Moduli of Lagrangian immersions in pair-of-pants decompositions and mirror symmetry

Siu-Cheong Lau
Boston University

December 2017

Joint work with Cheol-Hyun Cho and Hansol Hong
Outline

- Overview.
- Construction for pair-of-pants decompositions.
- An application to wall-crossing.
Section 1

Overview and background
Moduli theory in the B-side

- Moduli theory for vector bundles was well established.

**Theorem (Donaldson, Uhlenbeck-Yau)**

*A slope-semistable holomorphic vector bundle admits a Hermitian Yang-Mills metric.*

- GIT and stability conditions were essential to the construction.
- **Bridgeland** developed a general mathematical theory of stability conditions for triangulated categories.
- **Toda** developed foundational techniques to construct Bridgeland stability conditions for derived categories of coherent sheaves.
- Moduli spaces undergo birational changes (such as flops) in a variation of stability conditions.
- How about moduli of Lagrangians in the mirror A-side?
Moduli theory in the B-side

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Ingredients for moduli theory of Lagrangians

- **Complexification.** The naive moduli spaces are affine manifolds with singularities. Complexification is needed in order to compactify. Technically we need to work over the Novikov ring.

- **Quantum correction.** The canonical complex structures need to be corrected using Lagrangian Floer theory [Fukaya-Oh-Ohta-Ono]. The combinatorial structure of quantum corrections for SYZ fibrations was deeply studied by Kontsevich-Soibelman and Gross-Siebert.

- **Landau-Ginzburg model.** The moduli in general are singular varieties. They are described as critical loci of holomorphic functions.

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Why care about Lagrangian immersions

- Lagrangian immersions have a well-defined Floer theory by Akaho-Joyce.
- They are the main sources of wall-crossing phenomenon in the SYZ setting.
- The deformation space of a Lagrangian immersion is ‘bigger’ than its smoothing and covers a local family of Lagrangians (including singular Lagrangians).
- A Lagrangian immersion in the pair-of-pants was used by Seidel and Sheridan to prove homological mirror symmetry for Fermat-type hypersurfaces.
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Family Floer theory

▶ **Fukaya** proposed to study mirror symmetry by using $\text{CF}(L_b, \cdot)$ for fibers $L_b$ of a Lagrangian torus fibration.

▶ **Tu** took this approach to construct mirror spaces away from singular fibers.

▶ **Abouzaid** constructed family Floer functors for torus bundles and showed that the functor is fully faithful.

▶ We consider moduli of Lagrangians which are not necessarily tori.

▶ Instead of Fukaya trick, *we consider pseudo-isomorphisms between Lagrangian immersions which intersect with each other in a certain manner, and obtain the gluing information from cocyle conditions.* (In particular we do not need diffeomorphisms.)

▶ Lagrangian immersions help in gluing. The immersed sectors provide $\Lambda_+$ (or even $\Lambda_0$) space of deformations, which is much ‘bigger’ than the deformation space of flat $\mathbb{C}^\times$-connections.
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Section 2

Pair-of-pants decompositions
Consider a pair-of-pants decomposition of a punctured Riemann surface.

Homological mirror symmetry was proved for punctured Riemann surface by Abouzaid-Auroux-Efimov-Katzarkov-Orlov, Bocklandt, and Heather Lee.

Our focus is on the construction of moduli rather than HMS. The construction comes with a natural functor which derives HMS.

We have a family of Lagrangian immersions as shown in the figure. We will take finitely many representing Lagrangians and glue their deformation spaces together.

\( \dim = 1 \) contains the essential ingredients. In higher dimensions, Seidel’s Lagrangians are replaced by Sheridan’s Lagrangians.
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First consider a pair-of-pants, with the Lagrangian immersion constructed by Seidel.

\[ \text{CF}(L, L) = \text{Span}\{1, X, Y, Z, \bar{X}, \bar{Y}, \bar{Z}, \text{pt}\} \]

We proved that \( xX + yY + zZ \) is weakly unobstructed for \( x, y, z \in \mathbb{C} \). It is important that the areas of the two triangles are the same.

Thus the local moduli is \((\mathbb{C}^3, W)\), where \( W = xyz \).

The pair-of-pants can be compactified to \( \mathbb{P}^1_{a,b,c} \). We used this to construct and compute the mirror, and derived homological mirror symmetry. In an ongoing work with Amorim we prove closed-string mirror symmetry. For elliptic orbifolds the coefficients of \( W \) are modular forms \([L.-Zhou]\).
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Consider the four-punctured sphere as shown above.

We need to glue the deformation spaces of the two Seidel Lagrangians $S_1$ and $S_2$.

