Higher-dimensional Heegaard Floer homology and Hecke algebras

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Joint work with Yin Tian and Tianyu Yuan

Goal

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Describe the appearance of various Hecke algebras such as the finite Hecke algebra, the affine Hecke algebra and the double affine Hecke algebra (DAHA) (of type A) in symplectic geometry.

1. Introduction and main result

Let $\Sigma \neq S^2$ be a closed oriented surface or a closed oriented surface with finitely many points removed.

The cotangent bundle
$$T^*\Sigma = \sqcup_{q\in\Sigma} T^*_q\Sigma$$
is a prototypical symplectic manifold: the phase space of a particle in Σ. The symplectic (i.e., closed nondegenerate) 2-form is locally given by $\omega = dx \wedge dp$, where x is the position and p is the momentum.

Introduction

In physics one writes down a quadratic-at-infinity Hamiltonian function

$$H : T^*\Sigma \to \mathbb{R} \qquad \text{e.g.} \quad H(x,p) = \frac{1}{2}|p|^2$$

and studies the dynamics of the corresponding Hamiltonian vector field.

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Remark

For $H(x,p) = \frac{1}{2}|p|^2$, the Hamiltonian vector field $X_H = p\partial_x$ is the geodesic flow with respect to the Riemannian metric g on Σ, if $|p|$ is measured with respect to g.

Introduction

The cotangent fiber $T_q^*\Sigma$ over $q \in \Sigma$ is a Lagrangian submanifold of $T^*\Sigma$, i.e., $\omega|_{T_q^*\Sigma} = 0$.

The wrapped Floer homology group $CW(T_q^*\Sigma)$ of $T_q^*\Sigma$ is generated by time-1 trajectories of the Hamiltonian vector field X_H starting and ending at q.

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The time-1 trajectories are in bijection with $\pi_1(\Sigma, q)$. (Here we are using the fact that $\Sigma \neq S^2$ and the metric on Σ has curvature 0 or -1.) Hence

$$CW(T^*_q\Sigma) = \mathbb{Z}[\pi_1(\Sigma, q)].$$

Statement of main theorem

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Theorem

$$CW(\sqcup_{i=1}^{\kappa} T^*_{q_i}\Sigma) \simeq \mathrm{H}_\kappa(\Sigma, \boldsymbol{q}).$$

1. The left-hand side is the wrapped higher-dimensional Heegaard Floer homology group of κ distinct cotangent fibers, representing trajectories of κ distinct particles.
2. The right-hand side is the type A Hecke algebra for Σ, called a surface Hecke algebra.

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Type of Hecke algebra $\mathrm{H}_\kappa(\Sigma, \boldsymbol{q})$	Surface Σ
finite Hecke algebra	plane $\mathbb{R}^2$
affine Hecke algebra	cylinder $\mathbb{R} \times S^1$
DAHA	torus T^2

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The plan for the rest of the hour is to explain what both sides of the isomorphism mean in more detail.

2. Hecke algebras

The most basic Hecke algebra is the finite Hecke algebra $\mathrm{H}_\kappa(\mathbb{R}^2, \boldsymbol{q})$, which is the algebra over $\mathbb{Z}[\![\hbar]\!]$ given by generators $T_1, \dots, T_{\kappa-1}$ and relations

$$\begin{aligned} T_i T_j &= T_j T_i, \quad |i-j| \geq 2, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad i < \kappa - 1 \\ T_i - T_i^{-1} &= \hbar\, \mathrm{id}\,. \end{aligned}$$

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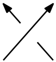
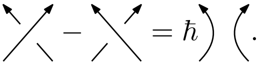
Diagrammatically, T_i is given by



and the last equation, called the HOMFLY skein relation, is given by:



(1)

Double affine Hecke algebras

Double affine Hecke algebras (DAHAs), due to Cherednik, have an incomprehensible algebraic description (at least to me). Fortunately, there exists particularly nice topological description due to Morton-Samuelson (apparently also known to Cherednik):

Start with the braid group $\mathrm{Br}_\kappa(\Sigma, \boldsymbol{q})$ of Σ with κ strands. A braid is a 1-dimensional submanifold of $\Sigma \times [0,1]$ whose boundary is $\boldsymbol{q} \times \{0,1\}$ and which is transverse to each $\Sigma \times \{t\}$. (Equivalently, $\mathrm{Br}_\kappa(\Sigma, \boldsymbol{q})$ is π_1 of the configuration space $\mathrm{Conf}_\kappa(\Sigma)$ of κ points on Σ with basepoint $\boldsymbol{q}$.)

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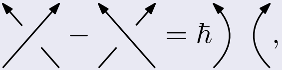
Also fix a marked point $\star$ disjoint from $\boldsymbol{q}$. Let $\mathrm{Br}_{\kappa,1}(\Sigma, \boldsymbol{q}, \star)$ be the subgroup of $\mathrm{Br}_{\kappa+1}(\Sigma, \boldsymbol{q} \sqcup \{\star\})$ consisting of braids whose last strand connects $\star$ to itself by a straight line in $[0,1] \times \Sigma$.

