# An application of power operations to quantum groups

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# I) History:

**1999:** Nakajima $\quad U_q(\hat{\mathfrak{g}}) \curvearrowright K_{eq}(\mathrm{Nakaj}(n))$

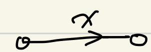
Varagnolo $\quad Y_\hbar(\mathfrak{g}) \curvearrowright H_{eq}(\mathrm{Nakaj}(n))$.

**2011:** Kontsevich–Soibelman

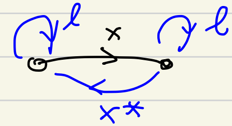
Cohomological Hall alg. (3d COHA $\mathcal{H}(\underset{\text{quiver}}{\Gamma}, \underset{\text{potential}}{W})$) $\curvearrowright H_{eq}\big(\underbrace{\text{moduli of framed reps of } \Gamma, \ \varphi_{trW}}_{\text{critical cohomology}}\big)$

**2016:** G. Zhao–Y. Davison. (dim reduction)

$Q$: quiver (ADE)




$\hat{Q}$: triple quiver




$W = \ell \cdot [x, x^*]$.

**Then:** (1) $Y_\hbar(\mathfrak{g}_Q) \hookrightarrow D(\mathcal{H}(\hat{Q}, W))$ alg hom.

& $Y_\hbar(\mathfrak{g}_Q) \xrightarrow{\simeq} D^{sph}(\mathcal{H}(\hat{Q}, W))$.

(2). 
$$\begin{tikzcd} Y_\hbar(\mathfrak{g}_Q) \arrow[rr, "N.V."] \arrow[dr] & & \mathrm{End}(H^*(\mathrm{Nak})) \\ & D(\mathcal{H}(\hat{Q}, W)) \arrow[ur, "K.S."'] & \end{tikzcd}$$

Oriented Cohomology theories $\longrightarrow$ Quantum groups

$$A^* \longmapsto \mathcal{U}^{A^*}(\mathfrak{g}) := \text{Drinfeld double of } A^*\text{-CoHA}:$$

$$\bigoplus_{\substack{v \\ \in \\ \mathbb{N}^I}} A^*_{\mathbb{C}^*}\Big( \underbrace{\mathrm{Rep}(\Pi_Q, v)}_{\text{preprojective alg}} \big/ GL_v \Big)$$

$\mathfrak{Rep}(\Pi_Q)$

## Integral Form.

Example: $A^* = K\text{-theory}$. $\mathfrak{g} = sl_2$.

$$\mathcal{U}_q(sl_2)^+ \subseteq \bigoplus_n \underset{\substack{\shortparallel \\ \mathbb{Z}[q^{\pm}]}}{K^*_{\mathbb{C}^*}(pt)} [z_1^{\pm}, z_2^{\pm}, \dots z_n^{\pm}]^{S_n}$$

over $\mathbb{Z}[q, q^{-1}]$.

$q = \varepsilon$ root of 1. $\qquad K_{\mathbb{Z}_p}(pt) = \mathbb{Z}[q^{\pm}]/q^p = 1$.

generators $E^{(n)} := \dfrac{E^n}{[n]!} \longmapsto 1_n$.

e.g.: $E \cdot E^{(n-1)} = [n]_q E^{(n)}$. $\rightsquigarrow$ $1_{(1)} * 1_{(n-1)} = \displaystyle\sum_{i=1}^{n} \sum_{j \mid j \neq i} \frac{z_i - z_j q}{z_i - z_j}$

Residue Thm $\longrightarrow$ $= \dfrac{1 - q^n}{1 - q} 1_n = [n]_q 1_n$.

