(Refined) Verlinde and Segre formula for Hilbert schemes of points

Lothar Göttsche
joint work with Anton Mellit

Gauge Theory, Moduli Spaces and Representation Theory, Kyoto 2023
In honor of the 60th birthday of Hiraku Nakajima

22.2.2023

Let S smooth projective surface over $\mathbb{C}$
Hilbert scheme of points:
$S^{[n]} = Hilb^n(S)$={zero dim. subschemes of length n on S}
$S^{[n]}$ is smooth projective, of dimension $2n$
Related to symmetric power $S^{(n)}$
Hilbert-Chow morphism

$$\omega : S^{[n]} \to S^{(n)}, Z \mapsto supp(Z)$$

is a resolution of singularities

Can also view $S^{[n]}$ as a moduli space of sheaves: $S^{[n]}$ is via correspondence $Z \mapsto I_Z$ isomorphic to the moduli space of stable rk 1 sheaves on S with Chern classes $c_1 = 0$, $c_2 = n$

Why care about it?

1. Simple example of moduli spaces of sheaves
   model case for all one wants to study about them
2. Building block of moduli spaces, used to study them
   many questions about moduli spaces of sheaves can be reduced to Hilbert schemes of points (e.g. wallcrossing, instanton counting).
3. Important example of higher dimensional varieties
   e.g. if S is K3 surface, then $S^{[n]}$ is hyperkähler
4. Enumerative applications, counting point configurations, curves and other things

$S^{[n]}$ is a fine moduli space:
Universal subscheme:

$$Z_n(S) = \{(x, [Z]) \mid x \in Z\} \subset S \times S^{[n]}$$

$p : Z_n(S) \to S^{[n]}$, $\quad q : Z_n(S) \to S$ projections
Fibre $p^{-1}([Z]) = Z$

Tautological sheaves:
V vector bundle of rank r on S
$V^{[n]} := p_* q^*(V)$ vector bundle of rank rn on $S^{[n]}$
$V^{[n]}([Z]) = H^0(V|_Z) = V \otimes_{\mathbb{C}} \mathcal{O}_Z$
These tautological bundles are useful for many applications of Hilbert schemes

Determinant bundles: $\det(V^{[n]}) \in \mathrm{Pic}(S^{[n]})$ generate $\mathrm{Pic}(S^{[n]})$

$Z_n(S) = \{(x,[Z]) \mid x \in Z\} \subset S \times S^{[n]}$
$p : Z_n(S) \to S^{[n]}, \quad q : Z_n(S) \to S$ projections

Tautological sheaves: V vector bundle of rank r on S
$V^{[n]} := p_* q^*(V)$ vector bundle of rank rn on $S^{[n]}$
$V^{[n]}([Z]) = H^0(V|_Z)$, in particular $\mathcal{O}_S^{[n]}([Z]) = H^0(\mathcal{O}_Z)$
Extends to Grothendieck group $K^0(S)$ by $(V - W)^{[n]} = V^{[n]} - W^{[n]}$

Line bundles on $S^{[n]}$: $\det(V^{[n]}) \in \mathrm{Pic}(S^{[n]})$, these generate $\mathrm{Pic}(S^{[n]})$
Want formulas for

$$\chi(S^{[n]}, \det(V^{[n]})) \quad \textbf{Verlinde formula}$$

$$\int_{S^{[n]}} c_{2n}(V^{[n]}) = \int_{S^{[n]}} s_{2n}(-V^{[n]}) \quad \textbf{Segre formula}$$

Verlinde formula: Via the correspondence $Z \mapsto \mathcal{I}_Z$ have
$S^{[n]} = M_S^H(1,0,n)$ (moduli sp. of rk 1 stable sheaves E with $\det(E) = 0$, $c_2(E) = n$)
Verlinde formula is rk 1 case of surface analogue of the celebrated Verlinde formula for curves.

Aim: For $V \in K^0(S)$ want formula for generating functions

$$I^{Verlinde}_{S,V}(x) = \sum_{n\geq 0} x^n \chi(S^{[n]}, \det(V^{[n]})), \quad \text{Verlinde formula}$$

$$I^{Chern}_{S,V}(x) = \sum_{n\geq 0} x^n \int_{S^{[n]}} c_{2n}(V^{[n]}), \quad \text{Segre formula}$$

For series of related numbers should study generating functions
They bring hidden relations between the numbers to the surface and can hint at structures behind them, e.g. Betti numbers of Hilbert schemes of points are explained by Nakajima's Heisenberg action.

