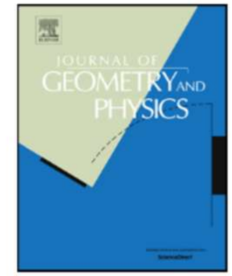
Where Quantum Topology
Meets
Geometric Representation Theory

2007-2008: IAS school of Mathematics

Journal of Geometry and Physics 168 (2021) 104311

Rozansky-Witten geometry of Coulomb branches and logarithmic knot invariants

Sergei Gukov, Po-Shen Hsin, Hiraku Nakajima, Sunghyuk Park, Du Pei et al.



J. Phys. A: Math. Theor. **53** (2020)

3d TQFTs from Argyres–Douglas theories

Mykola Dedushenko[1], Sergei Gukov[1], Hiraku Nakajima[2], Du Pei[1,3,4,*] and Ke Ye[1]

6d perspective: $M_3 \times S^1 \times \Sigma$

Rozansky-Witten theory with target $X = \mathcal{M}_H(G_c, \Sigma)$

Higgs bundles

K-theoretic ("4d") Coulomb branches

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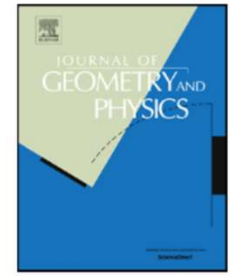

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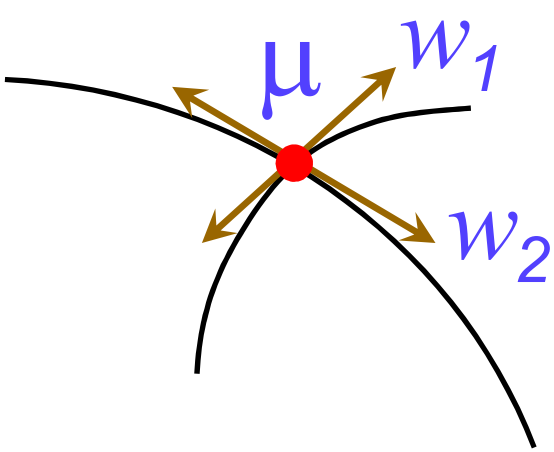

$$T_{\lambda\lambda} = t^{\mu(\lambda)}$$

$$(S_{0\lambda})^2 = \frac{K_X^{1/2}}{\text{K-theory Euler class}(T_\lambda X)}$$

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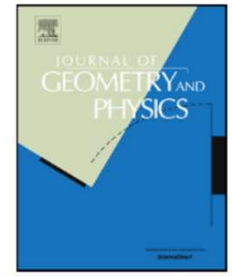

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6d perspective: $M_3 \times S^1 \times \Sigma$

Rozansky-Witten theory with target $X = \mathcal{M}_H(G_c, \Sigma)$

wild ramification

$\Sigma =$ 

Fibonacci MTC

$$S = \frac{2}{\sqrt{5}} \begin{pmatrix} \sin\frac{\pi}{5} & \sin\frac{2\pi}{5} \\ \sin\frac{2\pi}{5} & -\sin\frac{\pi}{5} \end{pmatrix}$$

$$T = \begin{pmatrix} e^{-\frac{\pi i}{15}} & 0 \\ 0 & e^{\frac{11\pi i}{15}} \end{pmatrix}$$

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Rozansky-Witten geometry of Coulomb branches and logarithmic knot invariants

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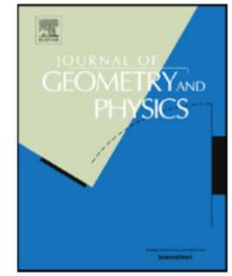

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3d TQFTs from Argyres–Douglas theories

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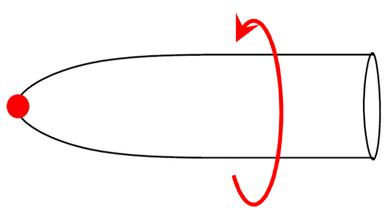
6d perspective: $M_3 \times S^1 \times \Sigma$

Rozansky-Witten theory with target $X = \mathcal{M}_H(G_c, \Sigma)$

$\Sigma = D^2$



- $U_q(\mathfrak{g})$ at generic q
- spherical DAHA
- $D^b\mathrm{Coh}(\mathcal{T})$

space of BFN triples (cf. $T^*\mathrm{Gr}_G$)

Quantum topology

$U_q(\mathfrak{g})$

Geometric representation theory

gauge theory,
ramification, quasimaps, ...

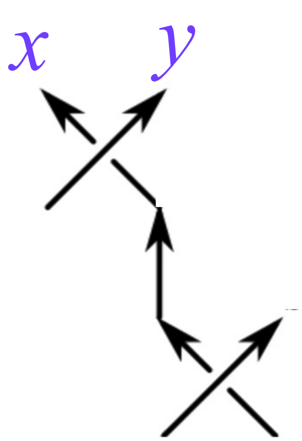
Coulomb branches,
affine Grassmannians

Theorem: Using the R-matrix for Verma modules, for all links of unknots (plumbings), torus links, positive braid links, fibered knots up to 10 Xs, and homogeneous braid links the two-variable series

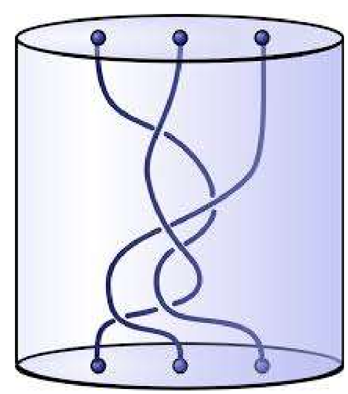
$$F_K(x,q) := \sum (R \cdots R)$$

is well defined and invariant under the required braid moves (cf. Reidemeister moves).

