached.
4. Over a singular point in ∂B insert a pinched T^2.

Polytopes to affine manifolds

Starting from a Fano polytope P, how can we construct B? We will encode the data used to define B in *degeneration data*, which depends on two auxiliary combinatorial notions. We fix a Fano polytope $P \subset N_{\mathbf{R}} \cong \mathbf{R}^3$ and a rational fan Σ in $M_{\mathbf{R}}$.

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Edge data

Edge data is a one-dimensional torus invariant cycle C on the toric variety X_P. Moreover we demand that C is supported on the collection of those torus invariant curves of X_P whose images under the moment map $X_P \to P^\circ$ are contained in a two-dimensional cone of Σ.

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Ray data

Ray data is a set $J := \{J(\rho) : \rho \in \Sigma(1)\}$ of collections of nef divisors on each torus invariant hypersurface X_ρ of X_Σ.

Degeneration data

This is our central combinatorial notion. We say the triple (Σ, C, J) forms degeneration data for P if:

1. J is *smooth*: all polyhedra of sections are standard simplices.
2. C and J are *compatible*: the numbers C assigns to edges neighbouring a vertex v of P° contained in a ray $\rho \in \Sigma(1)$ equal the edge lengths of the sum of the polyhedra in $J(\rho)$.
3. C is *smooth and convex*. If P is reflexive this means that C assigns a number to an edge E of P° which is either equal to $\ell(E^\star)$, the length of $E^\star$, or to $\ell(E^\star) - 1$.

Example

To show how this data is used to describe an affine structure we consider the prototypical example of $\mathbf{P}^3$. Let P be the convex hull of the standard basis and the point $(-1,-1,-1)$.

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Let Σ be the normal fan of P, let C be the sum of the torus invariant curves of $\mathbf{P}^3$ (assigning 1 to every edge of P°), and let J be the trivial Minkowski decomposition of each facet of P into itself.

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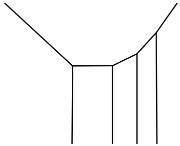
Affine structure

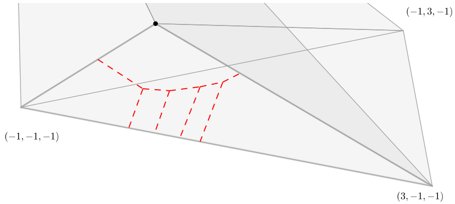
To describe an affine structure we first identify Δ. Note that each $\sigma \in \Sigma(2)$ defines a polyhedral subset c of P° and C defines a divisor on the toric variety $TV(c)$ (we call these pairs (c, D) slabs).

Example

Affine structure

Each toric variety $TV(c)$ is isomorphic to $\mathbf{P}(1,1,4)$ and the divisor described by C is in the class $\mathcal{O}(4)$ (corresponding to the base of the triangle c). Let Γ_c be the dual graph to a triangulation of the polyhedron of sections of $\mathcal{O}(4)$. We embed these graphs into c as shown below. Glue these curves together at points on the rays (positive nodes), trivalent points of each Γ_c become negative nodes.





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Smooth Minkowski decompositions

Given a reflexive polytope P and Minkowski decompositions of all of its facets into standard simplices, we can assign degeneration data as follows.

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We observe that the condition on P is very restrictive. In fact 89 of the 105 Fano threefolds are constructed this way. Note that the previous example is a special case of this construction, and that the slabs are always wps, with divisor class $\mathcal{O}(\ell(E)\ell(E^\star))$.

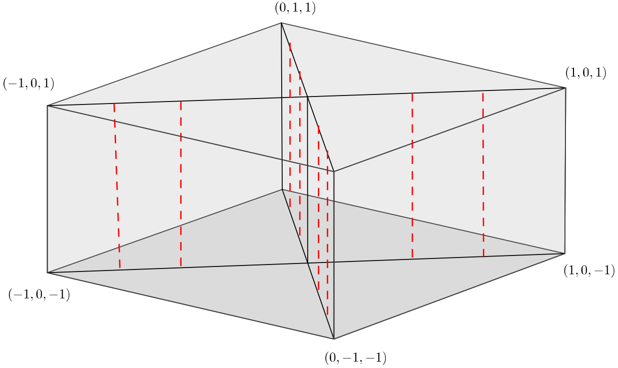
Two dimensional situation

Our constructions directly generalise a two-dimensional one, in which any of the 10 del Pezzo surfaces may be obtained by making *nodal trades* from a polygon. We summarise the two-dimensional affine manifolds important in our construction below.