Recall universal subscheme $Z_n(S) \subset S \times S^{[n]}$
can show: blowup of $S \times S^{[n]}$ along $Z_n(S)$ is

$$S^{[n,n+1]} = \{(Z, W) \in S^{[n]} \times S^{[n+1]} \mid Z \subset W\}$$

With projections to S, $S^{[n]}$, $S^{[n+1]}$. This allows to compute intersection numbers on $S^{[n]}$ recursively
Pull back class $\alpha \in H^*(S^{[n]})$ to $S^{[n-1,n]}$, push forward to $S \times S^{[n-1]}$, pull back to $S \times S^{[n-2,n-1]}$, etc until you arrive at S^n

This gives the following:

$$I^{Verlinde}_{S,V}(x) = \sum_{n\geq 0} x^n \chi(S^{[n]}, \det(V^{[n]})), \quad \text{Verlinde formula}$$

$$I^{Chern}_{S,V}(x) = \sum_{n\geq 0} x^n \int_{S^{[n]}} c_{2n}(V^{[n]}), \quad \text{Segre formula}$$

Universality (Ellingsrud-G-Lehn)

$$I^{Verlinde}_{S,V}(x) = A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3^{c_1(V)K_S - \frac{1}{2}K_S^2} A_4^{K_S^2},$$

$$I^{Chern}_{S,V}(x) = B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3^{c_1(V)K_S - \frac{1}{2}K_S^2} B_4^{K_S^2}$$

$A_1, \ldots, A_4, B_0, \ldots, B_4 \in \mathbb{Q}[[x]]$ universal power series (depending only on $k = \mathrm{rk}(V)$)

$$I_{S,V}^{Verlinde}(x) = A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3^{c_1(V)K_S - \frac{1}{2}K_S^2} A_4^{K_S^2}$$

Verlinde Series $I_{S,V}^{Verlinde}(x)$ (EGL (2001)): With the change of variables $x = -t(1-t)^{r^2-1}$ ($r = \mathrm{rk}(V)$) have

$$A_1(x) = (1-t), \quad A_2(x) = \frac{(1-t)^{r^2}}{1-r^2 t}.$$

and $A_3(x) = A_4(x) = 1$ for $|r| \leq 1$

Segre Series:

Lehn conjecture (1999): formula for $I_{S,-L}^{Chern}(x)$ for $L \in \mathrm{Pic}(S)$, he uses Nakajima's Heisenberg action

Proven by Marian-Oprea-Pandharipande, Voisin (2019)

MOP consider $I_{S,V}^{Chern}(x)$ for general $V \in K^0(S)$

$$I^{Chern}_{S,V}(x) = B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3^{c_1(V)K_S - \frac{1}{2}K_S^2} B_4^{K_S^2}$$

Theorem (MOP (2022))

Put $k = \mathrm{rk}(V)$, $r = k-1$, with change of variable $x = -y(1-ry)^{r-1}$

$$B_0(x) = \frac{(1-y)^{r+1}}{1-ry},\; B_1(x) = \frac{1-ry}{(1-y)^r},\; B_2(x) = \frac{(1-ry)^{2r}}{(1-y)(1-r^2y)}$$

Furthermore MOP determine $B_3(x)$, $B_4(x)$ as algebraic functions for $|k| \leq 2$.

(1) formulas are complicated, even when $K_S = 0$: multiplying out $A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)}$ or $B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)}$ and undoing the change of variables gives complicated formula

(2) A_1, A_2; B_0, B_1, B_2 are easier to study: can compute on K3 surface, then $S^{[n]}$ is hyperkähler and there are powerful tools

A_3, A_4, B_3, B_4 which involve K_S are much more difficult

$$I_{S,V}^{Verlinde}(x) = A_1^{\chi(\det(V))} A_2^{\frac{1}{2}\chi(\mathcal{O}_S)} A_3^{c_1(V)K_S - \frac{1}{2}K_S^2} A_4^{K_S^2},$$
$$I_{S,V}^{Chern}(x) = B_0^{c_2(V)} B_1^{\chi(\det(V))} B_2^{\frac{1}{2}\chi(\mathcal{O}_S)} B_3^{c_1(V)K_S - \frac{1}{2}K_S^2} B_4^{K_S^2}$$