# II) Power Operations

Cohomology theories $A^*$ with power operations called $H_\infty$-ring theories

skip. e.g.:

(a). $P_1 = id$, $P_0(x) = 1$

(b) $P_j(x) \wedge P_k(x) = res \Big|^{\Sigma_{i+j}}_{\Sigma_i \times \Sigma_j} P_{j+k}(x)$

(c) $\underbrace{P_j(P_k(x))}_{\text{Composition}} = res^{\Sigma_{jk}}_{\Sigma_k \wr \Sigma_j} (P_{jk}(x))$

(d) $P_j(x \wedge y) = res \Big|^{\Sigma_i \times \Sigma_j}_{\Sigma_j} (P_j(x) \wedge P_j(y))$

abelian $p$-gp., $|\pi| = r$

Let $\pi \subseteq S_n$, (e.g. $\pi = \mathbb{Z}/p \subseteq S_p$). (by ordering elts of $\pi$) $\pi \to S_r$

(i.e. a gp. that every elt has order equal to a power of $p$).

$$X^\pi := Map[\pi, X] \cong X^r$$

<u>Power operation</u>:

$$P_\pi : E^*(X) \longrightarrow E^{r*}(E\pi \times_\pi X^\pi) \text{ satisfies certain conditions.}$$

$$\begin{tikzcd} X^\pi \arrow[r, hook, "i"] & E\pi \times_\pi X^\pi \end{tikzcd} \qquad E^{r*}(E\pi \times_\pi X^\pi) \xrightarrow{i^*} E^{r*}(X^\pi)$$

$$z \longmapsto z^{\times r}$$

$$E^*(X) \longrightarrow E^{r*}(X^\pi)$$

## Examples:

① $H^*(-, \mathbb{Z}/p)$, Steenrod operations (Steenrod 62)

$$\beta : H^q(X, \mathbb{Z}/p) \longrightarrow H^{q+1}(X, \mathbb{Z}/p)$$

$$P^i : H^q(X, \mathbb{Z}/p) \longrightarrow H^{q+2i(p-1)}(X, \mathbb{Z}/p)$$

② K-theory, exterior powers, Adams operations (Atiyah 66)

Elliptic cohomology.

$$K(X) \longrightarrow K_{S_n}(X^n) \longrightarrow R(S_n) \otimes K(X) \xrightarrow{\chi(n\text{-cycle}) \otimes id} K(X)$$

$$E \longmapsto E^{\otimes n} \longmapsto [E^{\otimes n}] \longmapsto \chi \otimes (E^{\otimes n})$$

$\psi^n$ : Adams operations.

**prop:**

① $\psi^n : K(X) \to K(X)$ ring hom

② $\psi^n$ natural i.e. $f : X \to Y$, $\psi^n f^* = f^* \psi^n$.

③ $\psi^n(L) = [L]^n$. $\forall$ line bundle

④ $\psi^p(\alpha) = \alpha^p + p\beta$, for some $\beta \in K(X)$

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③ Complex cobordism (tom Dieck 68, Quillen 71): multiplicative but not additive.

④. • The theories $E_n^*$ of Hopkins-Miller (93) : $E_\infty$-ring struct $\Downarrow$
(Morava E-theory, integral lift)

Ando 95: $E_n^*$ has power operations.

Ando 95: $E(n)^*(pt) =$ (Lubin-Tate ring)
universal deformation of a
height $k$ f. g. l over $\mathbb{F}_{p^n}$

$$= \underbrace{W(\mathbb{F}_{p^n})}_{\text{Witt ring of } \mathbb{F}_{p^n}}[[ \omega_1, \dots \omega_{k-1} ]].$$

A complex orientation $x \in E^2(\mathbb{C}P^\infty) \rightsquigarrow$ a formal gp law $F$ on $E$
(coordinate on $F$.
or first chern class of a line bundle ).

$$c_1^E(L \otimes M) = F(c_1^E(L), c_1^E(M))$$

$B\pi \times X^{\pi} \xleftarrow{\Delta} X$

$$E^*(X) \xrightarrow{P_\pi} E^{2r*}(E\pi \times_\pi X^\pi) \xrightarrow{\Delta^*} E^*(B\pi \times X)$$

$$\cong E^*(B\pi) \otimes E^*(X)$$

$$\downarrow x^{tKR} \otimes 1$$

$$\tilde{E}^*(X) =: E^*(B\pi)[\tfrac{1}{S_\pi}] \otimes_{E^*} E^*(X),$$

$\mathbb{P}_\pi : E^*(X) \to \tilde{E}^*(X)$

Note: ① $X \mapsto \tilde{E}^*(X)$ is again a cohomology theory.

where: $S_\pi \subseteq E^*(B\pi)$ = multip. set gen. by $\{c_1^E(L) \mid L \circlearrowright \pi$ non-trivial line bundles$\}$

② $E^*(pt) \subseteq \tilde{E}^*(pt)$

· Cyclotomic extension

e.g.: $\pi = \mathbb{Z}/p$,

$K(pt) = \mathbb{Z}$.

