Mirror symmetry and big algebras

Tamás Hausel

Institute of Science and Technology Austria
http://hausel.ist.ac.at

Gauge Theory, Moduli Spaces and Representation Theory
In honor of the 60th birthday of Hiraku Nakajima
February 2023

FWF
Der Wissenschaftsfonds.

ISTA Institute of Science and Technology Austria

Motivation from mirror symmetry

- $\mathbb{M}_{\mathrm{G}}$ moduli of G-Higgs bundles on curve C; G cplx reductive
- classical limit (Donagi–Pantev 2012) of mirror symmetry: $\mathcal{S} : D^b(\mathbb{M}_{\mathrm{G}}) \sim D^b(\mathbb{M}_{\mathrm{G}^\vee})$
  generically Fourier-Mukai transform $\mathbb{M}_{\mathrm{G}} \overset{h_{\mathrm{G}}}{\to} \mathbb{A} \cong \mathbb{A}^\vee \overset{h_{\mathrm{G}^\vee}}{\leftarrow} \mathbb{M}_{\mathrm{G}^\vee}$
- (Kapustin–Witten 2007) dominant $\mu \in \Lambda^+(\mathrm{G}^\vee)$, $c \in C$
  Hecke operators:
  $$\begin{tikzcd} & D^b(\mathcal{H}^\mu_c) \arrow[dr] & \\ \mathcal{H}^\mu_c : \; D^b(\mathbb{M}_{\mathrm{G}}) \arrow[ur] \arrow[rr] & & D^b(\mathbb{M}_{\mathrm{G}}) \end{tikzcd}$$
  and
  Wilson operators: $\mathcal{W}^\mu_c : D^b(\mathbb{M}_{\mathrm{G}^\vee}) \overset{\otimes \rho^\mu(\mathbb{E}^\vee_c)}{\longrightarrow} D^b(\mathbb{M}_{\mathrm{G}^\vee})$
  intertwine: $\mathcal{H}^\mu_c \circ \mathcal{S} = \mathcal{S} \circ \mathcal{W}^\mu_c$
- test for $\mathcal{O}_{\mathbb{M}_{\mathrm{G}^\vee}} \in D^b_{coh}(\mathbb{M}_{\mathrm{G}^\vee})$:
  $\mathcal{H}^\mu_c(\mathcal{S}(\mathcal{O}_{\mathbb{M}_{\mathrm{G}^\vee}})) = \mathcal{H}^\mu_c(\mathcal{O}_{W_0^+}) = \mathcal{S}(\mathcal{W}^\mu_c(\mathcal{O}_{\mathbb{M}_{\mathrm{G}^\vee}})) = \mathcal{S}(\rho^\mu(\mathbb{E}^\vee)_c)$
- the Hecke transform of the Hitchin section $\mathcal{H}^\mu_c(\mathcal{O}_{W_0^+})$ is supported at a union of Lagrangian upward flows

Lagrangian upward flows in $\mathbb{M}$

- $\mathbb{M} := \mathbb{M}_{\mathrm{PGL}_n} \ni (E, \Phi); \Phi \in \mathrm{End}_0(E) \otimes K_C$
- $$\begin{array}{rccc} h: & \mathbb{M} & \to & \mathbb{A} := H^0(K_C^2) \times \cdots \times H^0(K_C^n) \\ & (E,\Phi) & \mapsto & \det(x-\Phi) \end{array}$$ Hitchin map
- $\mathbb{C}^\times \circlearrowright \mathbb{M}$ by $(E, \Phi) \mapsto (E, \lambda\Phi)$; *semiprojective*:
  1. $\mathbb{M}^{\mathbb{C}^\times}$ projective
  2. $\lim_{\lambda \to 0} \lambda\mathcal{E}$ exists for every $\mathcal{E} \in \mathbb{M}$
- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times} \rightsquigarrow W_\mathcal{E}^+ := \{\mathcal{F} \in \mathbb{M} | \lim_{\lambda \to 0} \lambda\mathcal{F} = \mathcal{E}\}$ *upward flow*
- (Bialynicki-Birula 1973): $W_\mathcal{E}^+ \subset \mathbb{M}$ locally closed $\cong T_\mathcal{E}^+\mathbb{M}$
- $\lambda^*(\omega) = \lambda\omega \rightsquigarrow W_\mathcal{E}^+ \subset (\mathbb{M}, \omega)$ is Lagrangian
- $\mathbb{M} = \coprod_{\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}} W_\mathcal{E}^+$
- $\mathcal{E} \in \mathbb{M}^{\mathbb{C}^\times}$ *very stable* $\Leftrightarrow W_\mathcal{E}^+$ closed $\Leftrightarrow h|_{W_\mathcal{E}^+} : W_\mathcal{E}^+ \to \mathbb{A}$ proper
- Motivating Problem: find coordinates s.t.

