Up and Down the Bow Construction

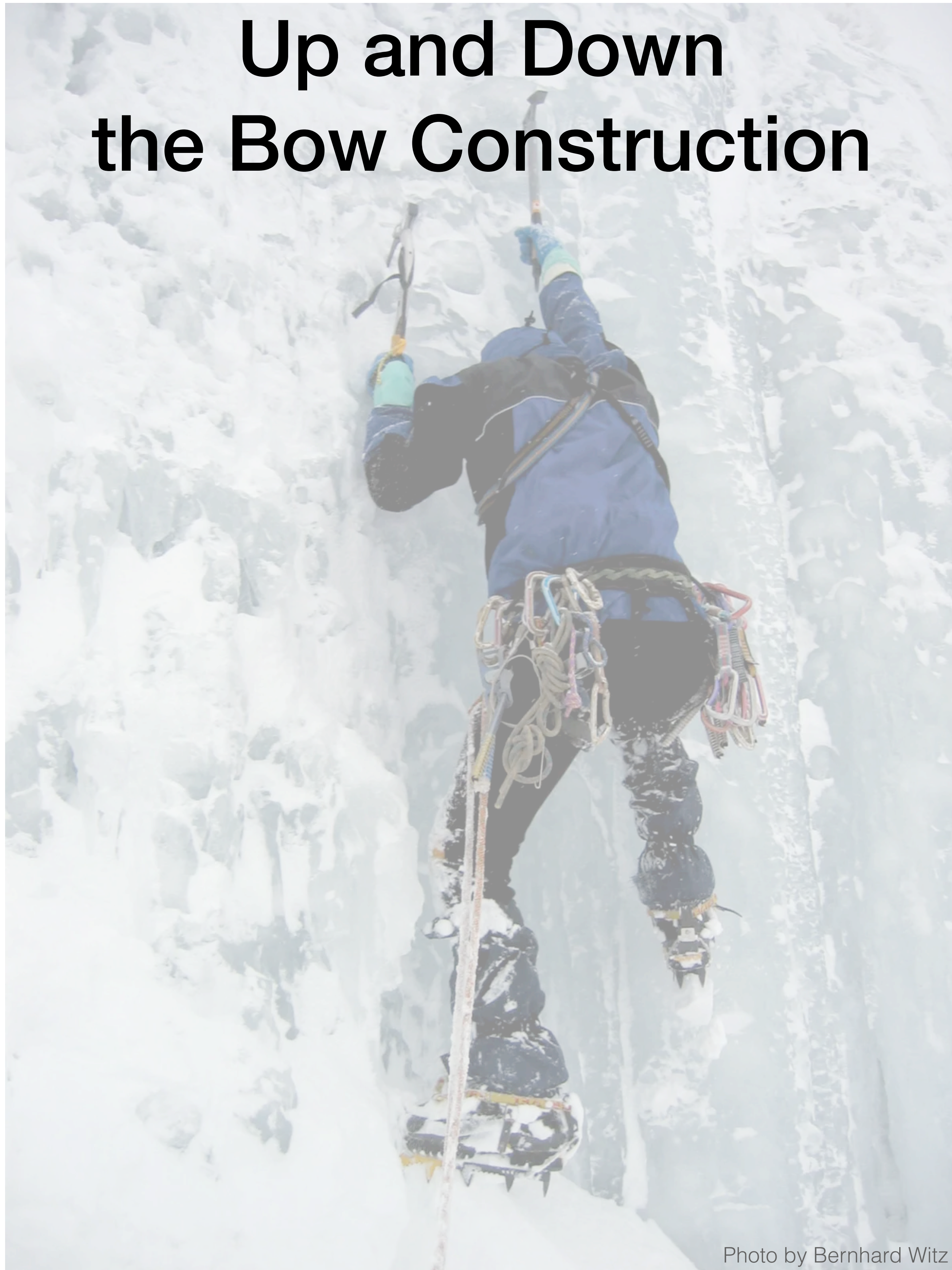

Photo by Bernhard Witz

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RIMS, Kyoto
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These results are obtained in collaboration with Mark Stern and Andres Larrain-Hubach:

- Instantons on multi-Taub-NUT spaces I: Asymptotic form and index theorem, J.Diff.Geom. 119 (2021) 1.
- Instantons on multi-Taub-NUT Spaces II: Bow Construction, J.Diff.Geom. 2023.
- Instantons on multi-Taub-NUT Spaces III: Down Transform, Completeness, and Isometry, to appear in 2023

This work derives from:

- M. F. Atiyah, N. J. Hitchin, V. G. Drinfeld and Y. I. Manin, “Construction of Instantons,” Phys. Lett. A 65, 185 (1978)
- W. Nahm, ”The Construction of All Self-dual Multimonopoles by the ADHM Method,” in Proceedings of Monopoles in Quantum Field Theory meeting Trieste, 1981.
- E. Corrigan and P. Goddard, “Construction of Instanton and Monopole Solutions and Reciprocity,” Annals Phys. 154, 253 (1984).
- P.B. Kronheimer and H. Nakajima, “Yang-Mills instantons on ALE gravitational instantons,” Math. Ann., 288(2):263– 307, 1990.
- Hiraku Nakajima, “Monopoles and Nahm’s equations,” in Einstein metrics and Yang-Mills connections (Sanda, 1990), volume 145 of Lecture Notes in Pure and Appl. Math. Dekker, New York, 1993.