Mysterious relation: Verlinde series $\longleftrightarrow$ Segre series:

Conjecture (Johnson, MOP)

Put $r = k - 1$, then

$$B_3^{(k)}(-y(1-ry)^{r-1}) = A_3^{(r)}(-y(1-y)^{r^2-1})$$
$$B_4^{(k)}(-y(1-ry)^{r-1}) = A_4^{(r)}(-y(1-y)^{r^2-1})$$

Here we mean that for the Segre (B) series we take $\mathrm{rk}(V) = k$ and for the Verlinde (A) series $\mathrm{rk}(V) = k - 1 = r$.

How can this be and where could shift k to $k-1$ come from?

Theorem

The Verlinde Segre correspondence is true:
$B_3^{(r+1)}(-y(1-ry)^{r-1})=A_3^{(r)}(-y(1-y)^{r^2-1}),\quad B_4^{(r+1)}(-y(1-ry)^{r-1})=A_4^{(r)}(-y(1-y)^{r^2-1})$

Therefore it is enough to determine A_3, A_4

Theorem

With $x=-y(1-y)^{r^2-1}$ *we have*

$$A_3^{(r)}(x)=\frac{1}{(1-y)^{\frac{r}{2}}}\exp\left(-\sum_{n>0}\frac{y^n}{2n}\mathrm{Coeff}_{x^0}\left(\frac{x^r-x^{-r}}{x-x^{-1}}\right)^{2n}\right)$$

Alternative formula: let $\alpha_i(y)$, $\quad i=1,\ldots,r-1$ branches of the inverse of $\frac{(x^{\frac{1}{2}}-x^{-\frac{1}{2}})^2}{(x^{\frac{r}{2}}-x^{-\frac{r}{2}})^2}=x^{r-1}+\ldots$
i.e. $x=\alpha_i(y)=\epsilon_{r-1}^i y^{\frac{1}{r-1}}+\ldots$ sol. of $(x^r-2+x^{-r})y=x-2+x^{-1}$.
Then

$$A_3(-y(1-y)^{r^2-1})=\frac{y^{\frac{1}{2}}}{(1-y)^{\frac{r}{2}}\prod_{i=1}^{r-1}\alpha_i(y)^{\frac{1}{2}}}.$$

Conjectural formula for A_4: Recall

$$A_3(-y(1-y)^{r^2-1}) = \frac{y^{\frac{1}{2}}}{(1-y)^{\frac{r}{2}} \prod_{i=1}^{r-1} \alpha_i(y)^{\frac{1}{2}}}.$$

Conjecture

With $x = -y(1-y)^{r^2-1}$, we have

$$\left(A_4(x)A_3(x)^r\right)^8 =$$
$$= \frac{(1-r^2y)^3}{(1-y)^{3r^2}} \prod_{i,j=1}^{r-1}(1-\alpha_i(y)\alpha_j(y))^2 \prod_{\substack{i,j=1\\ i\neq j}}^{r-1}(1-\alpha_i(y)^r\alpha_j(y)^r)^2$$

So complete Verlinde and Segre formula. Proven when $K_S^2=0$

Proposition (based on computations with Don Zagier)

This conjecture is true modulo x^{50} (until 49-th Hilbert scheme).

For $V \in K^0(S)$ of rank k define

$$I_{S,V}(x,z) := \sum_{n\geq 0}(-x)^n\chi\big(S^{[n]}, \det(\mathcal{O}_S^{[n]})^{-1}\otimes\Lambda_{-z}V^{[n]}\big) \in \mathbb{Z}[[x,z]]$$

where $\Lambda_{-z}W = \sum_{n\geq 0}(-z)^n\Lambda^n W$

$I_{S,V}(x,z)$ specializes to $I_{S,V}^{Verlinde}(x,z)$ and $I_{S,V}^{Chern}(x,z)$:

$$(-1)^{n(k-1)}\mathrm{Coeff}_{x^n z^{kn}}\big(I_{S,V}(x,z)\big) = \chi(S^{[n]}, \det((V-\mathcal{O}_S)^{[n]})$$

$$\lim_{\epsilon\to 0}\left(I_{S,V}\left(\frac{-(1+\epsilon)^k}{\epsilon^{k-2}}x, \frac{1}{1+\epsilon}\right)\right) = I_{S,V}^{Chern}(x,z)$$