DAHAs

Theorem (Morton-Samuelson)

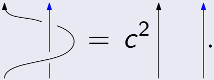
For $\Sigma = T^2$, the DAHA $\mathrm{H}_\kappa(\Sigma, \mathsf{q})$ is isomorphic to the quotient of the group algebra $\mathbb{Z}[\hbar^{\pm 1}, c^{\pm 1}][\mathrm{Br}_{\kappa,1}(\Sigma, \mathsf{q}, \star)]$ by two local relations:

1. *the HOMFLY skein relation*



(2)

2. *the marked point relation $P = c^2$*

$P :=$ 

(3)

Here the black lines are strands between basepoints in q and the straight blue line connects the marked point $\star$ to itself.
The product is given by the concatenation of braids.

3. Higher-dimensional Heegaard Floer homology

Given a genus g surface Σ, the classical Heegaard Floer homology of Ozsváth-Szabó is the Floer homology of two Lagrangians in the g-fold symmetric product

$$\mathrm{Sym}^g(\Sigma) := (\Sigma \times \cdots \times \Sigma)/S_g.$$

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It has been almost unreasonably effective at answering all sorts of low-dimensional topology questions: e.g., on knots and links, 3-manifolds, contact structures on 3-manifolds, surfaces in 4-manifolds, 4-manifolds, and symplectic structures on 4-manifolds.

Higher-dimensional Heegaard Floer homology (HDHF)

HDHF (developed with Vincent Colin and Yin Tian) replaces Σ by higher-dimensional symplectic manifolds X such as $T^*\Sigma$. In other words, we will be modeling Lagrangian Floer homology on the Hilbert scheme $\mathrm{Hilb}^{[\kappa]}(X)$ (or flag Hilbert scheme) of κ points on X.

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Remark

For us $T^*\Sigma$ is a symplectic manifold, not a holomorphic symplectic manifold, as is customary for people working on Hilbert schemes (cf. Nakajima's book). Also we do not view Σ as a Riemann surface.

Remark

It is well-known that DAHAs show up when studying sheaves/ equivariant K-theory on $\mathrm{Hilb}^{[\kappa]}(\mathbb{C}^2)$. The way DAHAs show up in our setting is slightly different in view of our remark above.

Setup for HDHF

Let $\boldsymbol{L}_0 = L_{01} \sqcup \cdots \sqcup L_{0\kappa}$ and $\boldsymbol{L}_1 = L_{11} \sqcup \cdots \sqcup L_{1\kappa}$ be κ-tuples of pairwise disjoint Lagrangian submanifolds in X. Suppose $\boldsymbol{L}_0 \pitchfork \boldsymbol{L}_1$.

Generators. The generators of the HDHF chain complex $\widehat{CF}(\Sigma, \boldsymbol{L}_0, \boldsymbol{L}_1)$ are κ-tuples of intersection points $\boldsymbol{y} = \{y_1, \dots, y_\kappa\}$, $y_j \in L_{0j} \cap L_{1\sigma(j)}$, $\sigma \in S_\kappa$ (i.e., we use each L_{0i}, L_{1i} exactly once).

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Differential. This is analogous to the cylindrical reformulation of Heegaard Floer homology due to Robert Lipshitz. We'll be counting holomorphic maps

$$u : \dot{F} \to \mathbb{R}_s \times [0,1]_t \times X,$$

where $\dot{F}$ is a compact Riemann surface with boundary + boundary punctures and can be of any genus. [Think of the target as a fibration over $\mathbb{R} \times [0,1]$ with fiber X.]

A little bit more about the differential

Given κ-tuples $\boldsymbol{x}$ and $\boldsymbol{y}$, let $\mathcal{M}(\boldsymbol{x}, \boldsymbol{y})$ be the moduli space of holomorphic maps

$$u : \dot{F} \to \mathbb{R}_s \times [0,1]_t \times X,$$

such that:

1. $\partial \dot{F}$ has 2κ boundary components and each component is mapped to a different $\mathbb{R} \times \{i\} \times L_{ij}$;
2. $\pi_X \circ u \to \boldsymbol{x}$ as $s \to +\infty$;
3. $\pi_X \circ u \to \boldsymbol{y}$ as $s \to -\infty$;
4. $\pi_{\mathbb{R}\times[0,1]} \circ u$ is a κ-fold branched cover.

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The differential is given by:

$$d\boldsymbol{x} = \sum_{\boldsymbol{y}, \chi \leq \kappa} \# \mathcal{M}^{\mathrm{ind}=0, \chi}(\boldsymbol{x}, \boldsymbol{y}) \hbar^{\kappa - \chi} \boldsymbol{y},$$

where $\chi = \chi(\dot{F})$.

