Vortices & VOA's

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RIMS, Kyoto 20 Feb. 2023

Thank you

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- to 中島さん, 入谷さん, & the organizing team, for putting this event together and having me here.

Building bridges

- between physics and mathematics, anchored in rigor

  $\rightsquigarrow$ for physicists, where to look & what language to use

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2014: Symplectic Duality

[Braden-Licata-Proudfoot-Webster]

from Higgs & Coulomb branches ??

+ boundary conditions

[Bullimore-TD-Gaiotto '15]

$\mathbb{C}[\mathcal{M}_{coul} \setminus \text{discriminant locus}]$ + HK structure

[Bullimore-TD-Gaiotto-Hilburn '16]

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[Bullimore-TD-Gaiotto-Hilburn '16]

vs.

[Nakajima '15]

[Braverman-Finkelberg-Nakajima '16]

$$\mathbb{C}[\mathcal{M}_{Coul}] := H^{BM}_{\bullet}[G_{[[z]]} \backslash R_{G,N}]$$

as a convolution algebra

$\rightsquigarrow$ opened up 3d $\mathcal{N}=4$ theories and 3d mirror sym to mathematics!

3d $\mathcal{N}=4$ gauge theories

Data (simple case) $\quad G_c \circlearrowright T^*N$

$G = (G_c)_{\mathbb{C}}$ reductive $\quad \circlearrowright \quad N \simeq \mathbb{C}^n$ cx-linear

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cf. Donaldson-Witten $\qquad$ Blau-Thompson
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Expect extended (3)-2-1-0 TQFT's $\qquad Z^A_{G,N} \qquad Z^B_{G,N}$

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$Z(\Sigma) \in \mathrm{Vect}$

$Z(\ominus) = \mathrm{Ops}$ $\quad$ is a (E_3-)commutative algebra

$Z(\circledcirc) : Z(\ominus) \otimes Z(\ominus) \to Z(\ominus)$

$$\mathcal{M}^{G,N}_{Coul} = \mathrm{Spec}\, Z^A_{G,N}(\ominus) \qquad\qquad \mathcal{M}^{G,N}_{Higgs} = \mathrm{Spec}\, Z^B_{G,N}(\ominus)$$

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Data (simple case) $\quad G_c \circlearrowright T^*N$ $\qquad\qquad \mu_{\mathbb{C}} : T^*N \to \mathfrak{g}^*$

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$$\mathcal{M}^{G,N}_{Higgs} = \operatorname{Spec} Z^{B}_{G,N}(\circledcirc) = \mathbb{C}[T^*N /\!\!/ G] = \mathbb{C}[\mu_{\mathbb{C}}^{-1}(0)]^G$$

$$\mathcal{M}^{G,N}_{Coul} = \operatorname{Spec} Z^{A}_{G,N}(\circledcirc)$$

- EOM in A twist are well behaved on $M^3 \approx \mathbb{C}_z \times \mathbb{R}_t$ (locally)

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Along $\mathbb{C}_z$, fields $\ni X(z,\bar z) \in N$, $Y(z,\bar z) \in N^*$, A a G connection...

algebraic EOM: $\{\bar\partial X = \bar\partial Y = 0\,,\ \mu_{\mathbb{C}}(X,Y) = 0\}/\text{hol}^c\ G\ \text{gauge}$

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- Replace  with a raviolo 

  $= D \underset{D^*}{\cup} D \qquad D = \mathrm{Spec}\, \mathbb{C}[[z]]$

  $\qquad\qquad\qquad\quad D^* = \mathrm{Spec}\, \mathbb{C}((z))$

3d $\mathcal{N}=4$ gauge theories

$G \circlearrowright T^*N \qquad X \in N \quad Y \in N^*$

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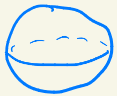
$\mathcal{M}^{G,N}_{\text{Coul}} = \operatorname{Spec} Z^{A}_{G,N}\big($  $\big)$

- Replace  with a raviolo  $= D \cup_{D^*} D$

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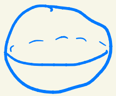
- Replace  with a raviolo  $= D \cup_{D^\times} D$

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$\overset{?}{=} H^\bullet \{$ G-bundles E on $D \cup_{D^\times} D$ and alg' sections (X,Y) of $E \times_G N$, $E \times_G N^*$ s.t. $\mu_{\mathbb{C}}(X,Y) = 0\}/\text{iso}$

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[Nak, BFN]

well defined (!)

and computable (!!)