$$h_\mathcal{E} := h|_{W_\mathcal{E}^+} : W_\mathcal{E}^+ \to \mathbb{A}$$

becomes explicit!

Examples of very stable Higgs bundles

- $E_0 := \mathcal{O}_C \oplus K_C^{-1} \cdots \oplus K_C^{1-n}$
- $a = (a_2, \ldots, a_n) \in \mathbb{A} = H^0(K_C^2) \times \cdots \times H^0(K_C^n)$
- $\Phi_a := \begin{pmatrix} 0 & 0 \ldots & 0 & a_n \\ 1 & 0 \ldots & 0 & a_{n-1} \\ \vdots & & \vdots & \\ 0 & \ldots 1 & 0 & a_2 \\ 0 & \ldots & 1 & 0 \end{pmatrix} : E_0 \to E_0 K_C$ companion matrix
- $\mathcal{E}_0 := (E_0, \Phi_0) \in \mathbb{M}^{\mathbb{C}^\times}$ *canonical uniformising Higgs bundle*
- upward flow $W_0^+ = \{(E_0, \Phi_a)\}_a$ *Hitchin section* $\Rightarrow$ very stable
- $c \in C$, $E_k := \mathcal{O}_C \oplus K_C^{-1} \ldots \oplus K_C^{-k}(c) \oplus \ldots K_c^{1-n}(c)$,

  $$s_c \in H^0(\mathcal{O}_C(c)),\ \Phi_k := \begin{array}{c} \quad k \\ \begin{pmatrix} 0 & \ldots 0 & \ldots 0 \\ 1 & \ldots 0 & \ldots 0 \\ \vdots & & \vdots \\ 0 & \ldots s_c & \ldots 0 \\ \vdots & & \vdots \\ 0 & \ldots & 1\ 0 \end{pmatrix} \end{array} : E_k \to E_k K_C$$

Theorem (Hausel–Hitchin 2022)

$\mathcal{E}_k := (E_k, \Phi_k)$ *is very stable.*

- proof by noticing $W_k^+ := W_{\mathcal{E}_k}^+ = \mathcal{H}_c^{\omega_k}(W_0^+)$
  ω_k kth fundamental character of SL_n, minuscule

Hitchin map as spectrum of equivariant cohomology

- G complex reductive; $EG \to BG$ universal principal G-bundle; $H^*_{\mathrm{G}} := H^*(BG;\mathbb{C}) \cong \mathbb{C}[\mathfrak{g}]^{\mathrm{G}} \cong \mathbb{C}[\mathfrak{t}]^{\mathrm{W}}$
- $\mathrm{G} \circlearrowright X$ variety; $X_{\mathrm{G}} := X \times EG/\mathrm{G}$ *Borel quotient*; $H^*_{\mathrm{G}}(X;\mathbb{C}) := H^*(X_{\mathrm{G}};\mathbb{C})$ *equivariant cohomology* H^*_{G}-algebra
- *equivariantly formal* $\Leftrightarrow H^*_{\mathrm{G}}(X;\mathbb{C})_0 \cong H^*(X;\mathbb{C}) \Leftrightarrow H^*_{\mathrm{G}}(X)$ is free over $H^*_{\mathrm{G}} \Rightarrow \mathrm{Spec}(H^{2*}_{\mathrm{G}}(X;\mathbb{C})) \to \mathrm{Spec}(H^{2*}_{\mathrm{G}}) \cong \mathfrak{t}/\!\!/\mathrm{W}$ is proper