Wrapped HDHF

"Wrapped" HDHF in our case means we take $\boldsymbol{L}_0 = \sqcup_{i=1}^{\kappa} T^*_{q_i}\Sigma$ and $\boldsymbol{L}_1 = \sqcup_{i=1}^{\kappa} \phi_H^1(T^*_{q_i}\Sigma)$, where ϕ_H^1 is the time-1 flow of the Hamiltonian vector field $X_H = p\partial_x$ from earlier.

$$CW(\sqcup_{i=1}^{\kappa} T^*_{q_i}\Sigma) := \widehat{CF}(\Sigma, \sqcup_{i=1}^{\kappa} T^*_{q_i}\Sigma, \sqcup_{i=1}^{\kappa}\phi_H^1(T^*_{q_i}\Sigma)).$$

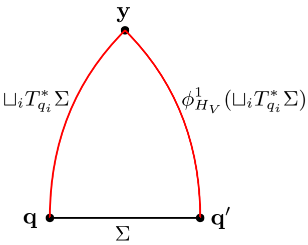
4. Some ingredients of the proof

We want to construct an algebra map

$$\mathcal{F} : \widehat{CF}(\boldsymbol{L}_0, \boldsymbol{L}_1) \to \mathrm{H}_\kappa(\Sigma, \boldsymbol{q})$$

which is an isomorphism on homology.

(1) In the case of $\kappa = 1$, this is due to Abbondandolo-Schwarz and Abouzaid. $\mathcal{F}$ is a transfer map which counts holomorphic triangles:



Here $\mathrm{H}_{\kappa=1}(\Sigma, \boldsymbol{q}) = \mathbb{Z}[\pi_1(\Sigma, q)]$ and $\mathcal{F}$ is an isomorphism on homology.

Some ingredients of proof

(2) In the case of $\kappa > 1$, we are counting holomorphic curves of higher genus and in a 1-parameter family of such holomorphic curves, the topological type of the curve may change. This only occurs along the zero section of $T^*\Sigma$ but not along $\boldsymbol{L}_0$ and $\boldsymbol{L}_1$ and the treatment of such degenerations is due to Ekholm-Shende and involves the HOMFLY skein relation.

Further developments

- Tian and Yuan can actually do an explicit holomorphic curve count which improves our main theorem for $\Sigma = \mathbb{R}^2$ to $\mathbb{Z}[\hbar]$-coefficients.
- H-Tian-Yuan can now give a Morse A_∞-model when $\Sigma = S^2$ and give a conjectural combinatorial description of the "dg Hecke algebra for S^2".
- Krutowski and Yuan are developing HDHF symplectic homology, which, together with Abouzaid's generating criterion in this case, shows that the HDHF Fukaya category $\mathrm{Fuk}_\kappa(T^*\Sigma)$ is equivalent to mod-$H_\kappa(\Sigma, \mathrm{q})$; cf. it is conjectured that the Fukaya category of the Hichin moduli space of Σ (which is close to $\mathrm{Hilb}^{[\kappa]}(T^*\Sigma)$ as a holomorphic symplectic manifold) is also mod-$H_\kappa(\Sigma, \mathrm{q})$.
- Reisin-Tzur showed that, if C is a nontrivial curve on $\Sigma = T^2$ and $C_1, \dots, C_\kappa$ are disjoint parallel copies, then $\mathrm{Hom}(\sqcup_i T^*_{C_i}\Sigma, \sqcup_i T^*_{q_i}\Sigma)$ is basically the polynomial representation of the DAHA.

dg Hecke algebra for S^2

Generators $T_1, \dots, T_{\kappa-1}$ and x_1, and relations

$$\begin{aligned} T_i^2 &= 1 + \hbar T_i, \quad 1 \le i \le \kappa - 1, \\ T_i T_{i+1} T_i &= T_{i+1} T_i T_{i+1}, \quad 1 \le i \le \kappa - 2, \\ T_i T_j &= T_j T_i, \quad 1 \le i, j \le \kappa - 1, |i - j| > 1, \\ x_1 T_j &= T_j x_1, \quad 2 \le j \le \kappa - 1, \\ T_1^{-1} x_1 T_1^{-1} x_1 + x_1 T_1^{-1} x_1 T_1 &= 0. \end{aligned}$$

Grading: $\deg(T_i) = 0$ and $\deg(x_1) = -1$.
The differential is given by $dT_i = 0$ and

$$dx_1 = T_1 T_2 \cdots T_{\kappa-2} T_{\kappa-1}^2 T_{\kappa-2} \cdots T_2 T_1 - c^2.$$

How does it fit in the grand scheme of things?

It is part of a TQFT (in progress) which assigns a category $\mathcal{C}(\Sigma)$ to a surface and a $\mathcal{C}(\Sigma)$-module $\mathcal{M}(X)$ to a 3-manifold X with boundary Σ.

Conjecture

This TQFT (or some variant of it) gives the 3-dimensional theory corresponding to Khovanov-Rozansky link homology (for links in S^3).

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The evidence (scant so far because of computational difficulties) is that there are a lot of formal parallels with the refined Chern-Simons invariants of Aganagic-Shakirov.

End of talk.

Happy Birthday, Hiraku!