S.G., D.Pei, P.Putrov, C.Vafa
S.G., C.Manolescu
S.Park (2020, 2021)
J.Chae
A.Gruen
:



complex weights



Surgery formula:

$\mathrm{rank}(G) = 1$

$$\widehat{Z}_b(S^3_{-p/r}(K)) = \oint\limits_{|x|=1} \frac{dx}{2\pi i x}(x^{\frac{1}{r}} - x^{-\frac{1}{r}}) F_K(x,q) \sum_n q^{\frac{r}{p}n^2} x^n$$

Theorem [Lickorish, Wallace]:

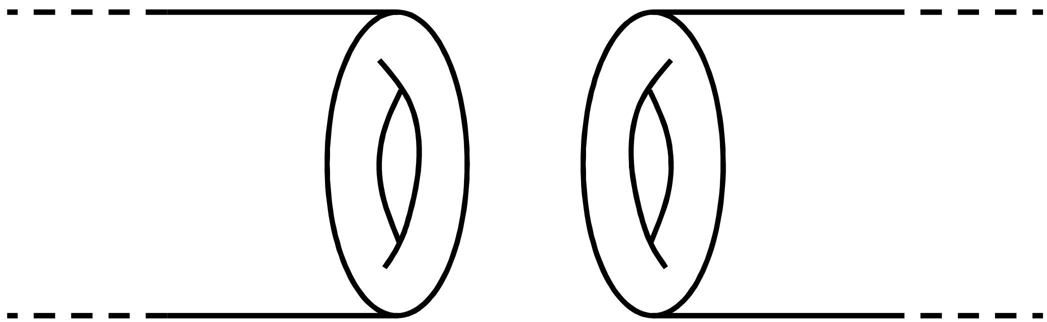
Every connected oriented closed 3-manifold arises by performing an integral Dehn surgery along a link in S^3.

$S^3_{-1}($  $) = S^3_{+1}($  $)$

Surgery formula:

$\mathrm{rank}(G) = 1$

$$\widehat{Z}_b(S^3_{-p/r}(K)) = \oint_{|x|=1} \frac{dx}{2\pi i x}(x^{\frac{1}{r}} - x^{-\frac{1}{r}}) F_K(x,q) \sum_n q^{\frac{r}{p}n^2} x^n$$



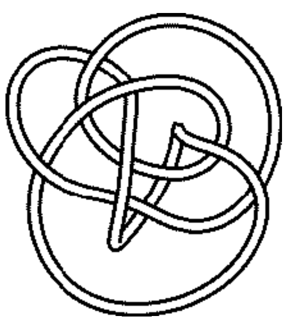
$$\mathcal{H}(T^2) = \mathbb{C}\left[\mathbb{T} \times \mathbb{T}^\vee\right]^W \quad \text{or} \quad \mathbb{C}\left[\frac{\Lambda \times \Lambda^\vee}{W}\right]$$

cf. $\mathcal{H}(T^2) = \mathbb{C}\left[\frac{\Lambda}{W \times k\Lambda^\vee}\right]$ for WRT invariants

$M_3 = S^3_{-1/2}($  $)$:

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 2017999742838833201212q^{500} + \ldots)$$

$M_3 = -S^3_{+5}(\mathbf{10_{145}})$:



$$\begin{aligned} b = 2: &\quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right) \\ b = 1: &\quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right) \\ b = 0: &\quad 2q^4 + 2q^5 + 4q^6 + 8q^7 + 14q^8 + \ldots \\ b = -1: &\quad q^{11/5}\left(-1 - q - 4q^2 - 7q^3 + \ldots\right) \\ b = -2: &\quad q^{14/5}\left(-1 + q + 2q^2 + 4q^3 + \ldots\right) \end{aligned}$$

<u>Conjecture:</u>

"conformal weight"

$$\chi_b(q) = \widehat{Z}_b(q) = q^{\Delta_b} \sum_n a_n q^n$$

Character of a logarithmic Vertex Algebra

M.Cheng, S.Chun, F.Ferrari, S.G., S.Harrison
M.Cheng, S.Chun, B.Feigin, F.Ferrari, S.G., S.Harrison, D.Passaro
S.Sugimoto
⋮