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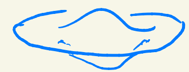
- Replace  with a raviolo  $= D \underset{D^*}{\cup} D \qquad D = \operatorname{Spec} \mathbb{C}[\![z]\!]$, $D^* = \operatorname{Spec} \mathbb{C}(\!(z)\!)$

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Product from convolution: $Z^{A}_{G,N}($  $)$.

Impact : mathematicians can directly access 3d $\mathcal{N}=4$ physics

Small sampling of examples:

[Webster '16] Symplectic duality for Higgs & Coulomb branch categories $\mathcal{O}$

[Braverman-Finkelberg-Nakajima '18]
[Webster '19] Coherent sheaves on resolutions of $\mathcal{M}_C$

[Hilburn-Kamnitzer-Weekes '20] BFN Springer theory

[Teleman '18]
[González-Mak-Pomerleano '22] $\mathbb{C}[\mathcal{M}_{cone}] \circlearrowright QH^\bullet$ and $SH^\bullet$ of G spaces

[Ginzburg-Kazhdan] Higgs branches of 4d Class S theories
cf [Arakawa '18] chiral algebras of Class S

[Cautis-Williams '18, '23] line operators in 4d $\mathcal{N}=2$ gauge thy

More on 3d TQFT's

Expect fully extended (3)-2-1-0 theories

$$Z^A_{G,N},\ Z^B_{G,N} : 3\text{-}Cob \xrightarrow[\text{sym mon}^l]{} 3\text{-category of dg 2-categories} \ldots$$

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- $Z(M^3)$ may not be finite   cf [Gukov-Hsin-Nakajima-Park-Pei-Sopenko '20] equivariant regularization for Z^B

but $Z(M^3 \setminus B^3) \circlearrowleft Z(S^2)$ may be nicer...

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- $\boxed{Z(S^1) \text{ a braided } \otimes \text{ category}}$ — cf [TD-Garner-Geracie-Hilburn '19] [Hilburn-Raskin '21]

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- $Z(pt)$ a 2-category   cf [Hilburn-Gammage-Mazel-Gee '22] also [Kapustin-Rozansky-Saulina '08] [Rezchikov-Doan '22, Wang '22]

Some physics of $Z(S^1)$

[TD-Gannon-Geraedts-Hilburn '19]

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cf. [Hilburn-Yoo]
[Hilburn-Raskin '21]

Proposed $Z^A_{G,N}(S^1) := D\text{-mod}\left(N((z))/G((z))\right)$

$\mathbb{1} = \mathcal{O}_{N[[z]]/G[[z]]}$ ✓

BUT: no $\otimes$, braiding

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Q: How to access $Z(S^1)$ concretely (and mathematically), as a BTC?

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Chern-Simons theory $G_{\mathbb{C}}$, $k: \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ admits a holomorphic boundary condition whose local operators form a VOA $WZW_{\mathfrak{g}}^{k}$

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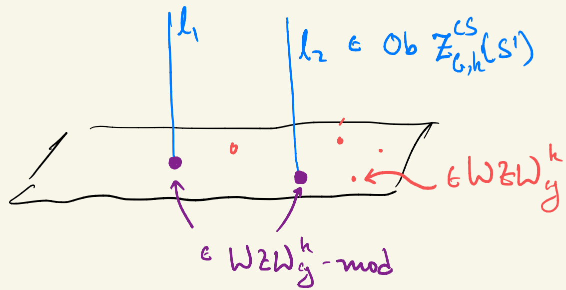
[Witten '89]
[Elitzur-Moore-Seiberg-Schwimmer '90]