Theorem (Hausel, 2022)

The Hitchin map on the minuscule upward flow W^+_k is modelled on spectrum of PGL_n-equivariant cohomology of the Grassmannian

$$\begin{array}{ccc} W^+_k & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n}),\mathbb{C})) \\ h_k \downarrow & \lrcorner & \downarrow \\ \mathbb{A} & \twoheadrightarrow & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}) \end{array}$$

- proof by fixed point scheme in $\mathrm{Gr}(\mathrm{k},\mathrm{n}) \cong \mathrm{Gr}^{\omega_{\mathrm{k}}}$ *affine Schubert*
- $\rightsquigarrow$ *multiplicity algebra* $\mathbb{C}[h_k^{-1}(0)] \cong H^{2*}(\mathrm{Gr}(\mathrm{k},\mathrm{n});\mathbb{C})$ (Hausel–Hitchin 2021)

Universal bundle of Kirillov algebras

- $X^*_+(\mathrm{G}) \ni \mu$-highest weight rep. $\rho^\mu : \mathrm{G} \to \mathrm{GL}(V^\mu)$
- *Kirillov algebra*:
  $C^\mu := (S(\mathfrak{g}) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \cong \mathit{Maps}(\mathfrak{g}, \mathrm{End}(V^\mu))^{\mathrm{G}}$
  associative $S(\mathfrak{g}^*)^{\mathrm{G}} \cong H^*_{\mathrm{G}}$-algebra
- (Kirillov 2000) C^μ commutative $\Leftrightarrow \rho^\mu$ weight multiplicity free; e.g. minuscule

Theorem (Hausel 2022)

For $\mathrm{G} \cong \mathrm{SL}_n$, ω_k fundamental $\exists$ universal bundle of algebra structure on $\rho^{\omega_k}(\mathbb{E})_c \cong \Lambda^k(\mathbb{E})_c$ along $W_0^+ \cong \mathbb{A}$ modelled on C^{ω_k}

$$\begin{array}{ccccccc} C^{\omega_k} & \hookrightarrow & \mathrm{End}(\Lambda^k(\mathbb{E})_c) & & \mathrm{Spec}(C^{\omega_k}) & \twoheadleftarrow & \mathrm{Spec}_{\mathbb{A}}(\Lambda^k(\mathbb{E})_c) \\ \uparrow & \llcorner & \uparrow & \rightsquigarrow & \downarrow & \llcorner & \downarrow \\ H^{2*}_{\mathrm{SL}_n} & \hookrightarrow & \mathbb{C}[\mathbb{A}] & & \mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) & \twoheadleftarrow & \mathbb{A} \end{array}$$

- construction by applying Kirillov M-operators to Φ_a and using cyclicity of C^{ω_k} (Panyushev 2004)
- $k = 1$ familiar bundle of algebra structure from BNR corr.

Mirror of equivariant cohomology is Kirillov algebra

- (Kapustin–Witten 2007) $\rightsquigarrow \mathcal{S}(\rho^\mu(\mathbb{E}^\vee)_c) = \mathcal{H}_c^\mu(\mathcal{O}_{W_0^+})$
- when $\mathrm{G} = \mathrm{PGL}_n$ and $\mu = \omega_k \in X_+^*(\mathrm{SL}_n)$ fundamental$\rightsquigarrow$ $\mathcal{H}_c^{\omega_k}(\mathcal{O}_{W_0^+}) = \mathcal{O}_{W_k^+}$ sheaf of algebras $\rightsquigarrow$ its mirror $\Lambda^k(\mathbb{E}^\vee)_c$ should aquire a bundle of algebra structure along dual Hitchin section from fiberwise Fourier-Mukai transform

