On the structure of Brieskorn lattices

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Brieskorn lattices

$f:(X,0):=(\mathbf{C}^{n+1},0)\to(\Delta,0)$ holo. with isol. sing.

$H''_f:=\Omega^{n+1}_{X,0}/\mathrm{d}f\wedge\mathrm{d}\Omega^{n-1}_{X,0}$ (Brieskorn lattice),

$\Omega_f:=\Omega^{n+1}_{X,0}/\mathrm{d}f\wedge\Omega^{n}_{X,0}$ $\big(\cong H''_f/\partial_t^{-1}H''_f\big)$.

Here $\partial_t^{-1}[\omega]:=[\mathrm{d}f\wedge\eta]$, $t\,[\omega]:=[f\omega]$ in H''_f,

with $\mathrm{d}\eta=\omega$ $(\omega\in\Omega^{n+1}_{X,0},\ \eta\in\Omega^n_{X,0})$.

These definitions come from Brieskorn's classical paper.
In this talk we consider only analytic sheaves.
This definition of Gauss-Manin connection is compatible with that of Gauss-Manin system explained later (by considering the differential of the G-M complex).
It should be stressed that the inverse of the Gauss-Manin connection ∂_t^{-1} is *always* well-defined.

Prop. H''_f free of rank μ over $\mathbf{C}\{t\}$, $\mathbf{C}\{\{\partial_t^{-1}\}\}$, where
$\mathbf{C}\{\{\partial_t^{-1}\}\}=\big\{\sum_{k\geqslant 0}a_k\partial^{-k}\ \big|\ \sum_k|a_k|r^k/k!<\infty\ (\exists\, r>0)\big\}$.

Def. A $\mathbf{C}$-linear section σ_0 of $pr_0:H''_f\to\Omega_f$ is **good** if

$$tI_{\sigma_0}\subset I_{\sigma_0}+\partial_t^{-1}I_{\sigma_0}\quad\text{with}\quad I_{\sigma_0}:=\operatorname{Im}\sigma_0\,,$$

i.e. $t\sigma_0=\sigma_0A_0+\partial_t^{-1}\sigma_0A_1$ $\big(A_0,A_1\in\operatorname{End}_{\mathbf{C}}(\Omega_f)\big)$,

or $t\,\mathbf{v}=A_0\mathbf{v}+\partial_t^{-1}A_1\mathbf{v}$ $\big(\mathbf{v}:={}^t(v_1,\dots,v_\mu)$ basis of $I_{\sigma_0}\big)$,

This can be proved easily by using Gauss-Manin system.
This $\big(\mathbf{C}\{\{\partial_t^{-1}\}\}\big)$ is the ring of microdifferential operators of non-positive degrees with constant coefficients.
The freeness over $\mathbf{C}\{t\}$ is closely related with negativity of roots of b-functions due to Kashiwara.
This is the expansion of action of t (only 2 terms appear).
Note that A_0 expresses the action of f on Ω_f.
Here A_1 and A_2 are identified with matrices of size (μ,μ).

and **very good** if moreover σ_0 *strictly compatible* with V,

i.e. $\sigma_0^{-1}(V^\alpha H''_f)=pr_0(V^\alpha H''_f)$ $(\forall\,\alpha\in\mathbf{Q})$.

Here $V^\alpha G_f:=\widehat{\bigoplus}_{\beta\geqslant\alpha}G_f^\beta$ $\big(G_f:=H''_f[\partial_t]$ G-M system$\big)$,

$G_f^\alpha:=\operatorname{Ker}\big(\partial_t t-\alpha\big)^r\subset G_f$ $(r\geqslant n)$ s.t. $G_f=\widehat{\bigoplus}_{\alpha\in\mathbf{Q}}G_f^\alpha$.

Note $G_f^\alpha\cong H^n(F_f,\mathbf{C})_{e^{-2\pi i\alpha}}$ $\big(:=\operatorname{Ker}(T_s-e^{-2\pi i\alpha})\big)$.

This means that the induced filtration by the section σ_0 coincides with the quotient filtration by the projection pr_0.
V is the filtration of Kashiwara-Malgrange indexed by $\mathbf{Q}$.
Localization of Br. lattice by ∂_t^{-1} coincides with G-M system.
We have this 'convergent' direct sum decomposition.
Each piece is isomorphic to an eigenspace of Milnor cohom.
This $\big(F_f\big)$ denotes the Milnor fiber of f around 0.