Intro:

"Bows form only the first step in generalizing quivers."

- Instanton on ALE <—> Quiver Kronheimer - Nakajima '90
- Instanton on ALF <—> Bow
- Instanton on ALG <—> Sling
- Instanton on ALH <—> Wall

Comment: 4 real dim. hyperkähler space with L^2 Riemann curvature = Tesseron

All tesserons come in ALE, ALF, ALG, ALG*, ALH, and ALH* types.

ALF spaces: A_{k-1} ALF = TN_k

D_k ALF are asymptotic to $A_{2k-5} / \mathbb{Z}_2$

(D_0=AH and D_1=Double cover(AH))

$$\begin{array}{l} \mathrm{TN}_k^\nu \leftarrow S^1 \\ \pi_k \downarrow \\ \mathbb{R}^3 = \mathrm{Im}\,\mathbb{H} \end{array}$$

$$g = V\,|dt|^2 + \frac{(d\tau + \pi_k{}^*\omega)^2}{V}$$

$$V = \ell + \sum_{\sigma=1}^{k} \frac{1}{2\,|t - \nu_\sigma|}, \qquad d\omega = *_3\, dV$$

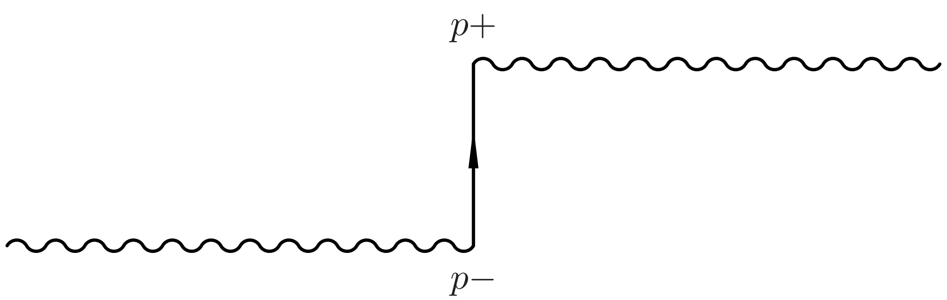
Yang-Mills Instanton: Hermitian bundle with connection $(\mathscr{E} \to \mathrm{TN}_k^\nu, A)$ satisfying:

1. $*F_A = -F_A$
2. F_A is L^2

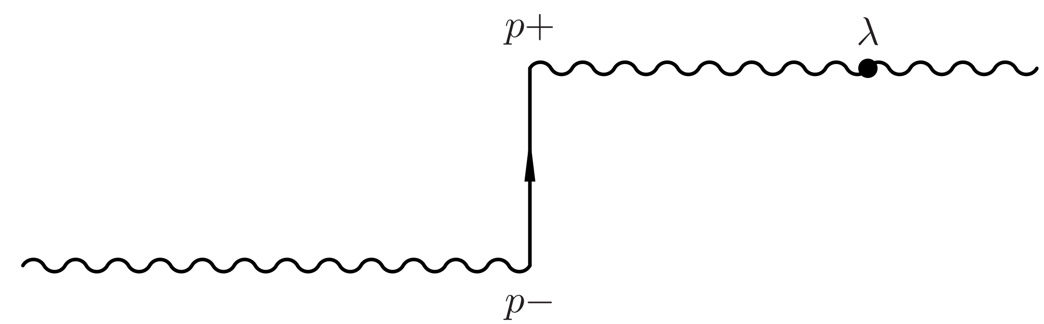
$\Leftrightarrow$ $\mathrm{Im}\, D_A^\dagger D_A = 0$ Dirac operator D_A

Bows

A **bow** is a collection of
1.oriented intervals and
2. arrows connecting their ends
(respecting the orientation).



A **bow representation** $\mathfrak{R}$ is
1. collection Λ of λ-points on the bow,
2. hermitian line bundles E on (Bow $-\Lambda$) (matching at λ points),
3. space W_λ for each constant rank λ point.