Note that in $\chi(S^{[n]}, \det((V-\mathcal{O}_S)^{[n]})$ the rank drops by 1

Universality says

$$I_{S,V}(x,z) = G_0^{c_2(V)}G_1^{\chi(\det(V))}G_2^{\frac{1}{2}\chi(\mathcal{O}_S)}G_3^{c_1(V)K_S-\frac{1}{2}K_S^2}G_4^{K_S^2}$$

for $G_0, G_1, G_2, G_3, G_4 \in \mathbb{Q}[[x,z]]$ depending only on $k = \mathrm{rk}(V)$.

$$I_{S,V}(x,z) = G_0^{c_2(V)} G_1^{\chi(\det(V))} G_2^{\frac{1}{2}\chi(\mathcal{O}_S)} G_3^{c_1(V)K_S - \frac{1}{2}K_S^2} G_4^{K_S^2}$$

Theorem

Let $k = rk(V)$, $r = k-1$. With the changes of variables

$$x = \frac{u(1-u)^r}{v(1-v)^r}, \quad z = \frac{v}{(1-u)^r}, \quad y = \frac{uv}{(1-u)(1-v)},$$

we have

$$G_0 G_1(x,z) = 1-y, \quad G_0 = \frac{(1-u-v)^{r+1}}{(1-v)^r((1-u)^r - v)},$$

$$G_2(x,z) = \frac{(1-\frac{u}{v})^2(1-v)^{r^2-1}((1-u)^r-v)}{(1-u-v)^{r^2}(1-u)^{r^2-1}(1-u-v-(r^2-1)uv)}$$

$$G_3(x,z) = A_3(-y(1-y)^{r^2-1}), \quad G_4(x,z) = A_4(-y(1-y)^{r^2-1})$$

Verlinde-Segre correspondence "explained" by the fact that $G_3(x,z)$ and $G_4(x,z)$ only depend on the variable y

Let X be a smooth projective variety with action of $T = \mathbb{C}^* \times \mathbb{C}^*$ with finitely many fixpoints, $p_1, \dots, p_e$, $d = \dim(X)$
Let E be equivariant vector bundle of rank r on X
Fibre $E(p_i)$ of X at fixp. p_i has basis of eigenvect. for T-action $E(p_i) = \bigoplus_{k=1}^r \mathbb{C} v_k$, with action $(t_1, t_2) \cdot v_k = t_1^{n_k} t_2^{m_k} v_k$, $n_k, m_k \in \mathbb{Z}$
Then the $n_k\epsilon_1 + m_k\epsilon_2 \in \mathbb{Z}[\epsilon_1, \epsilon_2]$ are called the **weights** of $E(p_i)$
Denote $u_{1,i}, \dots, u_{d,i}$ the weights of $T_{p_i}X$

$c_i^T(E(p_i)) = i$-th elementary symm. fctn in **weights** of $E(p_i) \in \mathbb{Z}[\epsilon_1, \epsilon_2]$

Let $P\big((c_i(E))_i\big)$ be a polynomial in Chern classes of E

Theorem (Bott residue formula)

$$\int_{[X]} P\big((c_i(E))_i, (c_j(T_X))_j\big) = \left(\sum_{k=1}^{e} \frac{P\big((c_i^T(E(p_k)))_i, (c_j^T(T_{p_k}X))_j\big)}{u_{1,k} \cdots u_{d,k}} \right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$

Sum in brackets is a **polynomial** *in* ϵ_1, ϵ_2.