* cf. sheaf counting on complex surfaces

Vol. 76, No. 2 DUKE MATHEMATICAL JOURNAL © November 1994

INSTANTONS ON ALE SPACES, QUIVER VARIETIES, AND KAC-MOODY ALGEBRAS

HIRAKU NAKAJIMA

To Professor Shoshichi Kobayashi on his 60th birthday

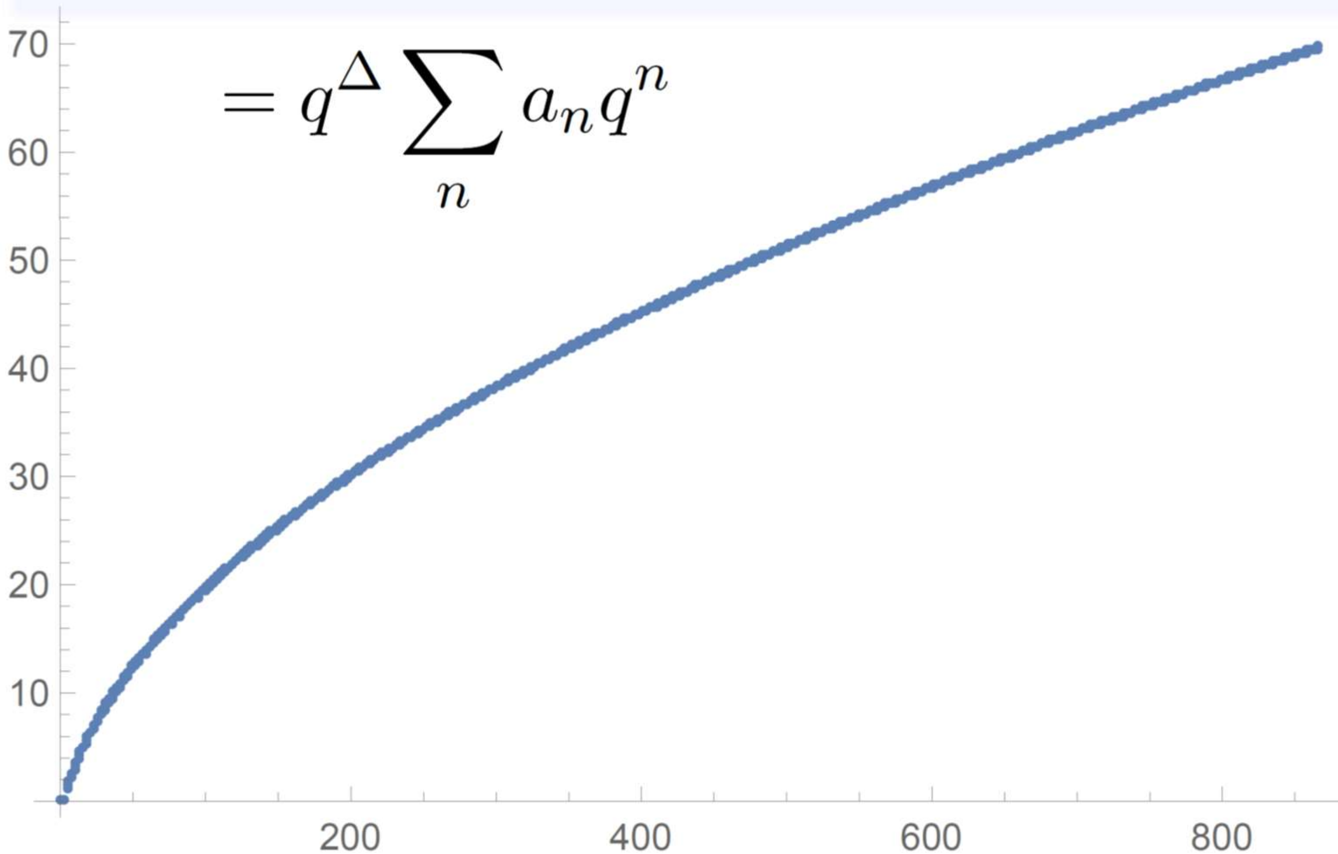
1. Introduction. In this paper we shall introduce a new family of varieties, which we call *quiver varieties*, and study their geometric structures. They have close relation to the singularity theory and the representation theory of the Kac-

Corollary (surprise #1):

$$a_n \sim \exp\, 2\pi \sqrt{\frac{1}{6} c_{\text{eff}}\, n}$$

$$\widehat{Z}(q) = q^{-\frac{1}{2}}(1 + q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + 12q^7 + \ldots$$
$$\ldots + 20179997428388332001212 q^{500} + \ldots)$$

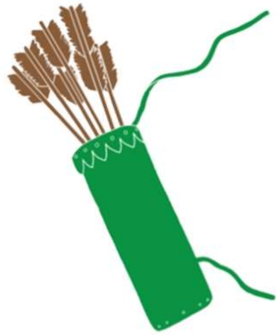
$$= q^{\Delta} \sum_n a_n q^n$$



John Cardy

Surprise #2:

$$C = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$$

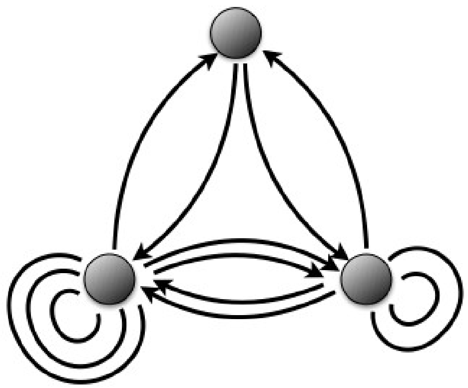


$$\widehat{Z}_b(M_3, q) = \sum_{d_i \geq 0} \frac{1}{(q)_{\boldsymbol{d}}} q^{\frac{1}{2}\boldsymbol{d}\cdot C\cdot\boldsymbol{d}+(\text{terms linear in } \boldsymbol{d})}$$

BPS quiver

DT theory, curve counting, Hall algebras, knot-quiver correspondence, ...