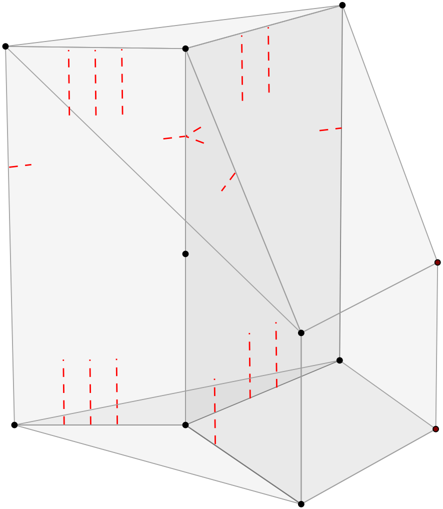
B	$\check{X}(B)$	Affine structure
Polygon	polarised toric variety	B is the image of the moment map
S^2	K3 surface	24 focus-focus singularities
Disc	Del Pezzo surface dP_d	$12 - d$ focus-focus singularities

Product constructions

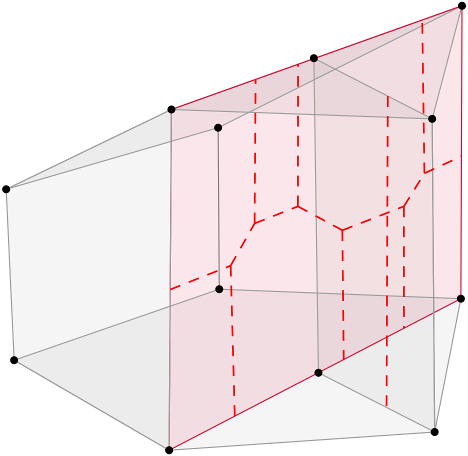
In fact the two-dimensional cases automatically give us 5 examples by taking products of non-toric del Pezzo surfaces with $\mathbf{P}^1$. The images below shows the base of a torus fibration on $dP_4 \times \mathbf{P}^1$. We see that the fan on which Δ is supported is *not* the normal fan of P.



Example: $\mathrm{MM}_{3\text{–}5}$



Example: $\mathrm{MM}_{4\text{–}2}$



Euler numbers

We can easily compute the Euler number of the manifold $\check{X}(B)$ by taking the sum of the Euler numbers of the fibres which do not contain a circle factor.

Special fibre	Euler number
Positive node	1
Negative node	-1
Point in $\Delta \cap \partial B$	1
Vertex of B	1

Formula for Euler number

In the case that B has smooth boundary, and is constructed using our first method, the Euler number obeys the following formula:

$$e(\check{X}(B)) = 24 + T - \sum_{E \in \text{edges}(P)} \ell(E)^2 \ell(E^\star),$$

Degree

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This is expected from the deformation of X_P to $\check{X}(B)$. We can give an entirely topological proof:

1. Construct a (specific) push-off D' of D so that $C := D' \cap D$ is a surface in D.
2. Find the genus of C by taking the image under C and computing the genus of the tropical curve.
3. Use Lefschetz theorem on $(1,1)$-classes and genus formula to compute $C^2 = 2g(C) - 2$.

Contraction map

We compute the Betti numbers of $\check{X}(B)$ via the contraction map $\xi\colon \check{X}(B) \to \check{X}_0(B)$, the union of toric varieties defined by decomposing P° along Σ. We tabulate the possible fibres over points in the interior of B.

codimension of $\pi_0(p)$	$p \in \tilde{\Delta}$	$\xi^{-1}(p)$
0	no	point
1	no	S^1
1	yes	point
2	no	T^2
2	yes	point or S^1
3	no	T^3

A Leray spectral sequence

Our aim is to compute $H^2(\check{X}(B), \mathbf{Q})$. If this is achieved all other Betti numbers will follow from the Euler number formula and simply connectedness of $\check{X}(B)$.