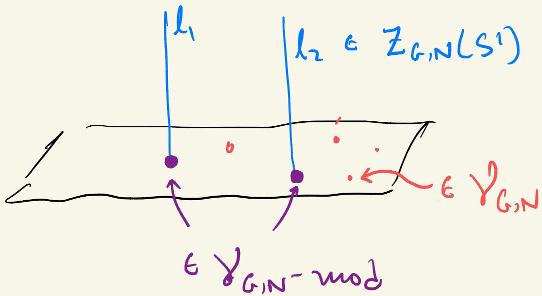


$$Z^{CS}_{G,k}(S^1) \simeq WZW_{\mathfrak{g}}^{k}\text{-mod}$$

Boundary VOA's

- 3d $\mathcal{N}=4$ gauge thy's in A & B twists also admit boundary VOA's $\mathcal{V}^A_{G,N}$, $\mathcal{V}^B_{G,N}$

  Perturbative definition: [Costello-Gaiotto '18]



- $\mathbb{1}$ = vacuum module

  $$\mathrm{Ext}^\bullet_{\mathcal{V}^A_{G,N}}(\mathbb{1},\mathbb{1}) \simeq \mathbb{C}[\mathcal{M}_{Coul}] \quad \checkmark$$

  [Costello-Creutzig-Gaiotto '18]

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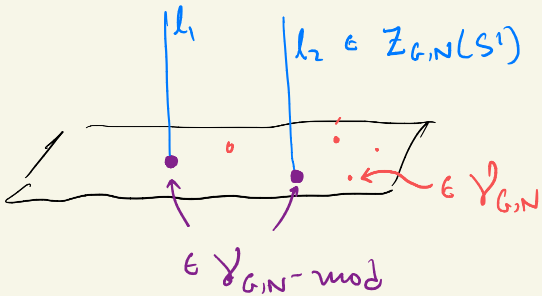

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Difficulties

- $\mathcal{V}^A_{G,N} := H^{\bullet}_{BRST}\left(\mathfrak{g}_{\mathbb{C}}, (\beta\gamma)^{T^*N} \otimes (bc)^{\#}\right)$ is hard to analyze

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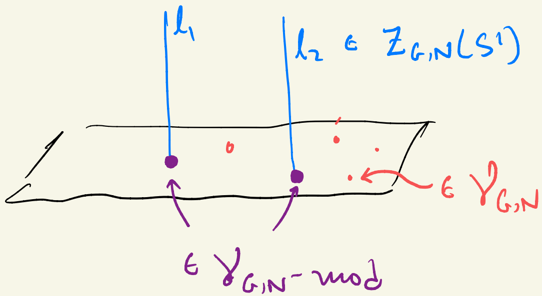

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  $A_{G,N} := \mathfrak{g} \oplus \mathfrak{g}^* \oplus \Pi N \oplus \Pi N^*$

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Overcame these for abelian theories.

Abelian boundary VOA's

[Ballin-Niu '22]
[Ballin-Creutzig-TD-Niu '23]

Setup: $G_c = U(1)^r \circlearrowright T^*N \quad N = \mathbb{C}^n$ faithfully

$n \times r$ charge matrix ρ

$$0 \to \mathbb{Z}^r \xrightarrow{\rho} \mathbb{Z}^n \xrightarrow{\tau} \mathbb{Z}^{n-r} \to 0$$

3d MS: $(\rho, \tau) \leftrightarrow (\tau^T, \rho^T)$

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[Ballin-Niu '22]
[Ballin-Creutzig-TD-Niu '23]

Setup: $G_c = U(1)^r \circlearrowright T^*N \quad N = \mathbb{C}^n$ faithfully

$n \times r$ charge matrix ρ

$$0 \to \mathbb{Z}^r \xrightarrow{\rho} \mathbb{Z}^n \xrightarrow{\tau} \mathbb{Z}^{n-r} \to 0$$