BPS spectrum



P.Kucharski, M.Reineke, M.Stosic, P.Sulkowski
T.Ekholm, A.Gruen, S.G., P.Kucharski, S.Park, M.Stosic, P.Sulkowski

Advanced Studies in Pure Mathematics 61, 2011
New Structures and Natural Constructions in Mathematical Physics
pp. 161–184

Fermionic formulas for characters of $(1, p)$ logarithmic model in $\Phi_{2,1}$ quasiparticle realisation

Boris Feigin, Evgeny Feigin and Il'ya Tipunin

The main result of the paper is formulated as follows.

Theorem 1.1. *The characters* (1.7) *can be written in the form*

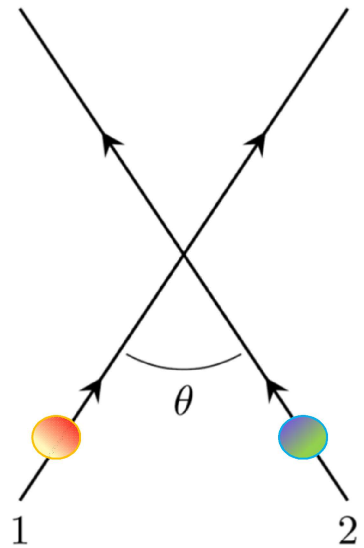
(1.8)
$$\chi_{s,p}(q) = q^{\frac{s^2-1}{4p}+\frac{1-s}{2}-\frac{c}{24}} \sum_{n_+,n_-,n_1,\ldots,n_{p-1}\geq 0} \frac{q^{\frac{1}{2}\mathbf{n}\mathcal{A}\cdot\mathbf{n}+\mathbf{v}_s\cdot\mathbf{n}}}{(q)_{n_+}(q)_{n_-}(q)_{n_1}\cdots(q)_{n_{p-1}}}$$

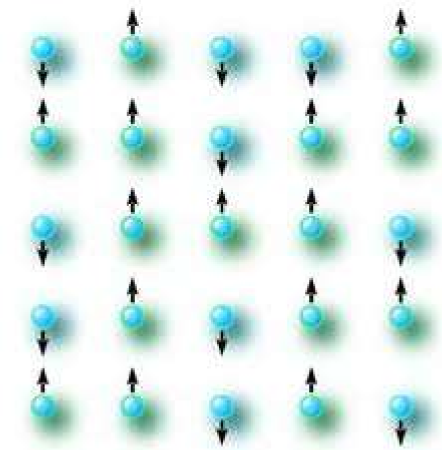
$U_q(\mathfrak{g})$

Quantum groups ⟷ Kazhdan-Lusztig ⟷ Vertex Algebras

Integrable lattice models

2d CFT





H.Bethe (1931)
:
A.Zamolodchikov, Al.Zamolodchikov (1979)
A.Zamolodchikov (1989)
Al.Zamolodchikov (1990)
F.Smirnov (1990)
N.Reshetikhin, F.Smirnov (1990)

➡ Yangian symmetry,
Bethe ansatz equation, ...

Fermionic Sum Representations for Conformal Field Theory Characters

R. Kedem,[1] T.R. Klassen,[2] B.M. McCoy,[1] and E. Melzer[1]

arXiv:hep-th/9301046

Recently it was found [1] that characters (or branching functions) of the coset conformal field theories $\frac{(G^{(1)})_1 \times (G^{(1)})_1}{(G^{(1)})_2}$, G a simply-laced Lie algebra, can be represented in the form

$$\sum_{\mathbf{m}}^{Q} \frac{q^{\frac{1}{2}\mathbf{m}B\mathbf{m}^t}}{(q)_{m_1}\cdots(q)_{m_r}} \quad , \tag{1.1}$$

Rogers-Ramanujan

S.Kerov, A.Kirillov, N.Reshetikhin (1986)

A.Kirillov, N.Reshetikhin (1988)

R.Kedem, B.McCoy (1993)

R.Kedem, T.Klassen, B.McCoy, E.Melzer (1993)

:

S.Dasmahapatra, R.Kedem, T.Klassen, B.McCoy, E.Melzer (1993)

R.Kedem, B.McCoy, E.Melzer (1993)

A.Berkovich, B.McCoy, A.Schilling, S.Warnaar (1997)

E. Frenkel, A. Szenes (1993)

:

W.Nahm, A.Recknagel, M.Terhoeven (1993)

B.Feigin, E.Feigin, M.Jimbo, T.Miwa, E.Mukhin (2009)

M.Finkelberg (ICM - 2018): Zastava characters, relation to monopole formulae?

