Symplectic Transformation

$$U : \mathcal{H}^{\mathcal{X}} \longrightarrow \mathcal{H}^{Y}, \quad \mathcal{X} = \mathbb{P}(1,1,1,3), \quad Y = \mathbb{F}_3$$

$$U = \begin{array}{c|c} & \begin{array}{cccccc} 1 & p & p^2 & p^3 & \mathbb{1}_{2/3} & \mathbb{1}_{1/3} \end{array} \\ \begin{array}{c} 1 \\ p_2/3 \\ p_2^2/9 \\ [\mathbb{P}^2] \\ \text{line} \\ \text{top} \end{array} & \left(\begin{array}{ccc|ccc} 1 & & & & & \\ & 1 & & & 0 & \\ & & 1 & & & \\ \hline 0 & 0 & 0 & 0 & \boxed{\alpha z} & \frac{\sqrt{3}\beta}{\pi} \\ -\frac{\pi^2}{3z^2} & 0 & 0 & 0 & \frac{\pi\alpha}{\sqrt{3}} & -\frac{\beta}{z} \\ \frac{8\zeta(3)}{z^3} & 0 & 0 & 1 & \frac{\pi\alpha}{3z} & \frac{\pi\beta}{\sqrt{3}z^2} \end{array}\right) \end{array}$$

positive powers in z!

exceptional set.

$$\alpha = \frac{2\sqrt{3}\pi}{3\Gamma(\frac{1}{3})^3}, \quad \beta = \frac{2\pi^2}{3\Gamma(\frac{2}{3})^3}, \quad \alpha\beta = \frac{1}{2}$$

Non-linear change of variables

$$J^{Y}(\tau, -z) = \mathbb{U}\, J^{\mathcal{X}}(t, -z)$$

$$\tau = \tau(t), \quad \tau = \tau_1 p_1 + \tau_2 p_2 \in H^2(Y)$$

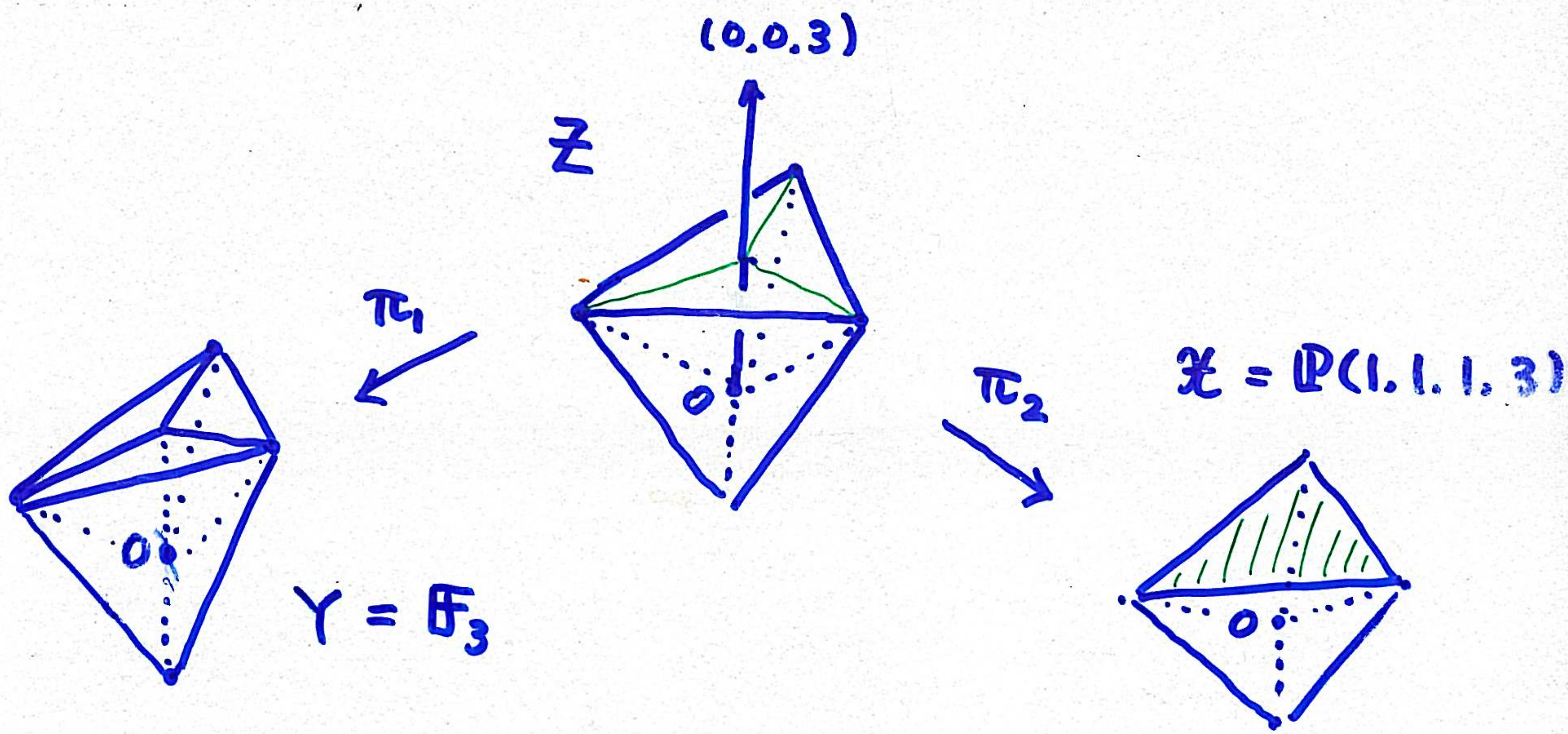
$$t = t_1 \mathbb{1}_{1/3} + t_2 (3p) \in H^2_{orb}(\mathcal{X})$$