$K(pt)(B\mathbb{Z}/p)[\frac{1}{c_1(L_\zeta)}]$ non-trivial

$= \mathbb{Z}[x]/(1-x^p)\ [\frac{1}{1-x}]$

$= \mathbb{Z}[\frac{1-x^p}{1-x}] = \mathbb{Z}[\zeta_p]$ add $p$-th roots of 1.

⑦

★ Thm (Ando's ?): $\mathbb{P}_H : E^*(X) \to \widetilde{E}^*(X)$

F ... $\widetilde{F}$.

subgp

Let $H \subseteq \mathrm{Hom}((\mathbb{Z}/p^k)^n, \mathbb{C}^\times)$

① $\exists$ a unique a formal gp law F on $E^*$, s.t:

$\mathbb{P}_H$ = the composition ~~(analogue of Adams operations in K-theory)~~
exists.

② $\mathbb{P}_H$ is a ring homomorphism.

Euler class of $\underset{E}{\overset{L}{\downarrow}}$.

★ (3) $\mathbb{P}_H(c_1^E(L)) = f_H(c_1(L)) := \prod_{h \in H} (h +_F e(L)) \in \widetilde{E}^*(pt)[[t]]$.

★ (4) $\mathbb{P}_H(F) = \widetilde{F}$, where $\widetilde{F}$ formal gp law on $\widetilde{E}^*(pt)$ s.t $F \xrightarrow{f_H} \widetilde{F}$ hom. of gp law.

ie: $f_H(x +_F y) = f_H(x) +_{\widetilde{F}} f_H(y)$

In particular, $x, y = c_1$ of line bundles

$\mathbb{P}_H(x +_F y) = \mathbb{P}_H(x) +_{\widetilde{F}} \mathbb{P}_H(y)$

eg Full $H = \Lambda_k^* \cong (\mathbb{Z}/p^k)^n \cong {}_{p^k}F(D_k)$, where

Then: $\mathbb{P}_H^{p^k} : E_n(X) \to E_n(X)$, identity on $E_n(pt)$

$\mathbb{P}_H^{p^k}(eL) = [p^k]_F(eL)$.

## III) Quiver with automorphism.

$Q = (I, H)$ quiver, $\qquad \cancel{\left(h \mapsto [h] = \begin{matrix} h' \\ h'' \end{matrix} \in I\right)}$

$a$: <u>admissible</u> automorphism of $Q$.

i.e: $\quad a : I \to I$ permutation

$\quad a : H \to H$

s.t $\forall h \in H$, $\quad [a(h)] = a[h]$.

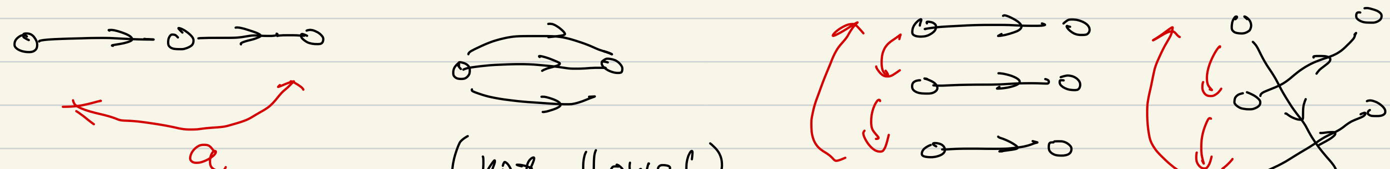
<u>Assumptions</u>:

① No edge join two vertices in the same $a$-orbits

② $\exists$ <u>min $n$</u>, s.t $\quad a^n = 1$ on both $I$ & $H$, uniform $n$.