There are two main processes: smoothing and gauge change.

Let’s pretend to work over $\mathbb{C}$ at this stage. The gluing we need is

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(C^3, W) \xrightarrow{\text{smoothing}} (\mathbb{C} \times \mathbb{C}^2) \xrightarrow{\text{gauge change}} (\mathbb{C} \times \mathbb{C}^2) \xrightarrow{\text{smoothing}} C^3.
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Choice of gauge change

- There is a vanishing sphere in a smoothing at an immersed point of $S$. In this case it is simply the union of two points.
- Put a flat $\mathbb{C}^\times$ connection on the smoothing $C$, which is acting by $t \in \mathbb{C}^\times$ when passing through the two points.
- The position of the two points are different for smoothings on the left and on the right.
- When a gauge point $T$ is moved across the immersed point $Y$, the $A_\infty$ algebras are related by $\tilde{y} = ty$.
- There are different ways of moving the gauge points to match them. This results in $\tilde{y} = t^ay$, $\tilde{z} = t^bz$ with $a + b = 2$. 
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▶ There are different ways of moving the gauge points to match them. This results in $\tilde{y} = t^a y$, $\tilde{z} = t^b z$ with $a + b = 2$. 
Smoothing

- We need to glue the deformation spaces of $S$ and $C$.
- They are given by $(x, y, z) \in \mathbb{C}^3$ and $(t, y_0, z_0) \in \mathbb{C}^\times \times \mathbb{C}^2$ respectively.
- Intuitively to match the superpotentials $xyz$ and $ty_0z_0$, we simply put $x = t$, $y = y_0$, $z = z_0$.
- We take $S_1$ to be the deformed Seidel Lagrangian which intersects $C$ as in the figure.
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Isomorphisms in smoothing

We consider $a_1 + b_1 \in CF((C, \nabla_t), (S, xX))$ and $c_2 + d_2 \in CF((S, xX), (C, \nabla_t))$.

Consider cocycle conditions on $a_1 + b_1$ and $c_2 + d_2$. Once the cocycle conditions are satisfied, they give isomorphisms between the two objects.
A paradox

- \( S \) is isomorphic to \( S_1 \), and \((S_1, xX)\) is isomorphic to \((C, \nabla t)\) for \( t = x \neq 0 \). Thus \((S, xX)\) is isomorphic to \( C \).
- But \( S \) and \( C \) are disjoint, and hence there is no morphism between them!
- We need to take a closer look at areas of holomorphic discs.
- Following Fukaya-Oh-Ohta-Ono, we shall use the Novikov ring \( \Lambda_0 = \left\{ \sum_{i=0}^{\infty} a_i T^{A_i} : 0 \leq A_0 \leq A_1 \leq \ldots \right\} \) to filter deformations into different energy levels.
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Area constraints for isomorphisms

- For the cocycle conditions on \((S_1, x_1 X_1) \rightarrow (C, \nabla_t)\), indeed we have \(t = T^{A_1 + \ldots + A_5 - A_7} x_1\).
- Note that \(t \in \mathbb{C}^\times\) while \(x_1 \in \Lambda_0\). When \(A_1 + \ldots + A_5 > A_7\), the condition is never satisfied.
- Hence \((S_1, x_1 X_1)\) can never be isomorphic to \((C, \nabla_t)\) if \(A_1 + \ldots + A_5 > A_7\).
- When \(A_1 + \ldots + A_5 = A_7\), the change of coordinates \(x_1 = t\) does not involve Novikov parameters.
- For the cocycle conditions on \((S, xX) \rightarrow (S_1, x_1 X_1)\), we have \(x_1 = T^A x\) for certain \(A > 0\). Thus \(\text{val}(x_1) \geq A > 0\) in order to have them to be isomorphic.
- The two regions \(\text{val}(x_1) = 0\) and \(\text{val}(x_1) \geq A > 0\) are disjoint. This solves the paradox.
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More pair-of-pants

- For the four-punctured sphere with two pants in the decomposition, we can choose \( S_1 \) as above in each pants such that the gluing \( x = t \) does not involve the Novikov parameter.

- Then the \( \mathbb{C} \)-valued mirror \( (\mathcal{O}(-1) \oplus \mathcal{O}(-1), W) \) embeds well in the \( \Lambda \)-valued mirror \( (\Lambda_0^3 \cup \Lambda_0^3, W) \).

- However suppose there is one more pants in the decomposition. Then we need to consider another Seidel Lagrangian \( S'_1 \).

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The moduli over $\Lambda$ is depicted above.