Bow representation data $\mathrm{Dat}(\mathfrak{R})$ consists of

1. Nahm data:
connection $\nabla_{\frac{d}{ds}}$ on E and
three endomorphisms
T_1, T_2, T_3 of E.

2. Bifundamental data:
For each arrow
$B_p : E_{p-} \to \mathcal{S} \otimes E_{p+}$
2-dim rep. of quaternions.

3. Fundamental data:
for each const. rank
λ-point:
$Q_\lambda : W_\lambda \to \mathcal{S} \otimes E_\lambda$

Let $\mathbb{T} := I_1 \otimes T_1 + I_2 \otimes T_2 + I_3 \otimes T_3$

Bow solution:

$\mathrm{Dat}(\mathfrak{R})$ is hyperkähler and the gauge group $\mathcal{G}$ of $\mathfrak{R}$ acts triholomorphically.
Its bow moment map equations (at level ν) are

$$i\nabla_{\frac{d}{ds}} T_1 = [T_2, T_3], \qquad \mathbb{T}(p-) + \mathrm{Im}\, iB_p B_p^\dagger = \nu_p, \qquad \mathbb{T}(\lambda+) - \mathbb{T}(\lambda-) = \mathrm{Im}\, iQ_\lambda Q_\lambda^\dagger$$

Bow Moduli Space

c.f. all ALE tesserons
as quiver moduli spaces:
Kronheimer '89

HKQ level set: $\mu^{-1}(\nu) := \{t - \mathrm{Im}\, bb^\dagger = \nu\} \curvearrowleft \mathscr{G}^{\mathfrak{s}}$

$$\mu^{-1}(\nu) \xrightarrow{\mathscr{P}} \mathrm{TN}_k^\nu = \mu^{-1}(\nu)/\mathscr{G}^{\mathfrak{s}}$$

Hyperkähler affine space: $\mathrm{Dat}(\mathfrak{s}) \ni (\nabla_{\frac{\partial}{\partial s}}, t, b)$

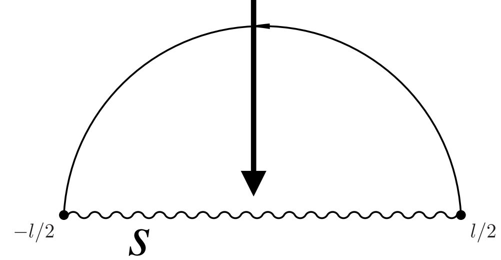
Moduli space $\mathscr{M}_{\mathfrak{s}}$ of the bow rep. $\mathfrak{s}$

A (small) bow representation $\mathfrak{s}$
(Herm. line bundle)

$e \curvearrowleft \mathscr{G}^{\mathfrak{s}}$

$\downarrow$

Bow Ψ point s



- Moduli space of a bow rep. is hyperkähler:

$$\mathscr{M}_{\mathfrak{s}} = \mathrm{Dat}\,(\mathfrak{s})///\mathscr{G}^{\mathfrak{s}}$$

Performing HKQ in stages, presents $\mathscr{M}_{\mathfrak{s}}$ as a finite HKQ.

- Focussing on one complex structure:

$$\mathscr{M}_{\mathfrak{s}} = \mu_{\mathbb{C}}^{-1}(\nu_{\mathbb{C}})/\mathscr{G}^{\mathfrak{s}}_{\mathbb{C}}$$

One can cut the bow into pieces to present $\mathscr{M}_{\mathfrak{R}}$
as a **finite** HK quotient.

This can be used to study its asymptotic, Ch ‘10

or realize it as a quiver variety in any given complex structure,

Nakajima & Takayama ‘16

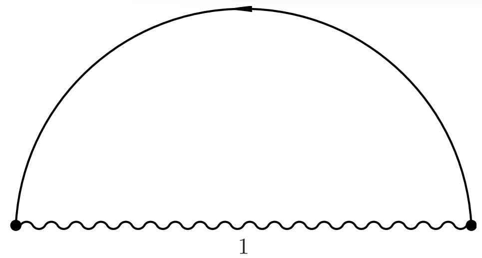
or adjust the level to obtain a non-commutative deformation of $\mathscr{M}_{\mathfrak{R}}$

R. Bielawski, Y. Borchard and S. A. Ch ‘22

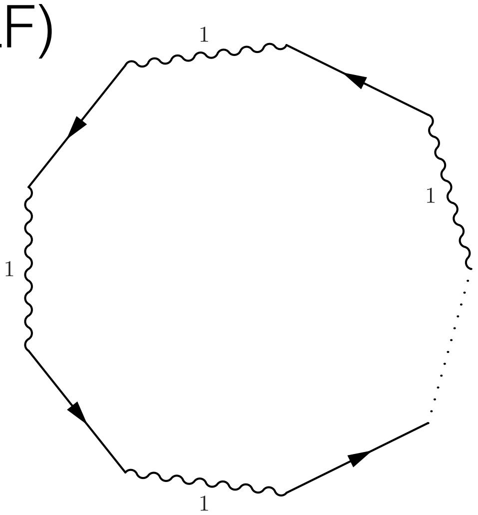
in which case the Bow construction delivers Instantons on non-commutative $\mathscr{M}_{\mathfrak{R}}$.