Warning: In Lusztig's setup, "$n$" doesn't need to be uniform.

e.g:




(not allowed) $\qquad$ (not allowed) $\qquad$ ok $\qquad$ ok.

(with more arrows also ok).

**Define:** $Q^\circ = \{ I^\circ, H^\circ \}$

$I^\circ = I/a$ $\quad$ $H^\circ := H/a$

$= a\text{-orbits of } I.$ $\quad$ $= a\text{-orbits of } H.$

$a \curvearrowright \mathrm{Rep}(Q).$

**Lemma:** $\mathrm{Rep}(Q^\circ) = \mathrm{Rep}(Q)^a$. under the assumptions.

Pf: Embedding of $\mathrm{Rep}(Q^\circ)$ into $\mathrm{Rep}(Q)^a$

$$\underset{\mathrm{Rep}(Q^\circ)}{\underset{\in}{V_1^\circ \xrightarrow{x_{h^\circ}} V_2^\circ}} \hookrightarrow \begin{array}{c} V_1^\circ \circ \xrightarrow{x_{h^\circ}} \circ V_2^\circ \\ V_1^\circ \circ \xrightarrow{y_{h^\circ}} \circ V_2^\circ \\ V_1^\circ \circ \xrightarrow{z_{h^\circ}} \circ V_2^\circ \end{array}$$

- $V_i := V_{i^\circ}$ as long as $i \in$ a-orbit
- $x_h := x_{h^\circ}$. as long as $h \in$ a-orbit of $h^\circ$

But if $V \in \mathcal{V}^a$.

Then the action of $\langle a \rangle$ on $\mathrm{Rep}(Q, V)$ is by permutation.

ie:

$$\begin{tikzcd} \circ \arrow[r] & \circ \\ \circ \arrow[r] & \circ \\ \circ \arrow[r] & \circ \end{tikzcd}$$

$(x, y, z) \longmapsto (y, z, x) \longmapsto (z, x, y)$

this is the action of $\langle a \rangle$.

use the f.g.l. $F$.

$$\Delta^*: E^*_{\langle a\rangle}(\mathcal{R}ep(Q)) \longrightarrow E^*_{\langle a\rangle}(\mathcal{R}ep(Q^0)) \to \tilde{E}^*(\mathcal{R}ep(Q^0))$$

use the fml gp law $\tilde{F}$ given by $\mathbb{P}_{\langle a\rangle}$.

$$\begin{tikzcd} & \tilde{E}^*(\mathcal{R}ep(Q^0)) \\ & E^*(\mathcal{R}ep(Q^0)) \arrow[u, "\mathbb{P}_{\langle a\rangle}"'] \end{tikzcd}$$

$\mathbb{P}_{\langle a\rangle}$: alg. hom.

$$e^{E}(V_0) \mapsto \mathbb{P}(e^{E}(V_0)) = e^{\tilde{E}}(V_0)$$

$$\| \\ e^{E}(V_0 \otimes \mathbb{C}[\langle a\rangle])$$

## Main Thm :

① $E^*_{\langle a\rangle}(\mathrm{Rep}(Q)) \xrightarrow{\Delta^*} E^*_{\langle a\rangle}(\mathrm{Rep}(Q^\circ))[\frac{1}{S_a}] \longrightarrow \tilde{E}^*(\mathrm{Rep}(Q^\circ))$ f.g.l. use. $\tilde{F}$

formal gp. law $F$. is an algebra hom. (alg. str: Hall alg. structures)

② $E^*_{\langle a\rangle}(\mathrm{Rep}(\Pi_Q)) \xrightarrow{\Delta^*} \tilde{E}^*(\mathrm{Rep}(\Pi_{Q^\circ}))$ is an alg hom.