Thm. $\big\{$very good sections σ_0 of $pr_0:H''_f\to\Omega_f\big\}$
$\overset{1:1}{\Longleftrightarrow}\big\{$good splittings of F on $H^n(F_f,\mathbf{C})\big\}$.

Here F is Hodge filtration, F_f is Milnor fiber, and *good splitting* means an opposite filtration U to F (i.e. $F^{p+1}\oplus U_p=H^n(F_f)$) *stable by monodromy* T.

Prop. $NU_p\subset U_{p-1}\,(\forall\,p)\Longleftrightarrow A_1$ semisimple $(N:=\log T_u)$.

This is the main result of an old paper on Brieskorn lattice.
F is the Hodge filtration of the canonical mixed Hodge structure on the Milnor cohomology.
Here only the stability by the monodromy T is assumed; i.e. this stronger condition (in Prop) is **not** assumed; so A_1 is not necessarily semisimple.
T_u is the unipotent part of the monodromy.

A good splitting of F corresponds to $E_f^\alpha\subset G_f^\alpha$ $\big(\alpha\in(0,n+1)\big)$

s.t. $E_f^\alpha\simeq\mathrm{Gr}_V^\alpha\Omega_f$, $G_f=\bigoplus_\alpha KE_f^\alpha$ $\big(K:=\mathbf{C}\{\{\partial_t^{-1}\}\}[\partial_t]\big)$.

For $\mathbf{C}$-bases $\{e_{\alpha,i}\}$ of E_f^α, there is a $\mathbf{C}$-basis $\{v_{\alpha,i}\}$ of $\operatorname{Im}\sigma_0$

s.t. $v_{\alpha,i}=e_{\alpha,i}+\sum_{j,k>0,\beta>\alpha}c^{\alpha,i}_{\beta,j,k}\,\partial_t^k e_{\beta,j}$ $\big(c^{\alpha,i}_{\beta,j,k}\in\mathbf{C}\big)$.

It is very difficult to determine all the $c^{\alpha,i}_{\beta,j,k}$ in general.

Note $\dim E^\alpha=\#\{\,i\mid\alpha_{f,i}=\alpha\,\}$ $\big(\alpha_{f,i}$ eigenvalues of $A_1\big)$.

Here we use the isomorphism between G_f and Milnor cohom.
K is the ring of microdifferential operators with const. coeff.
This basis $(v_{\alpha,i})$ is unique by these conditions: $k>0,\beta>\alpha$.
These $(c^{\alpha,i}_{\beta,j,k})$ are called the *structure constants* of the basis.
Eigenvalues of A_1 are called the *exponents* associated to σ_0.
If σ_0 is *very good*, they coincide with Steenbrink exponents, but they may depend on σ_0 if it is not very good.
But A_0 (expressing action of f on Ω_f) is independent of σ_0.

Basic example. Set $R:=\mathbf{C}\{\{\partial_t^{-1}\}\}$, $K:=R\,[\partial_t]$,

$M^{(0)}:=R\,\partial_t^{-1}e_1+R\,\partial_t^{-1}e_2+R(e_1+e_2)\subset M:=Ke_1\oplus Ke_2\,,$

$v_1:=e_1+e_2$, $v_2:=\partial_t^{-1}e_2$ with $\partial_t t\,e_i=\beta_i\,e_i$ $(i=1,2)$.

Then $\partial_t t\,v_1=\beta_1v_1+(\beta_2-\beta_1)\partial_t v_2\,,\ \partial_t t\,v_2=(\beta_2+1)v_2\,,$

$$A_1=\begin{pmatrix}\beta_1&0\\0&\beta_2+1\end{pmatrix}\ \left[A_1=\begin{pmatrix}\beta_2&0\\0&\beta_1+1\end{pmatrix}\ \text{if}\ v_2=\partial_t^{-1}e_1\right]$$

This is really a fundamental example.
Note first that this expression is redundant: either this $(R\,\partial_t^{-1}e_1)$ or this $(R\,\partial_t^{-1}e_2)$ is unnecessary.
If this $(R\,\partial_t^{-1}e_1)$ is removed, we get these generators (v_1,v_2) and this matrix (by these calculations).
If the other one is removed we get this matrix.