$$\Rightarrow \begin{cases} \tau_1 = -\dfrac{\sqrt{3}\beta}{\pi} t_1 + 3\alpha \dfrac{\partial F^{\mathcal{X}}_{orb}}{\partial t_1} \\ \\ \tau_2 = t_2 + \dfrac{\beta}{\sqrt{3}\pi} t_1 - \alpha \dfrac{\partial F^{\mathcal{X}}_{orb}}{\partial t_1} \end{cases}$$

where $3 \dfrac{\partial F^{\mathcal{X}}_{orb}}{\partial t_1} = \dfrac{1}{2} t_1^2 - \dfrac{t_1^5}{3^2 \cdot 5!} + \dfrac{t_1^8}{3 \cdot 8!} - \dfrac{1093}{3^5\, 11!} t_1^{11} + \cdots$

↑ genus 0 orbifold GW potential

① · Mirror Explanation of Borisov-Horja

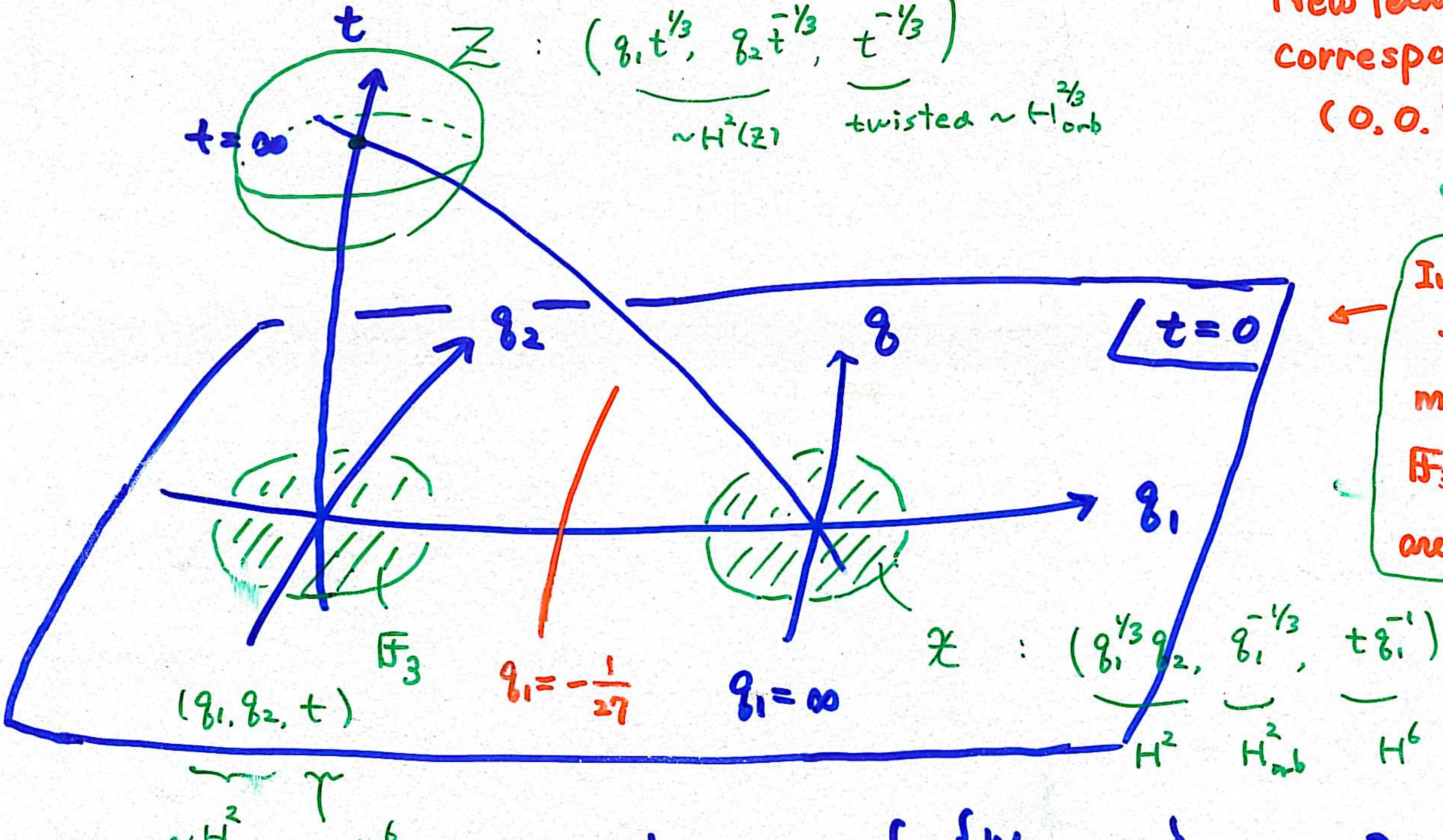


locally $\quad \mathcal{O}_{\mathbb{P}^2}(-3) \xleftarrow[\pi_1]{} \mathcal{O}_{\mathbb{P}^2}(-1)/\mathbb{Z}_3 \xrightarrow[\pi_2]{} \mathbb{C}^3/\mathbb{Z}_3$

$\overline{Y} \cong \overline{Z}$ as varieties

② <u>Mirror</u> $$W_{q_1,q_2,t} = x_1 x_3 + x_2 x_3 + \frac{x_3 q_1}{x_1 x_2} + \boxed{t\, x_3^3} + \frac{q_2}{x_3}$$

New term corresponding to (0, 0, 3)

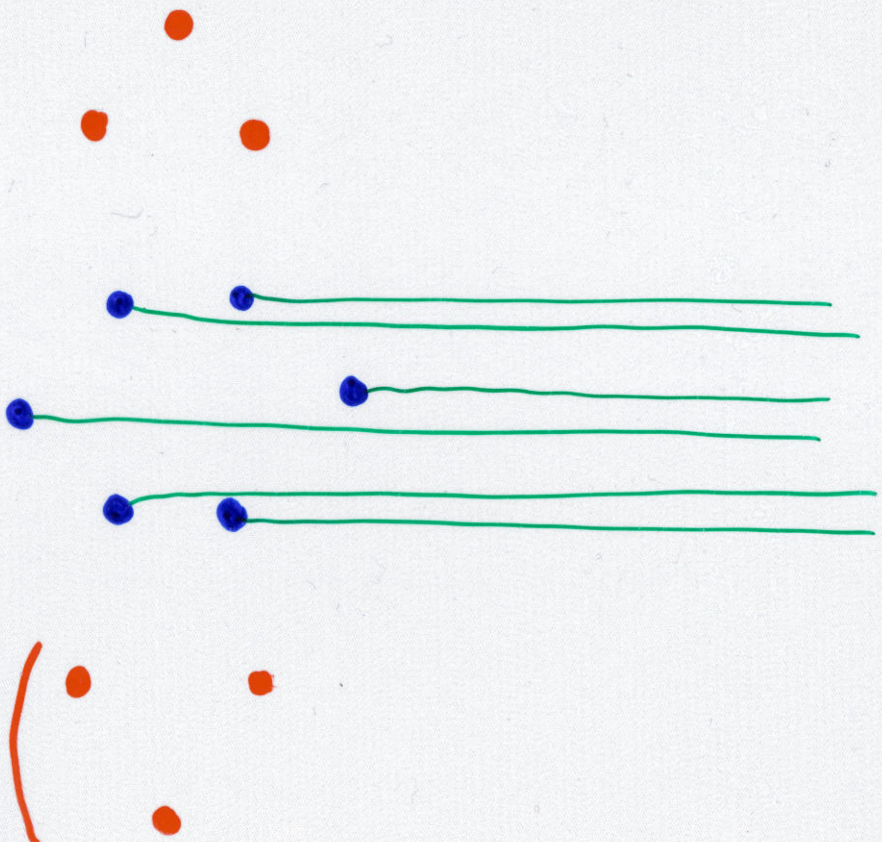


In the limit $t \to 0$, mirrors of $\mathbb{F}_3$ or $\mathcal{X}$ are recovered.