TN_k as a Bow Rep. Moduli Space

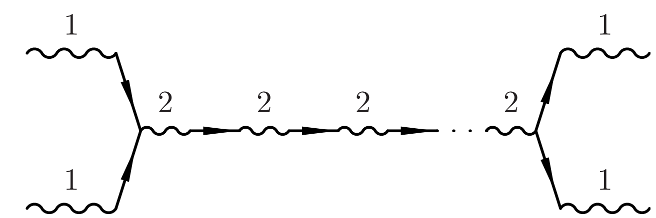
Taub-NUT (A_0 ALF)



multi-Taub-NUT (A_k ALF)

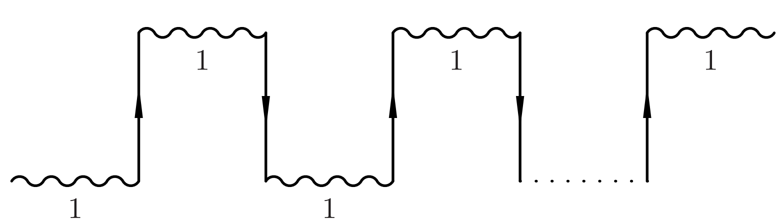


D_k ALF



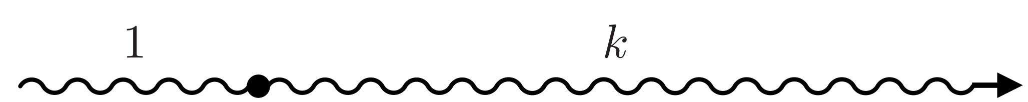
Note: there are other bows with representations with the same moduli space.

E.g. Cheshire bow rep:



Ch-Blair '10

and Ray bow rep.



Ch-Kapustin '98

all have TN_k as their moduli space.

Level set as a Poincaré bundle

- For any point on the bow $s \in$ Bow, one has exact sequence $1 \to \mathscr{G}_s \to \mathscr{G} \to U(e_s) \to 1$

  where $\mathscr{G}_s := \{g \in \mathscr{G} \,|\, g(s) = 1\}$

  and the partial quotient $\mu^{-1}(\nu)/\mathscr{G}_s$ forms a $U(e_s)$ principal bundle over the moduli space:

$$
\begin{array}{c}
\mu^{-1}(\nu)/\mathscr{G}_s \leftarrow U(e_s) \\
\downarrow \\
\mathrm{TN}_k^\nu
\end{array}
$$

  This is the Tautological principal bundle at s.
  Its associated bundle is $L_s \to \mathrm{TN}_k^\nu$, and
  it comes equipped with a tautological instanton connection a_s!

- Together, the family of tautological bundles form a bundle over $\mathrm{TN}_k^\nu \times$ Bow:

$$
\begin{tikzcd}
 & \mathscr{G} \curvearrowright & \underline{e} := \pi_2^* e \arrow[d] & & \\
 & & \mu^{-1}(\nu) \times \mathrm{Bow} \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] & & \\
L \arrow[r] & \mathrm{TN}_k^\nu \times \mathrm{Bow} & & \mathrm{Bow} & e \arrow[l]
\end{tikzcd}
$$

Section of L = $\mathscr{G}$-equivariant section of $\underline{e}$.

Note: all of the above applies to ANY bow rep. (except, its moduli space might not be TN).

Down Transform

"A space knows its bow;
an instanton knows its bow representation."

CLHS'21

Analytic results:

1. Any instanton (with asymp. holonomy) on ALF space has asymptotic form
$$A = \bigoplus_{\lambda\in\Lambda}\left(\pi_k^*\eta_\lambda - i(\lambda + \frac{m_\lambda}{2|t|})\frac{\omega}{V}\right) + O(|t|^{-2}),$$
where $\exp(2\pi i\frac{\lambda}{\ell})$ are the asymptotic values of the holonomy eigenvalues and m_λ are the first Chern numbers of the corresponding eigen-linebundles.

2. Index of the associated Dirac operator is
$$\mathrm{ind}_{L^2}D_A = \sum_{\lambda\in\Lambda}\left((\{\lambda/l\} - \frac{1}{2})(m_\lambda - k\lfloor\lambda/l\rfloor) - \frac{k}{2}\{\lambda/l\}^2\right) + \frac{1}{8\pi^2}\int \mathrm{tr}\, F\wedge F.$$
3. Harmonic spinors decay exponentially, if no $\lambda = 0$.

Given an instanton $(\mathscr{E}, A)$, its Dirac operator is $D_A : \Gamma(S \otimes \mathscr{E}) \to \Gamma(S \otimes \mathscr{E})$

The spin bundle S splits into chiral parts $S = S^+ \oplus S^-$.

Hyperkählerity => S^+ is trivial. Moreover, the Clifford action of the three Kähler forms $I_i := Cl(\omega_i)$ is covariantly constant and I_1, I_2, I_3 form quaternionic units!