Geometric example. Let $f=x^5+y^4+x^3y^2$. Then

$H''_f=\bigoplus_{(i,j)\in\Lambda}R\,v_{i,j}\subset G_f=\bigoplus_{(i,j)\in\Lambda}Ke_{i,j}$ with

$\partial_t t\,e_{i,j}=\alpha_{i,j}\,e_{i,j}$ $\big(\alpha_{i,j}:=\frac{i}{5}+\frac{j}{4}\big)$, $\Lambda:=[1,4]\times[1,3]$,

$v_{1,1}:=e_{1,1}+\partial_t^{-1}e_{4,3}\,,\quad v_{i,j}:=e_{i,j}$ $\big((i,j)\neq(1,1)\big)$.

Set $M^{(0)}:=R\,v_{1,1}\oplus R\,v_{4,3}\subset M:=Ke_{1,1}\oplus Ke_{4,3}\subset G_f\,.$

Then $(\beta_1,\beta_2+1)=\big(\frac{9}{20},\frac{31}{20}\big)$, $(\beta_2,\beta_1+1)=\big(\frac{11}{20},\frac{29}{20}\big)$.

This is one of the simplest examples of non weighted homogeneous singularities.
Steenbrink (or standard) exponents are given by these $\big(\alpha_{i,j}:=\frac{i}{5}+\frac{j}{4}\big)$ as is well-known.

This $\big(M^{(0)}\big)$ is a direct factor of Brieskorn lattice.
If we take these generators, we get these $\big(\frac{9}{20},\frac{31}{20}\big)$, (they are equal to $\alpha_{1,1}$ and $\alpha_{4,3}$), and $\big(\frac{11}{20},\frac{29}{20}\big)$ in other case.

Prop. Pairings of $H^n(F_f)$ compatible with monodromy T
$\overset{1:1}{\Longleftrightarrow}$ $S : G_f \times G_f \to K := \mathbf{C}\{\{\partial_t^{-1}\}\}[\partial_t]$ s.t.
$PS(u,v) = S(u,Pv) = S(P^*u,v) \quad (P\in K;\ \partial_t^* := -\partial_t)$,
$\big[t, S(u,v)\big] = S(u,tv) - S(tu,v)$.
(The last two imply $S(G_f^\alpha, G_f^\beta) \subset \mathbf{C}\,\partial_t^{\alpha+\beta}$ for $\alpha+\beta\in\mathbf{Z}$.)

This follows from the isomorphism $G_f^\alpha \cong H^n(F_f)_{e^{-2\pi i\alpha}}$ by restricting the pairing S to $G_f^\alpha \times G_f^\beta$.
P^* comes from transformation between left and right D-mod.
For G-M system we use microlocal dual as explained below.
This $\big(S(G_f^\alpha, G_f^\beta)\big)$ vanishes unless $\alpha+\beta$ is an integer.

This is closely related to $\mathbf{D}(G_f) = \mathrm{Hom}_K(G_f, K)$, where
$\mathbf{D}(G_f) := \mathrm{Ext}^1_{\mathcal{E}_{\Delta,0}}\big(G_f, \mathcal{E}_{\Delta,0}\big)$ (dual as $\mathcal{E}_{\Delta,0}$-module).
Note Canonical self-duality $\Longleftrightarrow$ Higher residue pairing S_K satisfying $S_K(H''_f, H''_f) \subset R\,\partial_t^{-n-1}$ $\big(R := \mathbf{C}\{\{\partial_t^{-1}\}\}\big)$.
(This gives strong restrictions on structure constants $c^{\alpha,i}_{\beta,j,k}$.)

Here Fourier transformation is implicit.
This is the microlocal dual of the Gauss-Manin system.
This canonical pairing is defined by vanishing cycle functor.
This inclusion also follows from the filtered self-duality.

Def. A good splitting of F is called *compatible with self-duality* if $S_K(E^\alpha, E^\beta) = 0$ for $\alpha+\beta\ne n+1$.

This can be stated in terms of the corresponding splitting of the Milnor cohomology and the canonical pairing on it.

Thm. Any very good section (i.e. a good splitting of F) compatible with self-duality gives a unique primitive form.