Let S be a smooth toric surface, i.e. S has action of T with finitely many fixpoints $p_1, \ldots, p_e$
Near each fixpoint have affine T-equivariant coordinates x_i, y_i
The action of T on S lifts to an action on $S^{[n]}$
$Z \in S^{[n]}$ is T-invariant $\iff$ $Z = Z_1 \sqcup \ldots \sqcup Z_e \quad supp(Z_i) = p_i$, and the Z_i are T-invariant
Z_i is T-invariant $\iff$ $I_{Z_i} \in k[x_i, y_i]$ is gen. by monomials i.e.

$$I_{Z_i} = (x_i^{n_0}, y_i x_i^{n_1}, ..., y_i^r x_i^{n_r}, y_i^{r+1}) \quad (n_0, \ldots, n_r) \text{ partition}$$

$\Longrightarrow$ Fixpoints on $S^{[n]}$ are in bijection to e-tuples $(P_1, \ldots, P_e)$ of partitions, adding up to n
$V^{[n]}(Z) = V^{[n_1]}(Z_1) \oplus \ldots \oplus V^{[n_e]}(Z_e)$,
$T_{S^{[n]}}(Z) = T_{S^{[n_1]}}(Z_1) \oplus \ldots \oplus T_{S^{[n_e]}}(Z_e)$
The weights of the action on $V^{[n_i]}(Z_i)$ and $T_{S^{[n_i]}}(Z_i)$ are given in terms of the combinatorics of the partition P_i

By Universality enough to prove for S toric surface and V toric vector bundle on S i.e. $T=(\mathbb{C}^*)^2$ acts on S with finitely many fixpoints, action lifts to $V \implies$ can use localization

Denote $H_T^*(pt)=\mathbb{C}[\epsilon_1,\epsilon_2]$ equivariant cohomology

Let $p_1,\ldots,p_e$ fixpoints of T-action on S, denote $t_1^{(i)}$, $t_2^{(i)}$ wts on $T_S(p_i)$ and $v_1^{(i)},\ldots,v_k^{(i)}$ wts on $V(p_i)$ (each weight is of the form $n\epsilon_1+m\epsilon_2$)

Put

$$\Omega(x,z_1,\ldots,z_k,q,t):=\sum_{\lambda partitions}\frac{\prod_{i=1}^k\prod_{\Box\in\lambda}(1-q^{c(\Box)}t^{r(\Box)}z_i)}{\prod_{\Box\in\lambda}(q^{a(\Box)+1}-t^{l(\Box)})(q^{a(\Box)}-t^{l(\Box)+1})}x^{|\lambda|}$$

Identify partition with graph, and put $c(\Box)$ column, $r(\Box)$ row, $a(\Box)$ arm length, $c(\Box)$ leg length

Put $H = \log(\Omega)$
By Riemann-Roch and localization on $S^{[n]}$ have

$$I_{S,V}(x,z) = \left(\prod_{i=1}^{e} \Omega(x, e^{v_1^{(i)}}z, \dots, e^{v_k^{(i)}}z, e^{t_1^{(i)}}, e^{t_2^{(i)}})\right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$
$$= \exp\left(\sum_{i=1}^{e} H(x, e^{v_1^{(i)}}z, \dots, e^{v_k^{(i)}}z, e^{t_1^{(i)}}, e^{t_2^{(i)}})\right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$

So we "only" have to compute this.

Proposition

We can expand

$$H(x, z_1, \dots, z_k, e^{\epsilon_1}, e^{\epsilon_2}) = \sum_{d_1, d_2 \geq -1} H_{d_1,d_2}(x, z_1, \dots, z_k)\epsilon_1^{d_1}\epsilon_2^{d_2}$$

(not trivial could have deep pole in ϵ_1, ϵ_2)

Trick (first saw this in work with Nakajima-Yoshioka on instanton counting): Rewrite previous formula for $I_{S,V}(x,z)$: Inside exponential apply localization formula on S

$$I_{S,V}(x,z) = \exp\left(\sum_{i=1}^{e} H(x, e^{v_1^{(i)}}z, \dots, e^{v_k^{(i)}}z, e^{t_1^{(i)}}, e^{t_2^{(i)}})\right)\Bigg|_{\epsilon_1=\epsilon_2=0}$$

$$= \exp\Bigg(\Bigg(\sum_{i=1}^{e} \frac{1}{t_1^{(i)}t_2^{(i)}}\Big(H_{-1,-1}(x, e^{v_j^{(i)}}z) + (t_1^{(i)} + t_2^{(i)})H_{-1,0}(x, e^{v_j^{(i)}}z)$$

$$+ t_1^{(i)}t_2^{(i)}H_{0,0}(x, e^{v_j^{(i)}}z) + ((t_1^{(i)})^2 + (t_2^{(i)})^2)H_{-1,1}(x, e^{v_j^{(i)}}z)\Big)\Bigg)\Bigg|_{\epsilon_1=\epsilon_2=0}\Bigg)$$

$$= \exp\left(c_2(V)C_2 + c_1(V)^2C_{11} + K_Sc_1(V)D_1 + e(S)F + (K_S^2 - 2e(S))E\right)$$