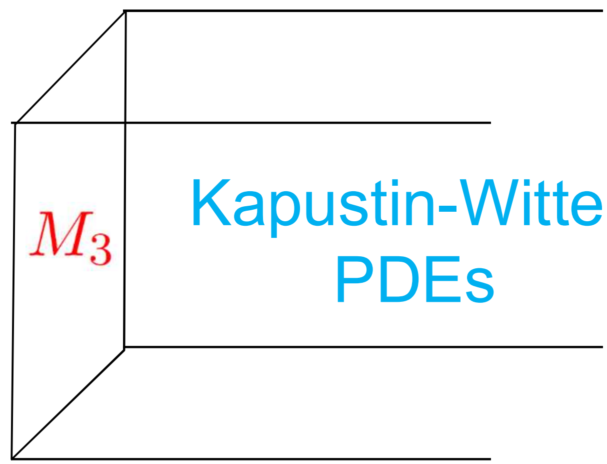
$$M_3 \times S^1 \times_q D^2$$

<u>Conjecture:</u> $\widehat{Z}_b(M_3;q)$ should admit a definition via moduli spaces in gauge theory *

S.G., D.Pei, P.Putrov, C.Vafa

$$\mathcal{B}_{\text{Nahm}}$$

$$\frac{\sum_{i=1}^{3} e^i_\mu t_i}{y} + \ldots$$



$$\mathcal{B}_b$$
$$\cap$$
$$\text{Spin}^c(M_3)$$

---

* Compactification and choice of chamber: K-theory / multiplicative / caloron version

Early clues: S.Chun, S.G., S.Park, N.Sopenko

More detailed analysis: S.G., P.-S.Hsin, D.Pei (to appear soon)

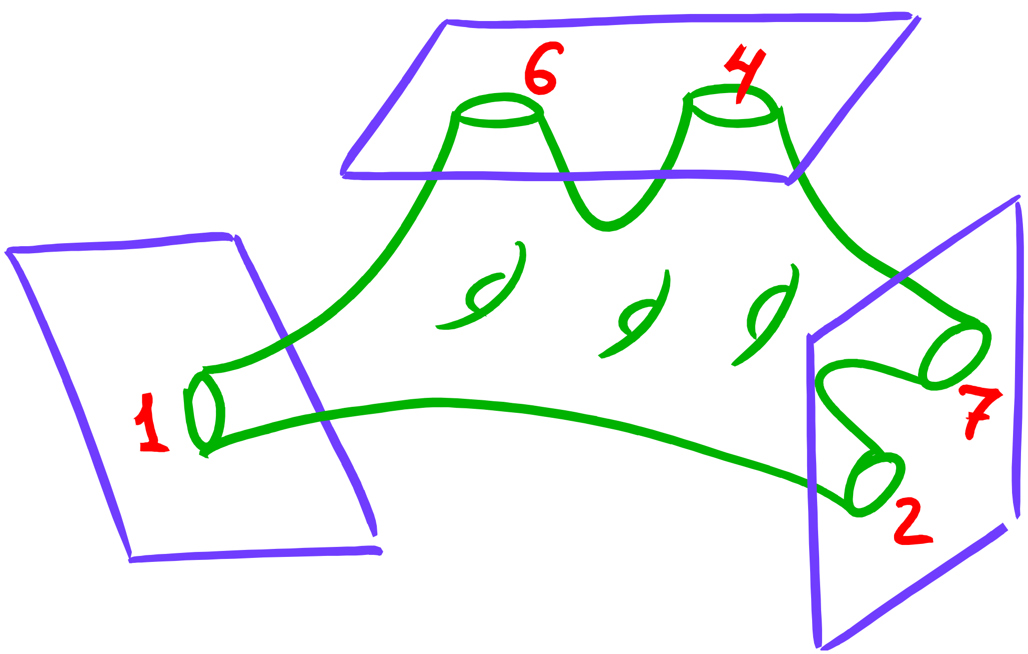
$$M_3 \times S^1 \times_q D^2$$

Conjecture: $\widehat{Z}_b(M_3;q)$ should admit a definition via moduli spaces in curve counting

S.G., D.Pei, P.Putrov, C.Vafa

$$\phi : (\Sigma, \partial\Sigma) \longrightarrow (X, L)$$

Σ genus g, with n boundary components



$$\partial\Sigma = \gamma_1 \sqcup \ldots \sqcup \gamma_n$$

$$\beta = \phi_*[\Sigma] \in H_2(X, L)$$

$$b_i = \phi_*[\gamma_i] \in H_1(L)$$

Some evidence: L.Diogo, T.Ekholm

:

$$\underbrace{S^1 \times D^2}_{\text{K-theory quasimaps}} \times \underbrace{M_3}_{\text{Rozansky-Witten theory}}$$

3d-3d

	K-theory quasimaps	Rozansky-Witten theory
Target:	$\mathcal{M}_{\text{flat}}(G, M_3)$ (complex G)	$\mathcal{T}$ for $\mathbf{N} = \mathfrak{g}$
Domain:	D^2	M_3
	3d $\mathcal{N}=2$	3d $\mathcal{N}=4$

$$\underline{M_3 = \mathbb{R} \times \Sigma \ :} \qquad \mathcal{M}_{\text{flat}}(G, \Sigma) \cong \mathcal{M}_H(G_c, \Sigma)$$