This was *not* shown in the old paper on Brieskorn lattice since the next proposition was not proved there.

For the proof we need the following

Prop. Minimal eigenvalue $\alpha_{f,1}$ of A_1 has multiplicity 1. The corresponding eigenvector $\overline{\zeta}_0$ generates Ω_f over $\mathcal{O}_{X,0}$.
(More precisely, $\mathrm{Gr}_V^\alpha\Omega_f$ are annihilated by $\mathfrak{m}_{X,0}\subset\mathcal{O}_{X,0}$.)
This is shown by using *microlocal Gauss-Manin system*.
(Varchenko's theory can be used if $\alpha_{f,1}<1$.)

For the proof we use microlocal G-M system and the theory of strict bifiltered complexes.
This is essential for uniqueness of generating eigenvector $\overline{\zeta}_0$.

Note Theorem does not hold for general good sections.

This was not stated explicitly in the old paper of Br. lattice.

In fact, there may be many eigenvectors $\overline{\zeta}_0$ of A_1 generating Ω_f over $\mathcal{O}_{X,0}$, and uniqueness does not necessarily hold,
e.g. $f = x^a + y^b + x^{a-3}y^{b-2} + x^{a-2}y^{b-2}$ $\big(a>b,\ \frac{3}{a}+\frac{2}{b}<1\big)$.

This seems to be a rather generic phenomenon; it may always occur if f is sufficiently complicated, although the calculation is not necessarily easy.

As for existence we have the semi-simplicity problem of A_1 (the eigenspace in strict sense can be contained in $\mathfrak{m}_{X,0}\,\Omega_f$),
e.g. $f = x^{10} + y^3 + x^2y^2 + z^6 + w^5 + z^4w^3$.

On the other hand, this is a quite rare case.
It is not very easy to construct this kind of example.
Here we use the compatibility of the Thom-Sebastiani isomorphism with the canonical self-duality isomorphism.

In the weighted homogeneous case, there is no problem since
Prop. *Any* good section is *very good* if f *weighted homog.*
(This easily follows from $\partial_t t\,\sigma_0 = \sigma_0 A_1$ since $A_0 = 0$.)

This means that the image of the section σ_0 is stable by the action of $\partial_t t$, and this is expressed by A_1.

Deformation

$F : Y := X\times S \to \Delta$ (miniversal) deformation of f.
$H''_{F,S} := \Omega^{n+1}_{Y/S,0}/\mathrm{d}F\wedge\mathrm{d}\Omega^{n-1}_{Y/S,0}$ (rel. Brieskorn module),
$\Omega_{F,S} := \Omega^{n+1}_{Y/S,0}/\mathrm{d}F\wedge\Omega^{n}_{Y/S,0}$ $\big(\cong H''_{F,S}/\partial_t^{-1}H''_{F,S}\big)$.
Def. An $\mathcal{O}_{S,0}$-linear section σ_S of $H''_{F,S}\to\Omega_{F,S}$ is **good** if
$tI_{\sigma_S}\subset I_{\sigma_S} + \partial_t^{-1}I_{\sigma_S}$, $\partial_{s_i}I_{\sigma_S}\subset I_{\sigma_S} + \partial_t I_{\sigma_S}$ $(I_{\sigma_S} := \mathrm{Im}\,\sigma_S)$.

Here we still consider in the analytic category, and use analytic sheaves.
We do not assume that the deformation is miniversal.
These are natural generalization to the deformation case.
Here it is better to consider the action of $\partial_t^{-1}\partial_{s_i}$, and say that its expansion has only two terms (this expression is convenient for proof of uniqueness of extensions of good sections to the deformation).

Thm. (Malgrange). Any good section σ_0 is uniquely lifted to a good section σ_S (by solving Birkhoff's R-H problem).

This is a well-known theorem of Malgrange.
New proofs were found by Sabbah and Hertling.

Thm. (K. Saito). Any good section σ_S is uniquely lifted to
$\sigma_S^\nabla : \Omega_f \to H''_{F,S}$ s.t. $\partial_{s_i}(\mathrm{Im}\,\sigma_S^\nabla)\subset\mathcal{O}_{S,0}\,\partial_t(\mathrm{Im}\,\sigma_S^\nabla)$.