Theorem (Hausel 2022)

$$
\begin{tikzcd}
\mathrm{Spec}(C^{\omega_k}) \arrow[d] \arrow[rrrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\Lambda^k(\mathbb{E}^\vee)_c) \arrow[l, two heads] \arrow[d] \arrow[r, phantom, "\cong"] & W_k^+ \arrow[d] \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n}),\mathbb{C})) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, two heads] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, two heads] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}
$$

- using (Panyushev 2004)'s $C^{\omega_k} \cong H^{2*}_{\mathrm{SL}_n}(\mathrm{Gr}(\mathrm{k},\mathrm{n});\mathbb{C})$
- generalises - partly conjecturally - to all $\mu \in X_*^+(\mathrm{G})$
- $\rightsquigarrow$ *classical limit of geometric Satake*

Construction of the big algebra

- $\{X_i\}$ basis for $\mathfrak{sl}_n$, $\{X^i\}$ dual basis wrt Killing form
- $$\begin{array}{rccc} D: \mathcal{C}^\mu & \to & \mathcal{C}^\mu = (S(\mathfrak{sl}_n^*) \otimes \mathrm{End}(V^\mu))^{\mathrm{G}} \\ A & \mapsto & \sum_i \rho^\mu(X^i) \frac{\partial(A)}{\partial X_i} \end{array}$$ Kirillov's D-operator
- $c_i \in \mathbb{C}[\mathfrak{sl}_n]^{\mathrm{SL}_n} \cong H^*_{\mathrm{SL}_n}$ invariant polynomial
  $t^n + c_2(a)t^{n-2} + \cdots + c_n(a)$ char poly of $a \in \mathfrak{sl}_n$
- $M_{i-1} := D(c_i)$ Kirillov's M-operators *medium operators*
- $\mathcal{M}^\mu := \langle 1_{V^\mu}, M_1, \ldots, M_{n-1} \rangle_{H^*_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *medium algebra* $\cong Z(\mathcal{C}^\mu)$
- $B_{k,i-k} := D^k(c_i) \in \mathcal{C}^\mu$ *big operators* of age k and degree $i-k$
- $\mathcal{B}^\mu := \langle 1_{V^\mu}, \{B_{k,i-k}\}_{k,i} \rangle_{H^*_{\mathrm{SL}_n}} \subset \mathcal{C}^\mu$ *big algebra* $\supset \mathcal{M}^\mu$

Theorem (Hausel–Zveryk, Hausel 2022)

$\mathcal{B}^\mu \subset \mathcal{C}^\mu$ is commutative, cyclic, finite-free $/H^*_{\mathrm{SL}_n}$ and maximal

- proof via a universal big algebra $\mathcal{B} \subset (S(\mathfrak{g}) \otimes U(\mathfrak{g}))^{\mathrm{G}}$ as a Gaudin algebra from (Feigin–Frenkel, 1992)
- cyclicity follows from (Feigin–Frenkel–Rybnikov 2006)
  in quantization of Mishchenko-Fomenko integrable systems
  $\rightsquigarrow \mathcal{B}^\mu$ is some sort of quantum integrable system

Geometric properties of $\mathcal{B}^\mu$

- $\mathrm{Gr} := \mathrm{PGL}_n((z))/\mathrm{PGL}_n[[z]]$ affine Grassmannian of PGL_n
- $\mathrm{Gr}^\mu := \overline{\mathrm{PGL}_n[[z]]z^\mu} \subset \mathrm{Gr}$ affine Schubert; $\mathrm{Gr}^{\omega_k} = \mathrm{Gr}(k,n)$
- (Bezrukavnikov–Finkelberg 2008) describe the graded $H^*_{\mathrm{PGL}_n}$-algebra $H^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ & $IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$ as module over it $\rightsquigarrow$

Corollary (Hausel 2022)