• moduli space of $\{W_{q_1,q_2,t}\}$ and 3 charts

③ Critical Values of $W_{q_1, q_2, t}$ near $t \sim 0$

12 crit values $<$ 6 converges / 6 diverges as $t \to 0$



$\sim 2\sqrt{q_2(1+3q_1^{1/3})}$ (

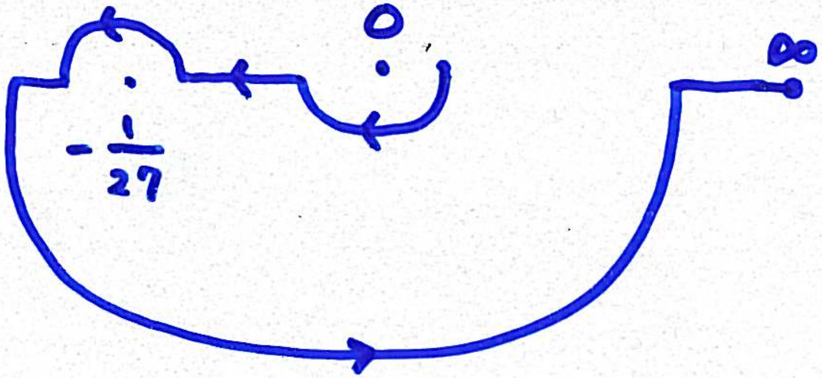
$\sim \dfrac{-2i}{3\sqrt{3t}}(1+3q_1^{1/3})^{3/2}$ (

divergent

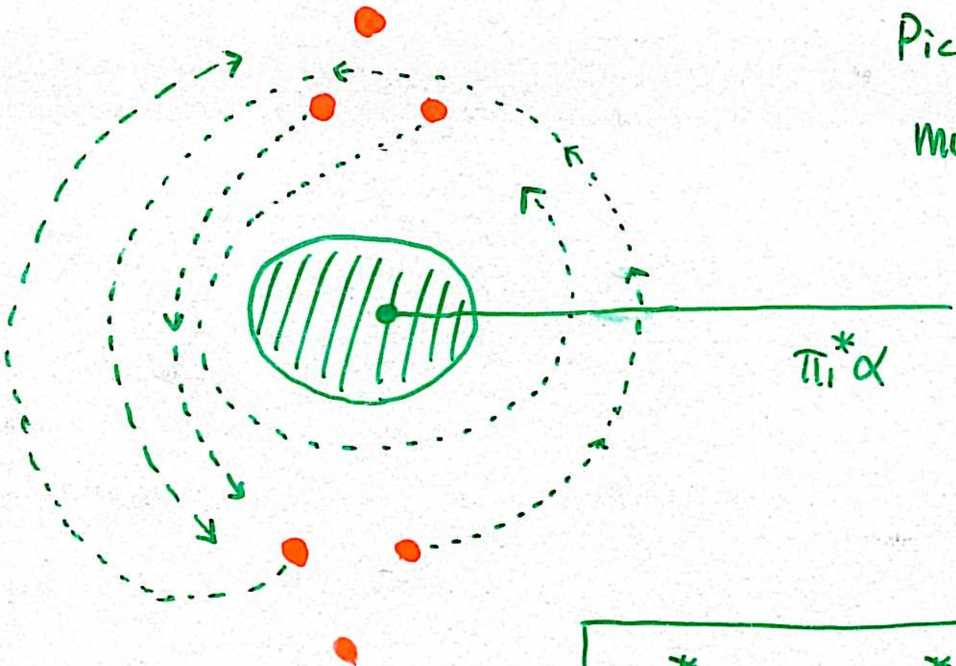
Lefschetz thimbles corresponding to $\pi_1^* K^0(\mathbb{F}_3)$ (if q_1, q_2 small)

or to $\pi_2^* K^0(\mathcal{X})$ (if $q_1^{-1/3}, q_1^{1/3} q_2$ small)