This is proved by using a connection on $\Omega_{F,S}$ induced by the action of ∂_{s_i} on $\mathrm{Im}\,\sigma_S$.

Note The associated primitive form can be defined by
$\zeta_S := \sigma_S^\nabla(\overline{\zeta}_0)$ for $\overline{\zeta}_0\in\Omega_f$ s.t. $A_1\overline{\zeta}_0 = \alpha_{f,1}\,\overline{\zeta}_0$.

Here we *assume* that σ_0 is *very good.*
For good sections, we take a generating eigenvector $\overline{\zeta}_0$, and replace $\alpha_{f,1}$ by the corresponding eigenvalue.

Formal Gauss-Manin systems

$\widehat{G}_{F,\widehat{S}} := H^{n+1}\big(\widehat{C}^{\bullet}_{F,\widehat{Y}/\widehat{S}}\big)$ with $\widehat{Y} = X \times \widehat{S}$ and

$$\widehat{C}^{\bullet}_{F,\widehat{Y}/\widehat{S}} := \big(\Omega^{\bullet}_{X,0}((\partial_t^{-1}))[[s_1,\dots,s_m]]\,\delta(t-F),\,\mathrm{d}\big) = \big(\Omega^{\bullet}_{X,0}((\partial_t^{-1}))[[s_1,\dots,s_m]],\,\mathrm{d}-\partial_t\,\mathrm{d}F\wedge\big),$$ where

$$\partial_{x_i}\delta(t-F) = -(\partial_{x_i}F)\,\partial_t\,\delta(t-F), \quad t\,\delta(t-F) = F\,\delta(t-F).$$

Define similarly $\widehat{G}_{f,\widehat{S}}$ with F replaced by f. Then

$$\widehat{G}_{f,\widehat{S}} = \widehat{G}_f[[s_1,\dots,s_m]].$$

Note $\widehat{G}_{F,\widehat{S}}$ free of rank μ over $\mathbf{C}((\partial_t^{-1}))[[s_1,\dots,s_m]]$.
Hence $\widehat{G}_{F,\widehat{S}} \ne \bigcup_i \partial_t^i \widehat{H}''_{F,\widehat{S}}$ $\big(\widehat{H}''_{F,\widehat{S}} \cong \bigoplus^{\mu} \mathbf{C}[[\partial_t^{-1}, s_1,\dots,s_m]]\big)$.
$\big($Recall $\mathbf{C}((\partial_t^{-1}))[[s_1,\dots,s_m]] \ne \mathbf{C}[[s_1,\dots,s_m]]((\partial_t^{-1}))$.$\big)$

Thm. (C. Li, S. Li, K. Saito). There are isomorphisms
$\Psi := e^{(F-f)\,\partial_t} : \widehat{G}_{f,\widehat{S}} \xrightarrow{\sim} \widehat{G}_{F,\widehat{S}}$, $\Phi := e^{(f-F)\,\partial_t} : \widehat{G}_{F,\widehat{S}} \xrightarrow{\sim} \widehat{G}_{f,\widehat{S}}$,
compatible with actions of $\mathbf{C}((\partial_t^{-1}))[[s_1,\dots,s_m]]$, t, ∂_{s_i}.
$\big($In fact, we have $e^{(F-f)\,\partial_t} : \widehat{C}^{\bullet}_{f,\widehat{Y}/\widehat{S}} \xrightarrow{\sim} \widehat{C}^{\bullet}_{F,\widehat{Y}/\widehat{S}}$ and its inverse $e^{(f-F)\,\partial_t}$, compatible with actions of t, ∂_{s_i}, etc.$\big)$

Cor. Any good section $\widehat{\sigma}_0$ of $\widehat{pr}_0 : \widehat{H}''_f \to \Omega_f$ is uniquely extended to a good section $\widehat{\sigma}_{\widehat{S}}$ of $\widehat{pr}_{\widehat{S}} : \widehat{H}''_{F,\widehat{S}} \to \Omega_{F,\widehat{S}}$.
Moreover, it is uniquely determined by the condition:

$$\mathrm{Im}\,\widehat{\sigma}_{\widehat{S}} = \widehat{H}''_{F,\widehat{S}} \cap \Psi\big((\mathrm{Im}\,\widehat{\sigma}_0)[\partial_t][[s_1,\dots,s_m]]\big) \text{ in } \widehat{G}_{F,\widehat{S}},$$

where $\mathrm{Im}\,\widehat{\sigma}_0 \subset \widehat{G}_f \subset \widehat{G}_{f,\widehat{S}} = \widehat{G}_f[[s_1,\dots,s_m]]$.
$\big($In the weighted homogeneous case, this is proved in [LLS].$\big)$