$$\begin{array}{ccc} E^*_{\langle a\rangle}(\mathrm{Rep}(\Pi_Q)) & \xrightarrow{\Delta^*} & \tilde{E}^*(\mathrm{Rep}(\Pi_{Q^\circ})) \\ \cup\!\! \mid & & \cup\!\! \mid \end{array}$$

In particular: $\mathcal{U}^E(Q) \xrightarrow{\text{alg hom}} \mathcal{U}^{\tilde{E}}(Q^\circ)$ ← gen. by $\left\{ 1_{n e_k} \;\middle|\; \begin{array}{l} k\in I \\ n\in\mathbb{N} \end{array} \right\}$

③. $\forall$ $w$ framing, s.t $p \mid w$

$$E^*_{\langle a\rangle}(\mathrm{Rep}(\Pi_Q)) \xrightarrow{\Delta^*} \tilde{E}^*_{\langle a\rangle}(\mathrm{Rep}(\Pi_{Q^\circ}))$$

$\circlearrowright$ $\circlearrowright$ mod. hom.

$$E^*_{\langle a\rangle}(\mathrm{Nak}_Q(\tfrac{w}{p},\dots,\tfrac{w}{p})) \longrightarrow \tilde{E}^*(\mathrm{Nak}_{Q^\circ}(\tfrac{w}{p}))$$

"                    "

**Proof:** ①.

$$x \overset{E}{*} y \overset{\text{def}}{=} q_*^E\left(x \cdot y \cdot e(V^\varphi)\right)$$

$$\Delta^*(x \overset{E}{*} y) = \Delta^* q_*^E\left(x \cdot y \cdot e^E(V^\varphi)\right)$$

$$= \tilde{q}_*^{\circ E}\left(\Delta^* x \cdot \Delta^*(y) \cdot \Delta^* e^E(V^\varphi)\right)$$

$$= q_*^{\circ \tilde{E}}\left(\Delta^*(x) \cdot \Delta^*(y) \cdot e^{\tilde{E}}(V_0^\varphi)\right),$$

where $\vartheta = \vartheta_0 \otimes \mathbb{C}[a]$

as $\langle a \rangle$ reps. $\mathbb{Z}/p$

$$= \Delta^*(x) \overset{\tilde{E}}{*} \Delta^*(y)$$

**Application:** Recover Lusztig's $q$Frob. map.

$E^* = K$-theory. $\mathfrak{g} = sl_2$ (illustration purpose).

$Q$ : $\cdot\ \cdot\ \cdot$ $a$ $\qquad Q^0$: $\bullet$

$p \mid n$:

$$\mathrm{Rep}(Q^0, n) \xleftarrow{\oplus} \mathrm{Rep}(Q, \tfrac{n}{p}, \tfrac{n}{p}, \ldots, \tfrac{n}{p}) \xleftarrow{\Delta} \mathrm{Rep}(Q^0, \tfrac{n}{p})$$

$p \nmid n$ otherwise get zero map.

I identify $q$ parameter with $\langle a \rangle$-parameter.

$$\Delta^* \circ \oplus^* : \underbrace{K_{\mathbb{Z}/p \subseteq \mathbb{C}^*}(pt)}_{\substack{\parallel \\ \mathbb{Z}[q]/(q^p=1)}} [z_1^{\pm}, \ldots, z_n^{\pm}]^{S_n} \longrightarrow \boxed{\mathbb{Z}\left[\frac{1-x^p}{1-x}\right]} [z_1^{\pm}, z_2^{\pm}, \ldots, z_{\frac{n}{p}}^{\pm}]^{S_{\frac{n}{p}}}$$

cyclotomic extension

$q \to \varepsilon$

$$\xrightarrow{\oplus^*} K_{\substack{\mathbb{Z}/p \\ \cap \\ \mathbb{C}^*}} \times \langle a \rangle \left( \coprod_{\langle a \rangle} pt / GL \right) \xrightarrow{\Delta^*}$$

quiver automorphism.