④ # Parallel Transport of Thimbles along the path



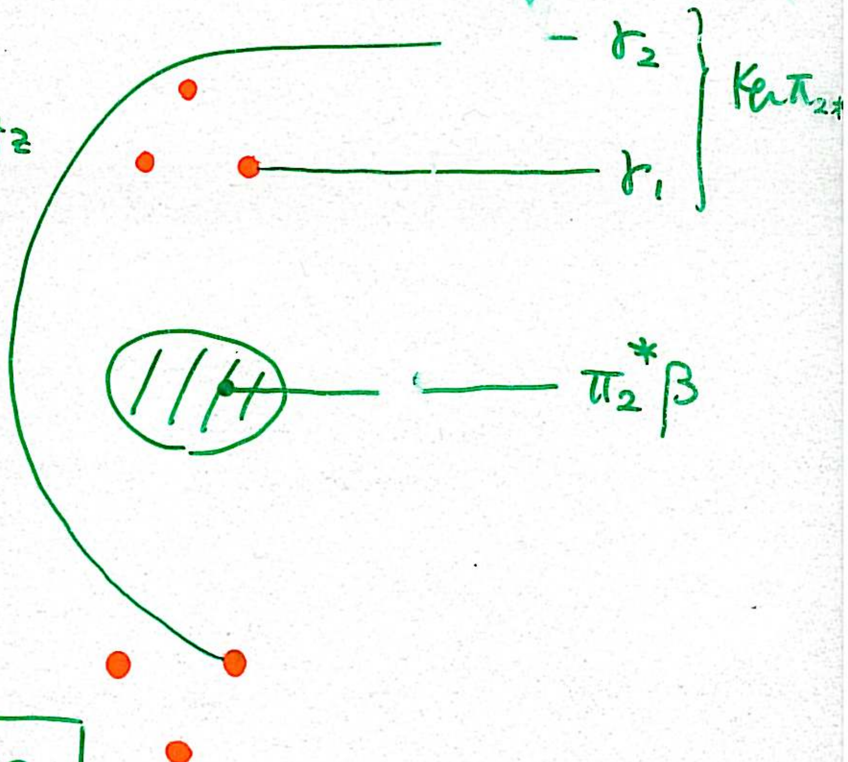
on the q_1-plane



Picard-Lefschetz monodromy

$\Longrightarrow$

$q_1 \to \infty$



$$\boxed{\pi_1^*\alpha = \pi_2^*\beta + \gamma_1 + \gamma_2}$$

$$\Longrightarrow \quad \beta = \pi_{2*}\pi_1^*\alpha$$

⑤ • Higher Genus Story

Givental's Quantization

Genus 0 : $(\mathcal{H}, \mathcal{L} = \text{graph of } d\mathcal{F}_0)$

$\rightsquigarrow$

All genera : $\left(\mathrm{Fock}(\mathcal{H}, \mathcal{H}_-), \exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}_g \right) \right)$

$\mathrm{Fock}(\mathcal{H}, \mathcal{H}_-)$ = "functions" on the quotient space $\mathcal{H}/\mathcal{H}_-$ (geometric quantization)

$\exp\left(\sum_{g=0}^{\infty} \hbar^{g-1} \mathcal{F}_g \right) = \mathcal{D}_X$: total descendant potential

• $\mathbb{U} : \mathcal{H}^{\mathcal{X}} \longrightarrow \mathcal{H}^Y$ should induce

$\widehat{\mathbb{U}} : \mathrm{Fock}(\mathcal{H}^{\mathcal{X}}, \mathcal{H}_-^{\mathcal{X}}) \longrightarrow \mathrm{Fock}(\mathcal{H}^Y, \mathcal{H}_-^Y)$

Thm (Coates-I-Tseng ; Coates-I).

Assume
- $g=0$ CRC holds for $Y \to \overline{\mathcal{X}}$
- Quantum cohomology of Y & $\mathcal{X}$ are semisimple

$$\Rightarrow \quad \hat{U}\,\mathcal{D}_{\mathcal{X}} = \mathcal{D}_{Y}$$ in a "suitable sense"

definition of $\mathrm{Fock}(\mathcal{H}, \mathcal{H}_-)$ is the real problem.

e.g. $\mathcal{X} = \mathbb{P}(1,1,1,3)$, $Y = \mathbb{F}_3$

Ingredients :
- Givental's formula for $\mathcal{D}_{\mathcal{X}}, \mathcal{D}_{Y}$
- Teleman's classification of semisimple theories.