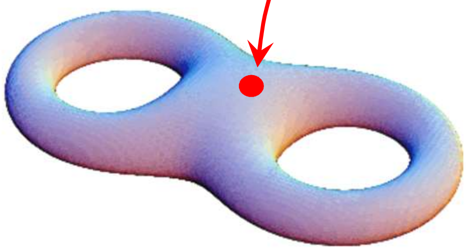
Complex coadjoint orbit

$$T^*\text{Fl} \to \mathcal{M}_H(G_c, \Sigma; \text{ramif.})$$

$$\downarrow$$

$$\mathcal{M}_H(G_c, \Sigma; \text{unramif.})$$

ramification



P.Kronheimer
C. Simpson
H. Konno
H. Nakajima
:
S.G., E.Witten

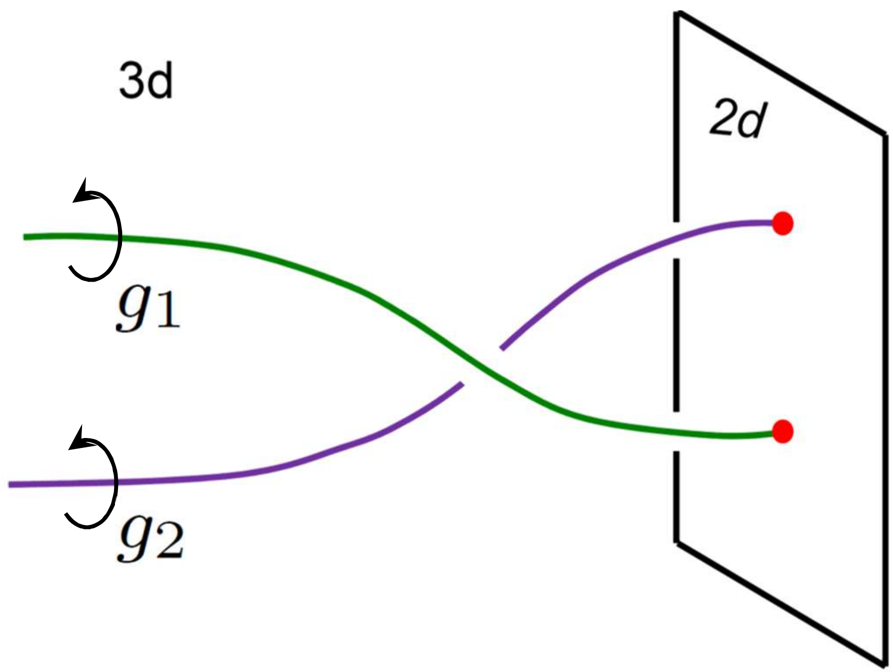
$$F_A + [\Phi, \Phi^*] = 0$$
$$\overline{D}\Phi = 0$$

$$\Rightarrow \begin{cases} \dfrac{Da}{Ds} = [b, c] \\ \dfrac{Db}{Ds} = [c, a] \\ \dfrac{Dc}{Ds} = [a, b] \end{cases}$$

Equivariant cohomology of based quasimaps to GL_n flags produces a universal Verma module for $U(\mathfrak{gl}_n)$

A.Braverman, M.Finkelberg
B.Feigin, M.Finkelberg, I.Frenkel, L.Rybnikov
M.Bullimore, T.Dimofte, D.Gaiotto, J.Hilburn, H.Kim
J.Hilburn, J.Kamnitzer, A.Weekes
Z.Zhou
:
G.Naisse, P.Vaz
R.Rouquier

complex weights



q ⟷ circle action on the domain

highest weight ⟷ $\mathbb{T}$ action on the target

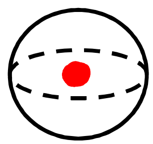
$$S^1 \times D^2 \times M_3$$

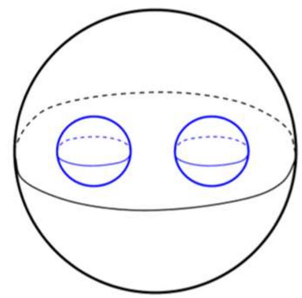
3d-3d

	K-theory quasimaps	Rozansky-Witten theory
Target:	$\mathcal{M}_{\text{flat}}(G, M_3)$ (complex)	$\mathcal{T}$ for $\mathbf{N} = \mathfrak{g}$
Domain:	D^2	M_3
	3d $\mathcal{N} = 2$	3d $\mathcal{N} = 4$

3d TQFT

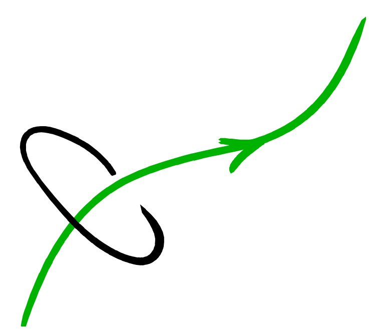
$$\mathcal{H}_{\mathrm{TQFT}}(S^2) = \text{ commutative ring } = \mathbb{C}[\mathcal{M}]$$





A raviolo would be better

$$\mathcal{H}_{\mathrm{TQFT}}(T^2) = \text{ line operators } = K^0(\mathcal{C})$$



3d TQFT

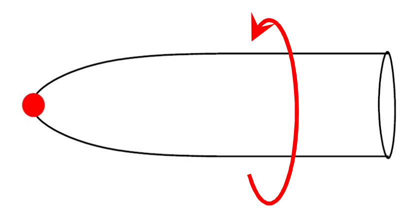
associated to $U_q(\mathfrak{g})$ at generic q

$$\widehat{Z}(S^1 \times S^2) = \mathrm{Ch}_{\mathbb{C}_q^* \times \mathbb{T} \times \mathbb{C}_t^*} \, \mathbb{C}[\mathcal{M}]$$