Think: $a: \mathbb{Z}/p \hookrightarrow GL_n$

$$1 \longmapsto \begin{bmatrix} 0 & 1 & & \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ 1 & & & 0 \end{bmatrix}$$

So: $a$: permutes the variables $z_1, \ldots z_n$

$$
\begin{tikzcd}
z_1, \dots, z_{\frac{n}{p}} \arrow[r, mapsto] \arrow[d, "a"'] & z_1, \dots, z_{\frac{n}{p}}. \\
z_{\frac{n}{p}+1}, \dots, z_{2\cdot\frac{n}{p}} \arrow[r, mapsto] \arrow[d, "a"'] & z_1\cdot\varepsilon, \dots, z_{\frac{n}{p}}\cdot\varepsilon. \\
\vdots \arrow[d, "a"'] & \vdots \\
z_{(p-1)\frac{n}{p}+1}, \dots, z_{p\cdot\frac{n}{p}} \arrow[r, mapsto] \arrow[uuu, bend right, "a"'] & z_1\varepsilon^{p-1}, \dots, z_{\frac{n}{p}}\cdot\varepsilon^{p-1}.
\end{tikzcd}
$$

<u>Cor</u>: ① This is an alg homomorphism

② Take the sub alg gen. by $\mathcal{L}_n$, then

Lusztig's q-Fr: $\mathcal{U}_\varepsilon(\mathfrak{g})^+ \xrightarrow[\text{q-Fr}]{} \mathcal{U}(\mathfrak{g})^+$

$$E^{(n)} \longmapsto \begin{cases} e^{(n/p)} & p \mid n \\ 0 & \text{otherwise.} \end{cases}$$

IV) **Motivation**

Alg. gp $G$ / $\mathbb{k}$, $\mathrm{char}(\mathbb{k}) = p$. & $\mathcal{U}_\varepsilon(\mathfrak{g})$, $\varepsilon^p = 1$. (Lusztig).

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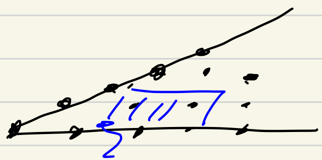
$X := \mathrm{Hom}(T, \mathbb{k}^\times) =$ char. lattice of $G$.

$$X^+ = \{\lambda \in X \mid \langle \lambda, \alpha_i^\vee \rangle \in \mathbb{N}\} \xleftrightarrow{1:1} \mathrm{Irr}(G)$$



$X^+$ $\quad \lambda \longmapsto L_\lambda / \mathbb{k}$.

$X^+_{red}$.

$$X^+_{red} := \{\lambda \in X^+ \mid 0 \leq \langle \lambda, \alpha_i^\vee \rangle < p\}$$

$$\underset{\substack{\circlearrowright \\ G}}{\mathrm{ch}(V)} := \sum_{\mu \in X} \dim V(\mu) e^\mu \in \mathbb{Z}[X].$$

T-wt space

$\mathrm{Fr}: G_{/\mathbb{k}} \longrightarrow G_{/\mathbb{k}}$

By Steiberg $\otimes$ Thm:

p-adic expansion

$\forall \lambda \in X^+$, $\quad \lambda = \lambda_0 + p\lambda_1 + \cdots + p^m \lambda_m$, $\lambda_i \in X^+_{red}$

$$\mathrm{ch}(L_\lambda) = \mathrm{ch}(L_{\lambda_0})\, \mathrm{ch}(L_{\lambda_1})^{[1]}\, \mathrm{ch}(L_{\lambda_2})^{[2]} \cdots\cdots$$

$p \gg 0$ Lusztig's Conj. $\longrightarrow \overset{?}{=} E^1_{\lambda_0} \cdot (E^1_{\lambda_1})^{[1]} (E^1_{\lambda_2})^{[2]} \cdots\cdots$

$\xi = \sum_{\lambda \in X} c_\lambda e^\lambda \in \mathbb{Z}[X]$

$\xi^{[h]} := \sum_{\lambda \in X} c_\lambda e^{p^h \lambda} \in \mathbb{Z}[X]$

Lusztig 15'