The $\mathbb{C}$-valued toric Calabi-Yau is not embedded, since the gluing $x_1 = T^A x'_1$ does not preserve $\mathbb{C}$.

On the other hand, the $\mathbb{C}$-valued critical locus of $W$ is embedded in the $\Lambda$-valued moduli. Exact Lagrangians are transformed to matrix factorizations supported on the $\mathbb{C}$-valued critical locus.
A compactification

- The gluing $x = t$, $y = y_0$, $z = z_0$ looks pretty trivial. Let’s compactify to get more interesting gluing.
- As an example, compactify the four-punctured sphere to a sphere. We still consider the moduli of double-circles.
- The gluing is further quantum-corrected by discs emanated from infinite divisors.
- The new gluing is $t = x$, $y_0 = y_1 + t_1^{-1}$, $z_0 = z_1 - t_1^{-1}$. $W = xyz - y_1 + x_1 + z_1$. (The Novikov parameter is suppressed for simplicity.)
- Unlike the case for anti-canonical divisors, gluing needs to be corrected upon compactification.
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Changing stability and flop

Let’s go back to the four-punctured sphere. We can take another pair-of-pants decomposition.

It corresponds to another choice of a quadratic differential (stability condition [Haiden-Katzarkov-Kontsevich]).

This results in a flop of the moduli space $(\mathcal{O}(-1) \oplus \mathcal{O}(-1), W)$.

We can also consider a quadratic differential with double zeros. Then the moduli is the non-commutative resolution of the conifold corresponding to a quiver ([Cho-Hong-L. 15]).

In [Fan-Hong-L.-Yau] we studied a 3d version of this (deformed conifold).
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Section 3

Wall-crossing
Immersed sphere

The two-dimensional immersed sphere leads to wall-crossing phenomena for SYZ Lagrangian fibrations.

**Fukaya** studied this immersion and studied the relation with the mirror equation. Here we realize it from cocycle conditions.

It has two degree-one immersed generators $U$ and $V$. It gives the deformations $b = uU + vV$ where $(u, v) \in (\Lambda_0 \times \Lambda_+ \cup (\Lambda_+ \times \Lambda_0)$.

There are constant holomorphic discs with corners $U, V, \ldots, U, V$. To ensure there are only finitely many terms under every energy level, we only allow one of $u, v$ having valuation zero.
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The constant discs with corners $U, V, \ldots, U, V$ contributing to $\text{pt}$ of $m^b_0$ cancel with that with corners $V, U, \ldots, V, U$. Thus $b = uU + vV$ is weakly unobstructed.
Consider the immersed sphere $S$ and a Chekanov torus $T$. $S$ is made by deforming the immersed fiber.

We glue the deformations $(u, v) \in \Lambda_0 \times \Lambda_+$ with the deformations $(x, y) \in (\mathbb{C}^\times \oplus \Lambda_+)^2$ of $T$. 
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The cocycle conditions give \(u = y\) and \(x = uv - 1\). Note that the second equation is on the region \(x = -1 + \Lambda_+\).

In other words, \(\text{CF}(T, T)\) gives an extension of \(\text{CF}(S, S)\) from \(v \in \Lambda_+\) to \(v \in \Lambda_0\) with \(uv \neq 1\).

Similarly we can glue the immersed sphere \(S\) (with \((u, v) \in \Lambda_+ \times \Lambda_0\)) with a Clifford torus \(T'\). It is \(v = y'\) and \(x' = uv - 1\).

The resulting moduli is \(\{(u, v) \in \Lambda_0 \times \Lambda_0 : uv \neq 1\}\).
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The resulting moduli is $\{(u, v) \in \Lambda_0 \times \Lambda_0 : uv \neq 1\}$.
With **Hansol Hong and Yoosik Kim**, we are applying this to construct the compactified mirror of $\text{Gr}(2, n)$.

Flag manifolds have Gelfand-Cetlin systems serving as Lagrangian torus fibrations.

Immersed spheres are important in that case because they appear as critical fibers of the superpotential.

For instance, by **Nohara-Ueda** for $\text{Gr}(2, 4)$ there is a critical point $(0, 0)$ which corresponds to a certain fiber $S^3 \times S^1$ of the Gelfand-Cetlin system.

The trouble is $S^3$ is rigid!
Consider the symplectic reduction picture on $S^2$. We push in one singular point and consider the corresponding moduli.

There is one monotone Lagrangian torus above and below the wall respectively. We glue the deformation spaces of an immersed sphere (times $T^2$) with that of the two monotone tori like in the last slide.

This recovers the Rietsch Lie theoretical mirror for $Gr(2, 4)$. 