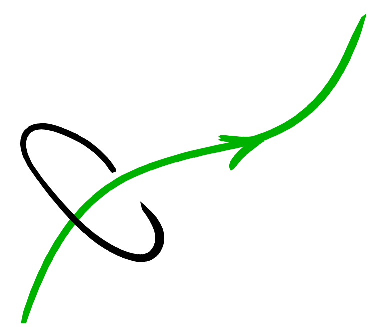
$$= \frac{(-tx;q)_\infty (-tq;q)_\infty (-tx^{-1};q)_\infty}{(qx;q)_\infty (q;q)_\infty (qx^{-1};q)_\infty}$$

$$= \text{ Hilbert series of } \mathcal{M}_H(G_c, \Sigma) \text{ for } \Sigma = D^2$$

A raviolo would be better



$$\mathcal{H}(T^2) = \mathbb{C}\left[\mathbb{T} \times \mathbb{T}^\vee\right]^W$$



3d TQFT

associated to $U_q(\mathfrak{g})$ at generic q

$$\widehat{Z}(S^1 \times S^2) = \mathrm{Ch}_{\mathbb{C}_q^* \times \mathbb{T} \times \mathbb{C}_t^*} \mathbb{C}[\mathcal{M}]$$

$$= \frac{(-tx;q)_\infty (-tq;q)_\infty (-tx^{-1};q)_\infty}{(qx;q)_\infty (q;q)_\infty (qx^{-1};q)_\infty}$$

$$= \text{ Hilbert series of } \mathcal{M}_H(G_c, \Sigma) \text{ for } \Sigma = D^2$$

x $\quad \mathbb{T}$ action (tri-holomorphic)

t $\quad$ "stretching fibers" (holomorphic) $\circlearrowleft \mathcal{M}$

q $\quad$ "loop rotation"

Expect: $\mathcal{M} =$ "$T^*\mathrm{Gr}_G$" $= \mathcal{T}$ for $\mathbf{N} = \mathfrak{g}$

infinite-rank vector bundle (smooth) over

$\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$ affine Grassmannian

$$G_{\mathcal{K}} = G((z)) \qquad G_{\mathcal{O}} = G[[z]]$$

$D = \mathrm{Spec}\,\mathbb{C}[[z]]$ formal disk

$D^\times = \mathrm{Spec}\,\mathbb{C}((z))$ formal punctured disk

x $\mathbb{T}$ action (tri-holomorphic)

t "stretching fibers" (holomorphic) $\circlearrowleft \mathcal{M}$

q "loop rotation"

Expect: $\mathcal{M} =$ "$T^*\mathrm{Gr}_G$" $= \mathcal{T}$ for $\mathbf{N} = \mathfrak{g}$

infinite-rank vector bundle (smooth) over

$\mathrm{Gr}_G = G_{\mathcal{K}}/G_{\mathcal{O}}$ affine Grassmannian

S.G., P.-S.Hsin, H.Nakajima, S.Park, D.Pei, N.Sopenko

$\widehat{Z}$ = Rozansky-Witten TQFT with target $T^*\mathrm{Gr}_G$

"Rozansky-Witten invariants via formal geometry"

"Rozansky-Witten invariants via Atiyah classes"

Examples: Lens spaces

$$\widehat{Z}(S^3;q,t) = \prod_{i=1}^{\operatorname{rank} G} \frac{1}{(q^{d_i}t^{d_i};q)_\infty}$$

Annals of Mathematics, **168** (2008), 175–220

The strong Macdonald conjecture and Hodge theory on the loop Grassmannian

By Susanna Fishel, Ian Grojnowski, and Constantin Teleman

11.12 Theorem. *For simply laced G, the vacuum vector $\omega \in \mathbf{H}_0$ gives an isomorphism*

$$(*) \qquad \omega\otimes : \{\mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]} \xrightarrow{\sim} \{\mathbf{H}_0 \otimes \mathrm{S}^p(\mathfrak{g}[[z]]dz)^*\}^{\mathfrak{g}[z]}.$$

Consequently, with $q = z^{-1}$,

$$\sum_{p\geq 0} t^p \dim_q \mathrm{Gr}_p \mathbf{H}_0^G = \prod_{k=1}^{\ell} \prod_{n>m_k} (1 - t^{m_k+1}q^n)^{-1}.$$

The raviolo subvariety $\mathcal{R}$ plays the role of the "affine Grassmannian Steinberg variety"

<u>Theorem:</u> $K^{G_{\mathcal{O}} \times \mathbb{C}^*}(\mathcal{R})$

admits an associative convolution product,
commutative at $q = 1$

A.Braverman, M.Finkelberg, H.Nakajima

<u>Theorem:</u> with the convolution product,

$$K^{G \times \mathbb{C}^*}(\mathrm{St}) \cong \mathcal{H}_{\mathrm{aff}}$$

N.Chriss, V.Ginzburg
G.Lusztig

cf. $K^{G}(\mathrm{St}) \cong \mathbb{Z}[W_{\mathrm{aff}}]$

$$K^{G_{\mathcal{O}}}(\mathrm{St}_G) \cong \mathbb{C}\left[\mathbb{T} \times \mathbb{T}^\vee\right]^W$$