Twisting by tautological bundles, we have a family of Dirac operators parameterized by the bow:

$$D_s := D_{A \otimes 1_{L_s} + 1_{\mathscr{E}} \otimes a_s} : \Gamma(S \otimes \mathscr{E} \otimes L_s) \to \Gamma(S \otimes \mathscr{E} \otimes L_s)$$

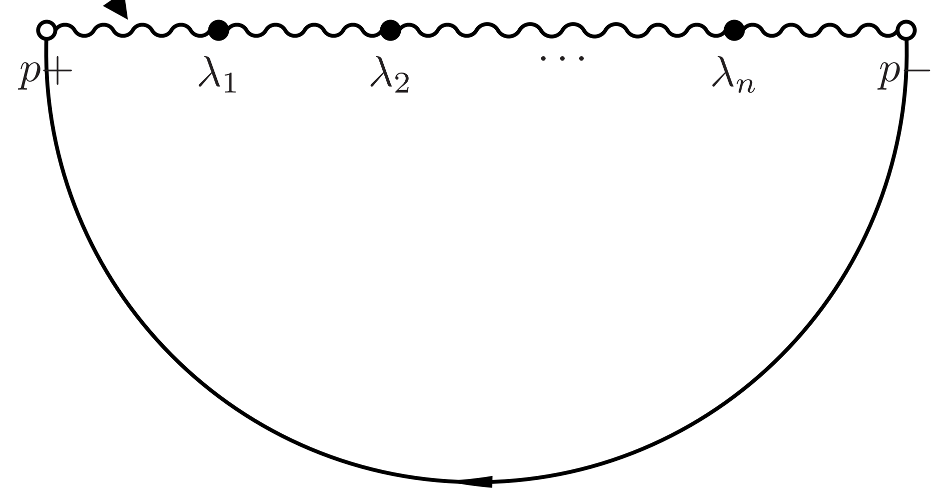
Importantly: anti-self-duality of both A and a_s is equivalent to

$$D_s^2|_{S^+} = \nabla_s^* \nabla_s \otimes 1_{S^+}$$

If $(\mathscr{E}, A)$ has no trivial factors, then $\nabla_s^* \nabla_s > 0$, thus $\ker_{L^2} D_s$ is entirely in S^-.

- Index bundle: $\mathscr{E}_s := \ker_{L^2} D_s$

Bounded harmonic sections: $W_\lambda := \ker_{L^\infty} \nabla_\lambda^* \nabla_\lambda$



- Induced Bow Solution: T, B, Q

1) W_λ comes with a map $\quad Q : W_\lambda \otimes S^+ \to \mathscr{E}_\lambda$, $\; f \mapsto D_\lambda f \quad$ equivalently, $Q \in \mathscr{E}_\lambda \otimes (S^+)^* \otimes W_\lambda^*$

Since $D_s D_s|_{S^+} = \nabla_s^* \nabla_s \otimes 1_{S^+}$, such $D_\lambda f$ is indeed in $\ker D_s$.

2) Using the orthogonal projection $\Pi_s : L^2(S^- \otimes \mathscr{E} \otimes L_s) \to \ker_{L^2} D_s$

Multiplication by t and b induces $\quad T_j := \Pi_s t_j \Pi_s \quad$ and $\quad B_\alpha = \Pi_s b_\alpha \Pi_s$

Prop: The resulting (T,B,Q) solves the bow moment map equations at level ν (determined by TN_k^ν) !

Up Transform

Bow Solution => Bow Dirac Operator => Index bundle over the level set => Instanton

Given a bow solution
(T, B, Q)
of a bow rep. $\mathfrak{R}$

$\downarrow$

$\mathfrak{D}_{\mathfrak{R}} : \Gamma(\mathscr{S} \otimes E) \to \Gamma(\mathscr{S} \otimes E) \oplus E_{p+} \oplus E_{p-} \oplus W_\lambda$

$$\mathfrak{D}_{\mathfrak{R}}\psi := \begin{pmatrix} (-\nabla_{\frac{d}{ds}} + i\mathbb{T})\psi \\ B_p^\dagger \psi(p-) \\ -B_p^{c\dagger}\psi(p+) \\ -Q_\lambda^\dagger \psi(\lambda) \end{pmatrix}$$

Any $(t, b) \in \mu^{-1}(\nu)$ is
a solution of the small bow rep.
and has associated Bow Dirac op. $\mathfrak{D}_{\mathfrak{s}}$.

Use its charge conjugate operator
$\mathfrak{D}_{\mathfrak{s}}^c$ to form a family

$$\mathfrak{D}_{(t,b)} := \mathfrak{D}_{\mathfrak{R}} \otimes 1_{e^*} + 1_E \otimes \mathfrak{D}_{\mathfrak{s}}^c$$

$$\mathfrak{D}_{(t,b)}\psi := \begin{pmatrix} \left(-\nabla_{\frac{d}{ds}} + i\mathbb{T} - it\right)\psi \\ B_p^\dagger \psi(p-) - b_p^\dagger \psi(p+) \\ -B_p^{c\dagger}\psi(p+) + b_p^{c\dagger}\psi(p-) \\ -Q_\lambda^\dagger \psi(\lambda) \end{pmatrix}$$

Index bundle over the level set
is $\mathscr{G}_{\mathfrak{s}}$ equivariant:

$$\ker \mathfrak{D}_{(t,b)}^\dagger \curvearrowleft \mathscr{G}_{\mathfrak{s}}$$
$$\downarrow$$
$$\mu^{-1}(\nu)$$

$\uparrow$

Quotient bundle $\mathscr{E} \to \mathrm{TN}_k^\nu$
with induced connection A
is an Instanton

Instanton Class in terms of Bow Rep. Ranks

1. rk $\mathscr{E}$ = dim $\ker \mathfrak{D}^{\dagger}_{(t,b)} = -\, ind \mathfrak{D}_{(t,b)} = |\Lambda|$.