$H^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{M}^\mu$ as $H^*_{\mathrm{PGL}_n}$-algebras
$\mathrm{End}_{H^*_{\mathrm{PGL}}(\mathrm{Gr}^\mu)}(IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \cong \mathcal{C}^\mu$
$IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu) \cong \mathcal{B}^\mu$ as $\mathcal{M}^\mu$-modules
$\rightsquigarrow$ (conj. unique) graded $H^*_{\mathrm{PGL}_n}$-algebra structure on $IH^*_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)$

Conjecture (Hausel 2022)

$$
\begin{tikzcd}
\mathrm{Spec}(\mathcal{B}^\mu) \arrow[d] \arrow[rrr, bend left, "\cong"] & \mathrm{Spec}_{\mathbb{A}^\vee}(\rho^\mu(\mathbb{E}_c^\vee)) \arrow[l, twoheadrightarrow] \arrow[d] \arrow[dl, phantom, "\llcorner"] \arrow[r, phantom, "\cong"] & \mathcal{H}_c^\mu(W_0^+) \arrow[r, twoheadrightarrow] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Spec}(IH^{2*}_{\mathrm{PGL}_n}(\mathrm{Gr}^\mu)) \arrow[d] \\
\mathrm{Spec}(H^{2*}_{\mathrm{SL}_n}) \arrow[rrr, bend right, "\cong"'] & \mathbb{A}^\vee \arrow[l, twoheadrightarrow] \arrow[r, phantom, "\cong"] & \mathbb{A} \arrow[r, twoheadrightarrow] & \mathrm{Spec}(H^{2*}_{\mathrm{PGL}_n})
\end{tikzcd}
$$

Further issues / problems

- $\mathrm{Spec}(\mathcal{M}^\mu) = \bigcup_{X_*^+ \ni \lambda \le \mu} \mathrm{Spec}(\mathcal{M}^\mu_\lambda)$ irreducible components.
  $\mathcal{M}^\mu_\lambda \cong H^*_{\mathrm{G}}(\mathrm{G}/P_\lambda, \mathbb{C}) \cong H^*_{\mathrm{G}}(\mathrm{Gr}^\lambda_\mu, \mathbb{C})$; $\mathrm{Gr}^\mu = \coprod_{\mu \le \lambda} \mathrm{Gr}^\mu_\lambda$ BB-dec.
  $\rightsquigarrow$ description of non-nilpotent directions in upward flows
- The mutliplicity algebra of $\mathcal{M}^\mu_\lambda \subset \mathcal{B}^\mu_\lambda$ endows
  $IH^*(W^\mu_\lambda) \cong \mathcal{B}^\mu_\lambda/(\mathcal{M}^\mu_{\lambda+})$ a graded ring
  ringifying Kazhdan-Lusztig polynomials $IP_t(W^\mu_\lambda)$.
  E.g. $IH^*(W^\mu_0) \cong \mathcal{B}^\mu/(\mathcal{M}^\mu_+)$. Can it be computed explicitly?
- $\theta : \mathrm{G} \to \mathrm{G}$ Cartan involution of real form $\rightsquigarrow \theta : \mathcal{B}^\mu \to \mathcal{B}^\mu$
  ${}^\theta\mathcal{B}^\mu := \mathcal{B}^\mu/(\mathcal{B}^\mu)^\theta_-$ coinvariant algebra
  models Hitchin map on upward flows in $\mathbb{M}^\theta$.
  E.g. for Hodge type $\theta = (-1)^{deg} : \mathcal{B}^\mu \to \mathcal{B}^\mu$
  What is its equivariant multiplicity $P_t(({}^\theta\mathcal{B}^\mu)_0)$?
  $P_1(({}^\theta\mathcal{B}^\mu)_0)$ should agree with the signature of the Poincaré duality pairing on $IH^*(\mathrm{Gr}^\lambda)$ (Lusztig–Yun 2013)
  general $\theta \rightsquigarrow P_1(({}^\theta\mathcal{B}^\mu)_0) \overset{conj}{=} |Sig(h_\lambda)|$
  when h_λ is a $\mathrm{G}(\mathbb{R})$-invariant Hermitian form on V_λ