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Leray spectral sequence

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$$\begin{array}{cccc}
\mathbf{Q} & & & \\
0 & \star & & \\
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We often make use of the maps i_k from the disjoint union of the codimension k toric non-boundary strata into $\check{X}_0(B)$.

A Leray spectral sequence

For example, consider $H^0(\check{X}_0(B), R^2\xi_\star\mathbf{Q})$. We have an injection,

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By left exactness of global sections we have an inclusion

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A Leray spectral sequence

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In fact $i_{2\star}i_2^\star R^2\xi_\star\mathbf{Q}$ is the direct sum of its restrictions to the one dimensional strata of the decomposition of P° induced by Σ. Computing this following the argument of Gross we have that,

$$\dim H^0(\check{X}_0(B), R^2\xi_\star\mathbf{Q}) = \dim H^0(\check{X}_0(B), i_{2\star}i_2{}^\star R^2\xi_\star\mathbf{Q}) = 0.$$

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We describe a general procedure for computing $b_2(\check{X}(B))$ in the case B is obtained from smooth Minkowski decompositions of the facets of P.

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Fix an orientation of every 1-dimensional space $M_{\mathbf{Q}}/\langle\sigma\rangle$ for $\sigma \in \Sigma(2)$. Then $\Gamma(\Sigma, C) \subset \mathbf{Q}^{\Sigma(2)}$ is (almost!) defined to be the subspace of elements $\gamma = (\gamma_1, \ldots, \gamma_n)$ where $n = |\Sigma(2)|$ such that $\sum_{i \in I} \gamma_i = 0$ whenever the elements of $\Sigma(2)$ corresponding to I appear as the support of singular locus near a point of $\Delta \cap \Sigma(1)$.

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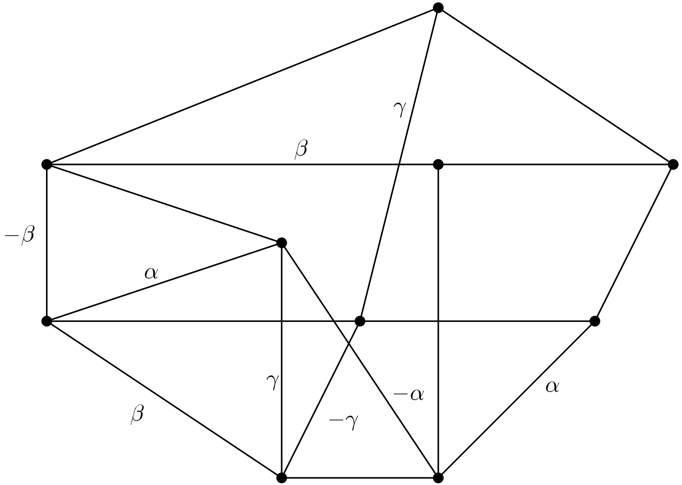
Computation of $b_2(\check{X}(B))$

In this context

$$b_2(\check{X}(B)) = \dim \Gamma(\Sigma, C) - 2$$

Example computation (V_{22})



Topological Classification

To prove that the rank one Fano varieties are homeomorphic to the manifolds $\check{X}(B)$ we construct we use the classification of 6-manifolds (Wall, Jupp). This says that 1-connected topological manifolds with torsion-free homology which admit a smooth structure are classified by the following invariants:

1. $b_3(X)$, the third Betti number,
2. $H^2(X, \mathbf{Z})$, the second integral cohomology group,
3. $w_2(X) \in H^2(X, \mathbf{Z}_2)$, the second Stiefel–Whitney class,
4. F_X, the cubic form on $H^2(X, \mathbf{Z})$,
5. $p_1(X) \in H^4(X, \mathbf{Z})$, the first Pontryagin class.