2. As $|t| \to \infty$ solutions of the Bow Dirac eq. concentrate near the λ-points; so the induces connection $\hat{A}$ has the form

$$\hat{A}(\frac{\partial}{\partial\tau}) = \operatorname{diag}_{\lambda} \frac{-i}{V}\left(\lambda + \frac{\hat{m}_\lambda}{2t}\right) + O(t^{-2}),$$

where the magnetic charges are $\hat{m}_\lambda = R(\lambda+) - R(\lambda-) + |\{p\,|\,p < \lambda\}|$.

3. Chern numbers of the resulting connection are

$$\frac{i}{2\pi}\int_{C_p} \mathrm{tr} F_A = R(p+) - R(p-) - |\{\lambda\,|\,\lambda > p\}| + \sum_{\lambda} \frac{\lambda}{\ell},$$

$$\frac{1}{2}\left(\frac{i}{2\pi}\right)^2 \int_{\mathscr{M}} \mathrm{tr} F_{\hat{A}} \wedge F_{\hat{A}} = -\frac{1}{2}\sum_{\lambda} \hat{m}_\lambda - R_0 + \sum_{\lambda} \frac{\lambda}{\ell}\hat{m}_\lambda - \frac{k}{2}\sum_{\lambda}\left(\frac{\lambda}{\ell}\right)^2$$

“Two Dirac equations know each other.”

Schematic Relation:

Instanton Dirac Operator D_s
(parameterized by a $s \in$ Bow)

$$\nabla^* \nabla \hat{\chi} = c^\tau \Psi$$

Frame of $\ker \mathfrak{D}_{(t,b)}$ on the Bow

frame of $\ker D_s$ on TN

Bow Dirac Operator $\mathfrak{D}_{(t,b)}$
(parameterized by $(t, b) \in \mu_{\mathfrak{g}}^{-1}(\nu)$)

Given an orthonormal frame of $\ker D_s$

$$D_s \Psi_\alpha = 0$$

$$c^\tau = Cl(\frac{d\tau + \omega}{V}) = [D_s, i\frac{d}{ds}]$$

Consider bounded solution χ_α of Poisson eqs

$$\nabla_s^* \nabla_s \chi_\alpha = c^\tau \Psi_\alpha \quad \text{and} \quad \nabla_s^* \nabla_s \beta_\alpha = \frac{b_p}{2|t - \nu_p|} c^\tau \Psi_\alpha$$

$$D_s^2 \chi_\alpha = i\frac{d}{ds} \Psi_\alpha$$

$$\kappa^C := \chi_\alpha(s) \otimes \Psi_\alpha^*(s) \in S^+ \otimes \mathscr{E} \otimes e^* \otimes E_s^*$$

$$D_s \chi_\alpha = \frac{d}{ds} \Psi_\alpha + \Psi_\beta T_{\beta\alpha}^0$$

$$D_s I_j \chi_\alpha = -t^j \Psi_\alpha + \Psi_\beta T_{\beta\alpha}^j$$

$$\nabla_\mu^A \begin{pmatrix} \chi \\ \beta \\ f \end{pmatrix} = \mathfrak{D}_{(t,b)} I_\mu^\dagger c^\tau \Psi$$

“Down harmony Eq”

Let $\kappa^c = \begin{pmatrix} \chi \\ \beta \\ f \end{pmatrix}_\alpha \otimes \Psi_\alpha^\dagger$, then

κ solves the Bow Dirac equation

Analogous calculation using Up transform give schematic relation for $\hat{\chi} \in \ker \mathfrak{D}_{(t,b)}$:

$$\Delta_{Bow} \Psi = \hat{\chi}$$

“Up harmony Eq”

Completeness

"The heaven and the Earth are in harmony."

Up ∘ Down = $\mathrm{Id}_{\mathrm{Inst}}$

$(\mathscr{E}, A) \to (E; T, B, Q) \to (\hat{\mathscr{E}}, \hat{A}) \simeq_{gauge} (\mathscr{E}, A)$

Down ∘ Up = $\mathrm{Id}_{\mathrm{Bow}}$

$(E; T, B, Q) \to (\mathscr{E}, A) \to (\hat{E}, \hat{T}, \hat{B}, \hat{Q}) \simeq_{gauge} (E; T, B, Q)$

Consider $\kappa^c = \begin{pmatrix} \chi \\ \beta \\ f \end{pmatrix}_\alpha (t; s) \otimes \Psi^\dagger_\alpha(t'; s)$ is valued in $(S^+ \otimes \mathscr{E} \otimes L^*_s)\,|_t \otimes E^*_s$,

its charge conjugate κ is valued in $S^+ \otimes \mathscr{E}^*_t \otimes L_s \otimes E_s$

and thus gives a map $\kappa : \mathscr{E}\,|_t \to S^+ \otimes E \otimes \underline{e}^*$ with image in $\hat{\mathscr{E}} := \ker \mathfrak{D}_{(t,b)}$.