Cor. Formal prim. form is the unique element $\widehat{\zeta}_{\widehat{S}} \in \widehat{H}''_{F,\widehat{S}}$ s.t. $\Phi(\widehat{\zeta}_{\widehat{S}}) = \widehat{\zeta}_0 \ \big(\mathrm{mod}\ \partial_t(\mathrm{Im}\,\widehat{\sigma}_0)[\partial_t][[s_1,\dots,s_m]]\big)$ in $\widehat{G}_{f,\widehat{S}}$,
where $\widehat{\zeta}_0 := \widehat{\sigma}_0(\overline{\zeta}_0) \in \widehat{G}_f \subset \widehat{G}_{f,\widehat{S}} = \widehat{G}_f[[s_1,\dots,s_m]]$.
$\big($It is *much easier* to calculate in $\widehat{G}_{f,\widehat{S}}$ than in $\widehat{G}_{F,\widehat{S}}$.$\big)$

Note In the weighted homogeneous case we have always

$$\widehat{\zeta}_0 = [\mathrm{d}x_0 \wedge \cdots \wedge \mathrm{d}x_n] \in \widehat{H}''_f.$$

Ex. Let $f = x_1^7 + x_2^3$, $F = f + \sum_{(i,j)\in[0,5]\times[0,1]} x_1^i x_2^j s_{i,j}$.
Then $\zeta_S/\mathrm{d}x_1\wedge\mathrm{d}x_2 \ \big(\mathrm{mod}\ \mathfrak{m}_{S,0}^7\big)$ is represented by

$$1 + \tfrac{1}{7^2}\,x_1 s_{5,1}^3 + \tfrac{2^2}{7^2\cdot 3}\,s_{4,1}s_{5,1}^2 - \tfrac{53}{7^4\cdot 3^2}\,x_1^2 s_{5,1}^6 - \tfrac{101}{7^4\cdot 5}\,x_1 s_{4,1}s_{5,1}^5 - \tfrac{2^6}{7^4\cdot 3}\,s_{4,1}^2 s_{5,1}^4 - \tfrac{19\cdot 2^2}{7^4\cdot 3^2}\,s_{3,1}s_{5,1}^5,$$

This agrees with a calculation using a computer in [LLS].

Note It is nontrivial to see which terms are in the expansion.
For instance, to verify whether $s_{5,1}^i$ is in the expansion,
one has to determine when $\partial_t^i\big[(x_1^5x_2)^i\mathrm{d}x_1\wedge\mathrm{d}x_2\big] \in H''_f\setminus\{0\}$.
This is given by the condition:

$$[5i/7] + [i/3] \geqslant i, \quad 5i+1 \notin 7\mathbf{Z}, \quad i+1 \notin 3\mathbf{Z}.$$

(Strictly speaking, however, this gives only candidates for it.)

Here the last 4 coefficients are obtained as follows:

$$-\tfrac{1}{6!}\cdot\tfrac{24\cdot 17\cdot 10\cdot 3\cdot 4}{7^4\cdot 3^2} + \tfrac{1}{3!}\cdot\tfrac{1}{7^2}\cdot\tfrac{10\cdot 3}{7^2\cdot 3} = -\tfrac{53}{7^4\cdot 3^2}$$
$$-\tfrac{1}{5!}\cdot\tfrac{23\cdot 16\cdot 9\cdot 2\cdot 4}{7^4\cdot 3^2} + \tfrac{1}{2!}\cdot\tfrac{1}{7^2}\cdot\tfrac{9\cdot 2}{7^2\cdot 3} + \tfrac{1}{3!}\cdot\tfrac{2^2}{7^2\cdot 3}\cdot\tfrac{9\cdot 2}{7^2\cdot 3} = -\tfrac{101}{7^4\cdot 5}$$
$$-\tfrac{1}{4!\cdot 2!}\cdot\tfrac{22\cdot 15\cdot 8\cdot 4}{7^4\cdot 3^2} + \tfrac{1}{2!}\cdot\tfrac{1}{7^2}\cdot\tfrac{8}{7^2\cdot 3} + \tfrac{1}{2!}\cdot\tfrac{2^2}{7^2\cdot 3}\cdot\tfrac{8}{7^2\cdot 3} = -\tfrac{2^6}{7^4\cdot 3}$$
$$-\tfrac{1}{5!}\cdot\tfrac{22\cdot 15\cdot 8\cdot 4}{7^4\cdot 3^2} + \tfrac{1}{2!}\cdot\tfrac{1}{7^2}\cdot\tfrac{8}{7^2\cdot 3} = -\tfrac{19\cdot 2^2}{7^4\cdot 3^2}$$