R.Bezrukavnikov, M.Finkelberg, I.Mirković

quantized K-theoretic Coulomb branches:

$K^{G(\mathcal{O})\times\mathbb{C}^*}(\mathrm{Gr}_G)$ ⇒ spherical nil-DAHA

M.Finkelberg, A.Tsymbaliuk

for $\mathbf{N} = \mathfrak{g}$ $\quad K^{G_{\mathcal{O}}\times\mathbb{C}^*}(\mathcal{R}) \cong sDAHA$

M.Varagnolo, E.Vasserot

A.Braverman, M.Finkelberg, H.Nakajima

⋮

S.G., P.Koroteev, S.Nawata, D.Pei, I.Saberi

cf. geometric Satake correspondence

$$\mathcal{H}(T^2) = \mathbb{C}\left[\mathbb{T} \times \mathbb{T}^\vee\right]^W$$

Introduction to a provisional mathematical definition of Coulomb branches of 3-dimensional $\mathcal{N} = 4$ gauge theories

Hiraku Nakajima

[VV10], affine flag varieties instead of affine Grassmannian, equivariant K-theory instead of equivariant Borel-Moore homology group were used, but it is basically understood as a special case of the Coulomb branch where $\mathbf{N}$ is the adjoint representation. The algebra constructed there is Cherednik double affine Hecke algebra (DAHA). If we use affine Grassmannian instead of flag, we get the spherical part of DAHA. We get the trigonometric version instead of the elliptic one if we use homology instead of K-theory. Our Coulomb branch for $\mathbf{N} = \mathfrak{g}$ is $\mathfrak{t} \times T^\vee/W$. It is a remarkable example, as the Coulomb branch does not receive quantum corrections.

In [BFM05, BF08], the case $\mathbf{N} = 0$ was considered. The Coulomb branch is the phase space of the Toda lattice for the Langlands dual group of G, or the moduli space of solutions of Nahm's equation on the interval. We omit further

Questions:

- Categorification of spherical DAHA?

- $\mathrm{Auteq}\, D(\mathcal{T}) \cong \pi_1(\{\text{parameters}\}) = ?$

cf. S.G., E.Witten

- Relations between $D(\mathcal{T})$ and $\mathsf{Rep}(S\ddot{H})$ cf. M.Finkelberg

and $U_q(\mathfrak{g})$-mod$^{\cdots}$ at generic q ?

Mirror symmetry, Hikita conjecture, ...

- R-matrix for quantum toroidal algebras A.Negut

:

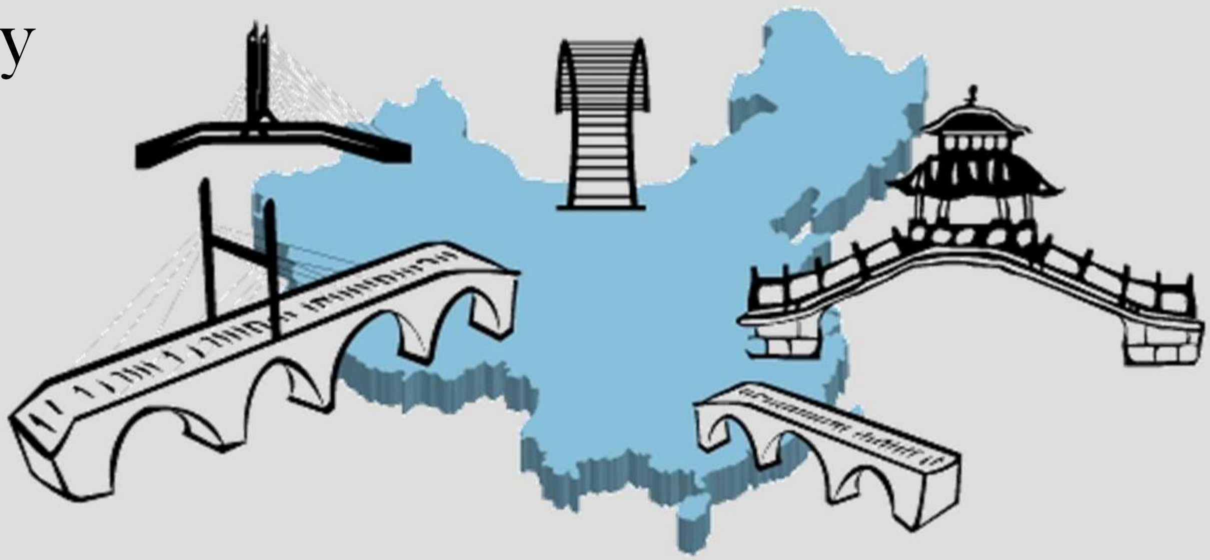
- Perfect crystal bases of Verma modules

S.-J.Kang, M.Kashiwara, K.Misra
S.-J.Kang, M.Kashiwara, K.Misra, T.Miwa, T.Nakashima, A.Nakayashiki

topology

enumerative geometry

mathematical physics



gauge theory

vertex algebra

quantum groups