It maps a fiber of $\mathscr{E}$ at [(t,b)] to a fiber of $\hat{\mathscr{E}}$ at (t,b).

Bow index theorem => κ is bijective.

Down harmony eq => Fiber isometry and covariantly constant.

Isometry

Monopole case: Nakajima '93

Down harmony eq:

$$\nabla^A_\mu \hat{\chi} = \mathfrak{D}^r_{(t,b)} I^\dagger_\mu c^\tau \Psi$$

Its variation is

$$\dot{A}_\mu \hat{\chi} + \nabla^A_\mu \dot{\hat{\chi}} = \dot{\mathbb{T}}\, I^\dagger_\mu c^\tau \Psi + \mathfrak{D}^r_{(t,b)} I^\dagger_\mu c^\tau \dot{\Psi}$$

directly relates the tangent vector $\dot{A}$ to $\mathscr{M}_{\text{Instanton}}$
to the corresponding tangent vector $\dot{\mathbb{T}}$ of $\mathscr{M}_{\text{Bow}}$

Using 1) the fact that Ψ is a frame of $\ker D_s$,
2) $\hat{\chi}^c$ is a frame of $\ker \mathfrak{D}_{(t,b)}$, and
3) some quaternionic identities

leads to the isometry relation

$$\|\dot{A}\|^2_{ALF} = \|(\dot{T}, \dot{B}, \dot{Q})\|^2_{Bow}$$

Which Bow is Best?

c.f. Witten 0902.09481
(via brane considerations)

- Distinct holonomy eigenvalues $\{e^{2\pi i\mu_1}, e^{2\pi i\mu_2}, \dots, e^{2\pi i\mu_n}\}$
  split the instanton bundle (on a complement of a compact set) $\mathscr{E}_{\mathrm{TN}_k\setminus B} = W_1 \oplus W_2 \oplus \dots \oplus W_n$

Lemma: The eigenvalues have the form

$$\mu_j(\vec{t}) = \frac{\lambda_j}{l} + \frac{\vartheta_j}{2|\vec{t}|} + O(|\vec{t}|^{-2})$$

comparing to the asymptotic form of our connection $a_j = (\lambda_j + \frac{m_j}{2t})\frac{d\tau + \omega}{V} - \frac{m_j}{k}\omega\,.$

$$l\vartheta_j = m_j + \frac{\lambda_j}{l}k$$

This combination is Independent of any gauge choice!

- Consider the neighborhood of infinity: $\mathrm{TN}_k \setminus B$ contracts to the lens space $S^3/\mathbb{Z}_k$.

$$\begin{tikzcd}
\text{Pullback Bundle: } W_j^* \arrow[r] \arrow[d] & W_j \arrow[d] \\
\text{Covering space: } S^3 \arrow[r] & S^3/\mathbb{Z}_k
\end{tikzcd}$$

is trivial

Thus, what distinguishes different line bundles W_j is $\mathbb{Z}_k$ action on the fiber.
There are only k types of line bundles W_j.
Changing the trivialization of S^3 acts by $e^{i\tilde{\tau}/k}: \begin{pmatrix}\lambda_j \\ m_j\end{pmatrix} \mapsto \begin{pmatrix}\lambda_j + l \\ m_j - k\end{pmatrix}$

- Moral: line bundles W_j are determined by k numbers $\{\hat{m}_j \,|\, \hat{m}_j = m_j \bmod k, 0 \le \hat{m}_j < k\}$

Choice of Twisting Family

- First Chern class: k real numbers $$c_1^\sigma = \frac{1}{2\pi i}\int_{C_\sigma} \operatorname{tr} F = \sum_j \frac{\lambda_j}{l} - (\mathfrak{c}_\sigma + n\mathfrak{s}_\sigma), \quad \mathfrak{s}_\sigma \in \mathbb{Z}$$

$$\mathfrak{c}_\sigma \in \{0,1,\ldots,n-1\}$$

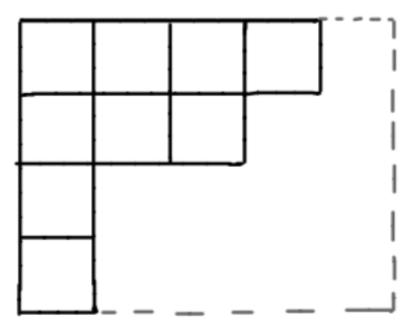
"Stiefel-Whitney classes" (obstructions of PSU(n) to SU(n) lifting)

Relabel the NUTs so that $0 \leq \mathfrak{c}_1 \leq \mathfrak{c}_2 \leq \ldots \mathfrak{c}_k < n$