Note that $\partial_t[x_1^i x_2^j\,\mathrm{d}x_1\wedge\mathrm{d}x_2]$ is equal to

$$\tfrac{i-6}{7}[x_1^{i-7}x_2^j\,\mathrm{d}x_1\wedge\mathrm{d}x_2] = \tfrac{j-2}{3}[x_1^i x_2^{j-3}\mathrm{d}x_1\wedge\mathrm{d}x_2] \text{ in } H''_f.$$

Here $\widehat{\ }$ in $\widehat{G}$ is for $((\partial_t^{-1}))$, and $\widehat{\ }$ in $\widehat{S}$ is for $[[s_1\dots,s_m]]$.
Convergent microlocal G-M system is defined similarly.
Usual G-M system is defined by replacing $((\partial_t^{-1}))$ with $[\partial_t]$.
Here it is better to consider delta function, i.e. the generator of the direct image as D-module by graph embedding.
This isomorphism is shown by using the following relation.
Similarly we can define the action of vector fields ∂_{s_i}.

Here ∂_{s_i} acts only on $[[s_1,\dots,s_m]]$, and trivially on $\widehat{G}_f$.

Note that the completion commutes with the cohomology of complex by using the Mittag-Leffler condition (using the acyclicity of the Koszul complex of $\mathrm{d}f\wedge$).
Note that the LHS is much bigger than the RHS.
This is related to the non-commutativity of $\varinjlim$ and $\varprojlim$.

This assertion easily follows from the above definition of formal Gauss-Manin system.

The compatibility with the action of t, ∂_{s_i} is proved by using the delta function $\delta(t-F)$ as explained above.

This uniqueness is essential for the calculation of formal primitive forms explained below.
It is easy to show that there is a unique section satisfying this condition by using the above theorem.
This is called the *opposite filtration* associated to $\widehat{\sigma}_0$.
Moreover, this condition is satisfied by *any* good extension $\widehat{\sigma}_{\widehat{S}}$ (using the condition on the action of ∂_{s_i} explained above together with the Taylor expansion of $\widehat{G}_f[[s_1,\dots,s_m]]$).

This easily follows from the definition of primitive form together with the above corollary.
An essentially equivalent assertion is stated in [LLS].
Note that we work in $\widehat{G}_{f,\widehat{S}}$ (instead of $\widehat{G}_{F,\widehat{S}}$) where the calculation is much easier.
This follows from the above proposition asserting that every good section is very good in weighted homog. case.

This calculation is possible only by using the exponential operator Φ in the above corollary.
Using the Taylor expansion of the exponential operator Φ, we can inductively determine the coefficients of the expansion of $\widehat{\zeta}_{\widehat{S}}$ in s (by comparing the coefficients).
One may wonder why only these monomials appear in the expansion.

This argument can be applied only in certain special cases.

This condition is equivalent in this case to that this element does not belong to the *opposite filtration* defined by σ_0 (so the argument is rather simplified in this case).
This number means how many times we can repeat ∂_t in H''_f.
(In fact, its coefficient may vanish after the calculation.)
The calculation is similar for other monomials of $s_{i,j}$.

The calculation is easy for the first 3 terms.
We explain the calculation for the last 4 terms, which have degree 6 in variables s_i.
These numbers on LHS come from the calculation of Taylor expansion of exponential operator $\Phi = e^{(f-F)\partial_t}$ and that of G-M connection explained *here*.