3d MS: $(\rho, \tau) \leftrightarrow (\tau^T, \rho^T)$

Results: $V_\rho^A = H^\bullet_{BRST}\left(\mathfrak{gl}(1)^r, \ (\beta\gamma)^{\otimes n} \otimes (bc)^{\otimes n} \right)$

has a free field realization allowing $H^\bullet_{BRST}$ to be taken

Abelian boundary VOA's

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$\mathcal{V}^B_\rho := \mathbb{Z}^r$ simple current extension of $\widehat{\mathcal{A}_{G,N}}$

(boundary monopole operators)

$$\mathcal{A}_{G,N} := \mathfrak{g} \oplus \mathfrak{g}^* \oplus \Pi N \oplus \Pi N^*$$

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satisfying all physical requirements

Abelian boundary VOA's

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$$V_\rho^A = H^\bullet_{BRST}\left(\mathfrak{gl}(1)^r, (\beta\gamma)^{\otimes n} \otimes (bc)^{\otimes n}\right)$$

has a free field realization allowing $H^\bullet_{BRST}$ to be taken

$V_\rho^B := \mathbb{Z}^r$ simple current extension of $\widehat{A_{G,N}}$

(boundary monopole operators)

Thm $V_\rho^A \simeq V_{\tau^T}^B$

Thm $\exists$ braided tensor categories of modules $\mathcal{C}_\rho^A$, $\mathcal{C}_\rho^B$

satisfying all physical requirements, w/ $\mathcal{C}_\rho^A \simeq \mathcal{C}_{\tau^T}^B$

Idea:
- start w/ KL category for $\widehat{A_{G,N}}$ (BTC w/ log intertwiners) [HLZ '10]
- pass through extensions using [Creutzig-Kanade-Linshaw '20]
  [Creutzig-McRae-Yang '20, '21]

Abelian boundary VOA's

[Ballin-Niu '22] [Ballin-Creutzig-TD-Niu '23]

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Thm $V_\rho^A \simeq V_{\tau^T}^B$

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satisfying all physical requirements, w/ $\mathcal{C}_\rho^A \simeq \mathcal{C}_{\tau^T}^B$

$\mathcal{C}_\rho^B$ includes Wilson lines $\mathbb{W}_\mu \qquad \mu \in \mathbb{Z}^r \simeq \mathrm{Char}(G)$

$$\mathrm{Ext}^d(\mathbb{W}_\mu, \mathbb{W}_{\mu'}) \simeq \mathbb{C}[T^*N]^{\deg d}_{\text{weight } \mu'-\mu}$$

Abelian boundary VOA's

[Ballin-Niu '22] [Ballin-Creutzig-TD-Niu '23]

Setup: $G_c = U(1)^r \circlearrowright_\rho T^*N \quad N = \mathbb{C}^n \qquad 0 \to \mathbb{Z}^r \xrightarrow{\rho} \mathbb{Z}^n \underset{\sigma}{\overset{\tau}{\rightleftarrows}} \mathbb{Z}^{n-r} \to 0$

Thm $\quad \mathcal{V}^A_\rho \simeq \mathcal{V}^B_{\tau^T}$

Thm $\quad \exists$ braided tensor categories of modules $\mathcal{C}^A_\rho$, $\mathcal{C}^B_\rho$

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$\mathcal{C}^A_\rho$ includes vertex lines $\mathbb{V}_\nu \qquad \nu \in \mathbb{Z}^{n-r}$, 3d-mirror to $\mathbb{W}$'s

$$\left\{\begin{array}{l} X(z) \sim z^{\sigma(\nu)} \\ Y(z) \sim z^{-\sigma(\nu)} \end{array}\right. \text{ near } z = 0$$

$\mathrm{Ext}^d(\mathbb{V}_\nu, \mathbb{V}_{\nu'})$ as expected from [BFN '16, 17]

還暦おめでとうございます！

Meallaibh ur Naidheachd !