Topological Classification

The Betti numbers have already been computed, for the form on H^2 we have:

Cubic form

The form F_X is determined in the case $b_2(\check{X}(B)) = 1$ by the value of $[\pi^{-1}(\partial B)]^3$ as well as the index of $[\pi^{-1}(\partial B)] \in H^2(\check{X}(B), \mathbf{Z})$.

The characteristic classes are computed in similar ways, using the restriction

Second Stiefel–Whitney class

$w_2(\check{X}(B)) = PD[\pi^{-1}(\partial B)] \mod 2$. We compute the cap product with D using the projection formula. Let $\theta \colon D \hookrightarrow \check{X}(B)$ be the inclusion, then,

$$\theta_\star\left(\theta^\star w_2(\check{X}(B)) \frown [D]\right) = w_2(\check{X}(B)) \frown \theta_\star[D]$$

Computing $w_2(\check{X}(B))$

We can compute

$$\begin{aligned}\theta^\star w_2(\check{X}(B)) &= w_2(T\check{X}(B)|_D) \\ &= w_2(TD \oplus \nu(D)) \\ &= w_2(TD) + w_2(\nu(D))\end{aligned}$$

Using the Whitney product formula and naturality. Observing that $w_2(TD) = 0$ and $w_2(\nu(D)) = e(\nu(D)) \mod 2$ we see that the cap product of $w_2(\check{X}(B))$ with D is exactly the cup of D with itself.

Since $b_2(\check{X}(B)) = 1$ this is sufficient to prove the result, so long as the index of D is odd. In general we can use the same argument with an integral lift of the second Stiefel Whitney class.

Topological Classification

Lemma

Given a Fano threefold X, $p_1(X)c_1(X) = -K_X^3 - 48$. This follows immediately from HRR and $p_1(X) = -2c_2(X) + c_1(X)^2$.

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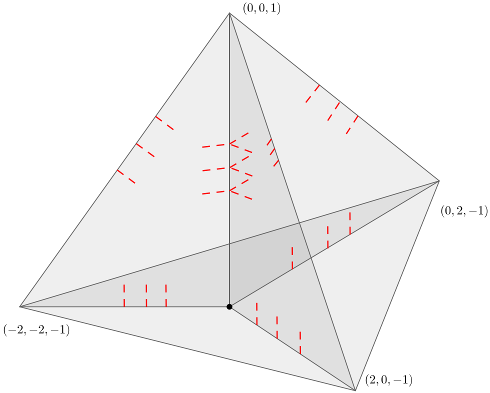
Computing $p_1(\check{X}(B))$

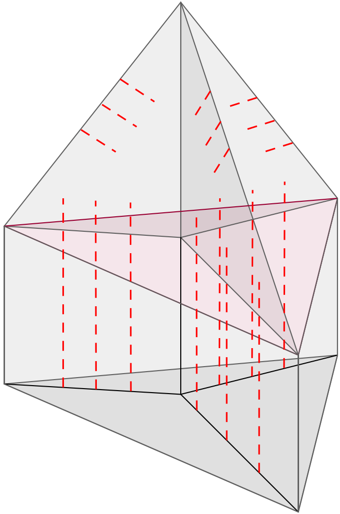
Noting that we can canonically identify $H^2(X, \mathbf{Z})$ and $H^2(\check{X}(B), \mathbf{Z})$ and cubic forms so that $[\pi^{-1}(\partial B)]$ is identified with $c_1(X)$, we use the projection formula

$$p_1(\check{X}(B)) \frown \theta_\star[D] = \theta_\star\left(\theta^\star(p_1(\check{X}(B))) \frown [D]\right),$$

and we have that $\theta^\star p_1(\check{X}(B)) = p_1(D) + p_1(\nu(D))$. However, D is diffeomorphic to a K3 surface so,
$p_1(D) = -2c_2(D) + c_1(D)^2 = -2c_2(D) = -48$. Moreover $p_1(\nu(D))$ is the Euler class of $\nu(D) \oplus \nu(D)$, which is precisely $[D]^3$.

Example: B_3 and its blow-up





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Mirror Symmetry

Tropical superpotential calculation in some of these examples. Relate SYZ mirror symmetry and Minkowski polynomials of Coates–Corti–Galkin–Golyshev–Kasprzyk.

The End