Put $H_{d_1,d_2,k}(x,z) = H_{d_1,d_1}(x,z,\dots,z)$, write $D_z = z\frac{\partial}{\partial z}$
Then $F(x,z) = H_{0,0,k}(x,z)$, $E(x,z) = H_{-1,1,k}(x,z)$,
$D_1(x,z) = -kD_zH_{-1,0,k}(x,z)$ and $C_2(x,z)$, $C_{11}(x,z)$ given by second partial derivatives of $H_{-1,-1,k}$

So we need to understand

$$H_{-1,-1,k}(x,z), \quad H_{-1,0,k}(x,z), \quad H_{0,0,k}(x,z), \quad H_{-1,1,k}(x,z).$$

Want to understand

$H_{-1,-1,k}(x,z), \quad H_{-1,0,k}(x,z), \quad H_{0,0,k}(x,z), \quad H_{-1,1,k}(x,z)$

Use two properties: **regularity** and **symmetry**

1. $f(x,z) \in \mathbb{C}[[x,z]]$ is d-regular (wrt k) if $f\left(x\epsilon^{2-k}(1+\epsilon)^k, \frac{1}{1+\epsilon}\right) \in \epsilon^d\mathbb{C}[[x,\epsilon]]$,
2. $f(x,z)$ is called symmetric if $f(x,z) = f(x^{-1}, xz)$.

Theorem

1. *$H_{d_1,d_2,k}(x,z)$ is $-d_1 - d_2$ regular for $d_1 + d_1 \leq 0$*
2. *$H_{d_1,d_2,k}(x,z) + \frac{B_{d_1+1}B_{d_2+1}}{(d_1+1)!(d_2+1)!}(Li_{1-d_1-d_2}(x) + kLi_{1-d_1-d_2}(z))$ is symmetric ($Li_d(x) = \sum_{n>0} x^n/n^d$ polylog).*

First part follows from the fact that $I^{Chern}_{S,V}$ is limit of $I_{S,V}$

Second part is input from symmetric function theory:

1. Express $\Omega(w, z_1, \ldots, z_k, q, t)$ in terms of generalized generalized MacDonald's polynomials $\widetilde{H}_\mu(X; q, t)$
2. Identities of MacDonald's polynomials give functional equation for $\Omega(w, z_1, \ldots, z_k, q, t)$. Put

$$\widetilde{\Omega}(w, z_1, \ldots, z_k; q, t) := \mathrm{Exp}\left[\frac{w + \sum z_i}{(1-q)(1-t)}\right]\Omega(w, z_1, \ldots, z_k; q, t)$$

then $\widetilde{\Omega}(w, z_1, \ldots, z_k; q, t) = \widetilde{\Omega}(w^{-1}, wz_1, \ldots, wz_k; q, t)$.

3. Take logarithm to get symmetry for H.

Symmetric and regular functions fulfill very strong constraints:

Theorem

Let $f(x,z)$ be a symmetric d-regular function (wrt k).

1. *if $d > 0$, then $f(x,z) = 0$.*
2. *if $d = 0$, there exists a unique $h(y) \in \mathbb{C}[[y]]$, such that*

$$f\left(\frac{u(1-u)^{k-1}}{(1-v)^{k-1}}, \frac{v}{(1-u)^{k-1}}\right) = h\left(\frac{uv}{(1-u)(1-v)}\right).$$

The functions $F(x,z)$, $E(x,z)$, $D_1(x,z)$, $C_2(x,z)$, $C_{11}(x,z)$ can be expressed in terms of symmetric regular functions
A symmetric regular function is determined by few of its coefficients,
Trick: assume $g(x,z)$ is 1-regular, then $f(x,z) := D_z g(x,z)$ is regular. If furthermore $f(x,z)$ is symmetric, then $g(x,0)$ determines $f(x,z)$

Happy birthday Hiraku.