$= \underbrace{\lim_{k \to \infty} E_\lambda^k :}$

$= E_\lambda^\infty.$

$E_\lambda^0 =$ Weyl char formula

$E_\lambda^1 = \sum_{\mu \in X^+} \mathbb{P}_{\mu,\lambda} E_\mu^0.$

$E_\lambda^2 = \sum_{\mu \in X^+} \mathbb{P}_{\mu, \frac{\lambda - \lambda_0}{p}} E^1_{\lambda^0 + p\mu}.$

$\vdots$

$n = 0$: $L_\lambda^0 =$ irrep. of $U(\mathfrak{g})$, $ch(L_\lambda^0) = E_\lambda^0.$

$n = 1$: For $U_\varepsilon(\mathfrak{g})$, $\varepsilon^1 = 1$: $\forall \lambda \in X^+.$

Let $L_\lambda^1 :=$ irrep. of $U_\varepsilon(\mathfrak{g})$, $\lambda \in X^+.$

Then: $ch(L_\lambda^1) = E_\lambda^1 = E^1_{\lambda_0} \cdot \left(E^0_{\frac{\lambda - \lambda_0}{p}}\right)^{[1]}$

where: $qFr : U_\varepsilon(\mathfrak{g}) \longrightarrow U(\mathfrak{g}).$

In particular,

$$\left[L^{(1)}_{p\lambda''}\right] = \left[L^{(0)}_{\lambda''}\right]^{[1]^\sigma}$$

~~② Steinberg tensor formula~~ $\cancel{\lambda = \lambda' + p\lambda''}$ s.t. $\cancel{\lambda' \in X^+_{\mathrm{red}}}$

$$\cancel{L^{(1)}_{\lambda} \underset{U_q(\mathfrak{g})\text{-mod}}{\overset{\sim}{=}} L^{(1)}_{\lambda'} \otimes \underline{\underline{L^{(1)}_{p\lambda''}}} \underset{\text{by ①}}{\cong} L^{(1)}_{\lambda'} \otimes \underbrace{\left(L^{(0)}_{\lambda''}\right)^{[1]}}_{\text{Frob.}}}$$

Problem / Hope:

① Iterate Frobenius map for quantum groups, i.e.

$$\cdots\cdots \to U^{(4)}_{\varepsilon_4}(\mathfrak{g}) \to U^{(3)}_{\varepsilon_3}(\mathfrak{g}) \longrightarrow U^{(2)}_{\varepsilon_2}(\mathfrak{g}) \xrightarrow{Fr} U_{\varepsilon}(\mathfrak{g}) \xrightarrow{\varepsilon Fr} U(\mathfrak{g})$$

② Hope the Morava $E_n^*$ - theoretical quantum gps will do the job.

②a. <u>p-th roots of 1 in $E_n^*$.</u>

$E_n^*(B\mathbb{Z}/p)$ "$\simeq$" $(\mathbb{Z}/p)^n$.
roughly

$= E_n^*(pt)[[x]] \Big/ [p]_F x$, where $[p]_F x = \underbrace{x +_F x +_F \cdots +_F x}_{p\text{-copies}}$.

$\underbrace{\qquad}_{\text{[HKR] } E_n^*(pt)\text{-free mod of rk } p^n.}$

(2b) Stapleton: $\exists$ an $E_n^*(pt)$-alg $C_{n,1}$ s.t.

$$\begin{tikzcd} \mathbb{G}'[p] \oplus (\mathbb{Z}/p)^1 \arrow[r] \arrow[d] & \mathbb{G}_n[p] \arrow[d] \\ \operatorname{Spec} C_{n,1} \arrow[r] & \operatorname{Spec} E_n^*(pt) \end{tikzcd}$$

height n-1 $\to \mathbb{G}'$; $\mathbb{G}_n \to$ height n

VS: [HKR]

$$\begin{tikzcd} (\mathbb{Z}/p)^n \arrow[r] \arrow[d] & \mathbb{G}_n[p] \arrow[d] \\ \operatorname{Spec} L \arrow[r] & \operatorname{Spec} E_n^*(pt) \end{tikzcd}$$

height 0 $\uparrow$