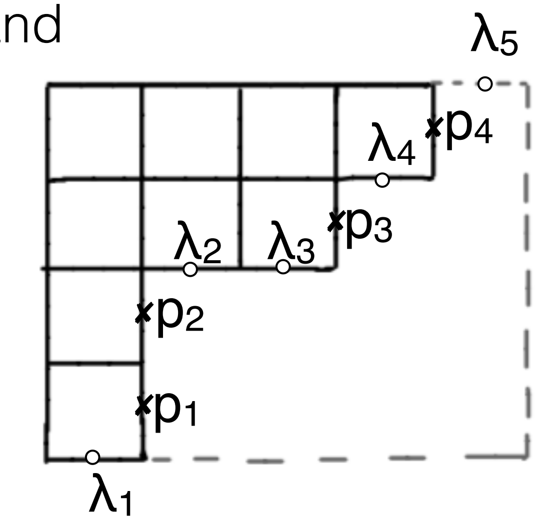
This labels the rows of a Young diagram that fits into an kxn rectangle

For example, for k=4, n=5 and $(\mathfrak{c}_1, \mathfrak{c}_2, \mathfrak{c}_3, \mathfrak{c}_4) = (1,1,3,4)$

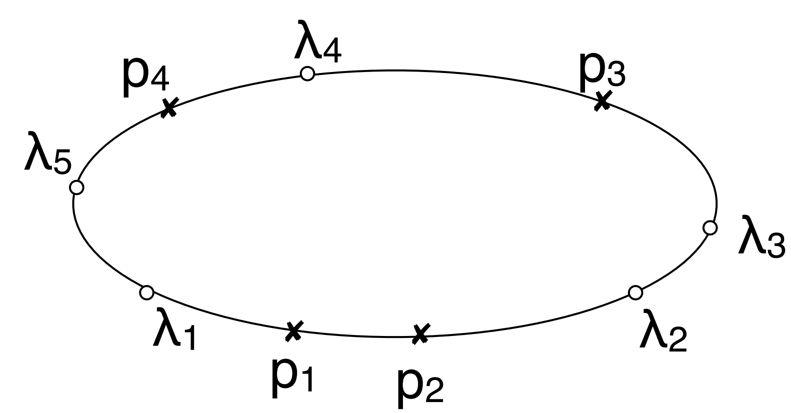
Y_{Bal}=



- View it as a closed path on an kxn torus.
  Mark
  - horizontal segments λ_j and
  - vertical segments p_σ

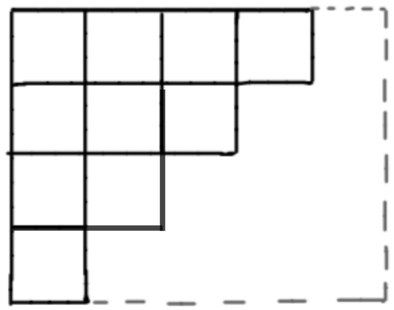


This gives the balanced bow representation (the rank is continuous at each p-point):



The set of magnetic charges also gives a Young diagram in nxk rectangle.

$$\left\{ \hat{m}_1, \hat{m}_2, \ldots, \hat{m}_n \,|\, \hat{m}_j \in \{1,2,\ldots,k-1\} \right\}$$

$Y_{\mathrm{Cob}}=$ 

Interior: (Chern numbers) Y_{Bal}

Infinity: (monopole charges) Y_{Cob}

Instanton Number

Since TN_k is not compact, Chern character value $ch_2[E,A]$ does NOT have to be integer.
Need another definition of the instanton number.

Let us focus on a single TN=TN_1:

TN≃$\mathbb{R}^4$ is contractible, so any bundle over it is trivial,
and connection one-form A is globally defined.

1. On a complement of a compact set, a gauge transformation $g : \mathrm{TN}\backslash B \to U(n)$

   transforms A to the diagonal form

$$A^g = -i\,\mathrm{diag}\begin{pmatrix} \lambda_1 + m_1 l & & & \\ & \lambda_2 + m_2 l & & \\ & & \cdots & \\ & & & \lambda_n + m_n l \end{pmatrix} \frac{d\tau + \omega}{V} + O(\frac{1}{t^2})\,.$$

   TN\B is contractible to S^3, thus the homotopy class of g is in $\pi_3(U(n)) = \mathbb{Z}$.

   Instanton Number is $m_0 := \deg\,[g] \in \pi_3(U(n))\,.$

2. Alternatively, holonomy splits the instanton bundle into orthogonal line bundles,
   giving a map $S^3_\infty \to U(n)/U(1)^n = N_n$

   Flag space

   Instanton Number is an element of $\pi_3(U(n)/U(1)^n) = \mathbb{Z}\,.$

Summary

- Quivers, Bows, Slings, and Walls give constructions of Instantons on ALE, ALF, ALG, and ALH tesserons.
- Down transform: Instanton Dirac index bundle on the bow.
  (The main problem is to identify ALL the necessary data to have a complete construction.)
- Up transform: Bow Dirac index bundle.
- Completeness: $\mathrm{Up} \circ \mathrm{Down} = \mathrm{Id}_{\mathrm{Inst}}$ and $\mathrm{Down} \circ \mathrm{Up} = \mathrm{Id}_{\mathrm{Bow}}$
- Isometry: $\mathscr{M}_{\mathrm{Inst}} = \mathscr{M}_{\mathfrak{R}}$.