⑦ • Local limit & Modularity

Recall $\mathbb{F}_3 \supset \mathcal{O}_{\mathbb{P}^2}(-3)$

$\mathbb{P}(1.1.1.3) \supset \mathbb{C}^3/\mathbb{Z}_3$

} local Calabi-Yau's

$(\mathcal{H}, \mathcal{L}) \xrightarrow[q_2 \to 0]{} (\mathcal{H}_{fin}, \mathcal{L}_{fin})$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \| $

$\qquad\qquad\qquad\qquad\qquad\qquad\quad H^*(\mathbb{F}_3)$

$\frac{\infty}{2}$ VHS

+

real structure

$\mathcal{H}_{\mathbb{R}}$

(the volume of fibers $\to \infty$)

finite dimensional polarized VHS

mirror $\cong$

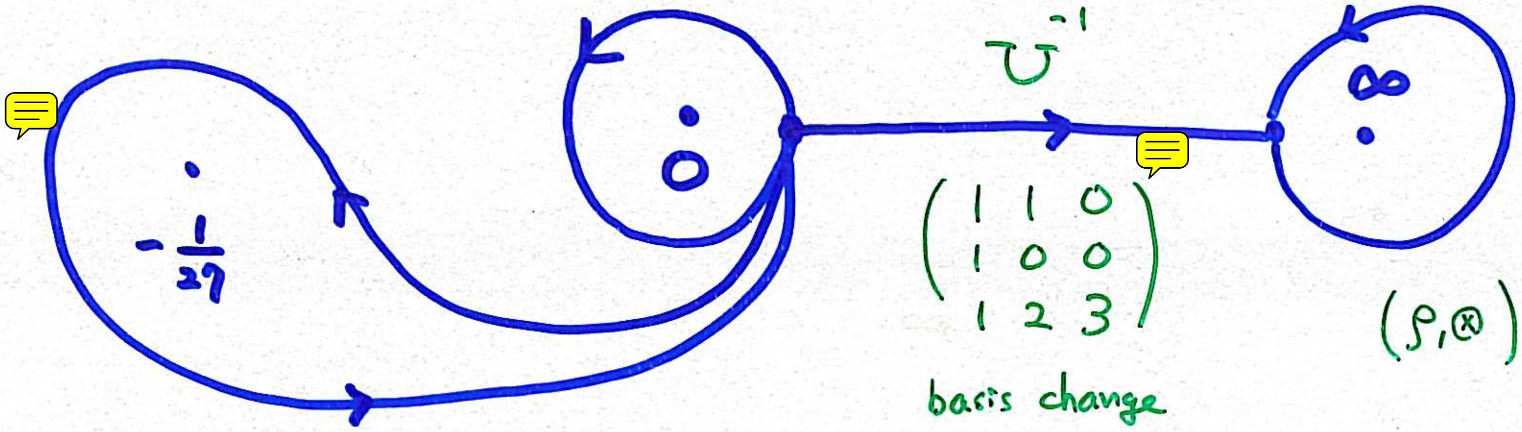
PVHS of family of elliptic curves

• Potentials & quantization $\hat{U}$ descends to the finite dim'al situation.

• Monodromy group of finite dim'al VHS

(in the integral basis)

$$(\otimes \mathcal{O}(-1)) = \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix}$$



$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 2 & 3 \end{pmatrix}$$

basis change

$$(\rho_1 \otimes) = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -3 & -6 \\ 0 & 7 & 12 \\ 0 & -3 & -5 \end{pmatrix} = \text{twist by } \mathcal{O}_{\mathbb{P}^2}(-2)$$

• Basis $\{\mathcal{O}_0, \mathcal{O}_0 \otimes \rho_1, \mathcal{O}_0 \otimes \rho_2\}$

• Basis $\{\mathcal{O}_{pt}, \mathcal{O}_{\ell}, \mathcal{O}_{\mathbb{P}^2}\}$

• Monodromy group

$$\Gamma_0(3) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in PSL_2(\mathbb{Z}) : c \equiv 0 \mod 3 \right\}$$

9 **Thm** (Aganagic - Bouchard - Klemm : CI work in progress)

The Gromov - Witten potentials of $\mathcal{O}_{\mathbb{P}^2}(-3)$ are quasi - modular functions for $\Gamma_0(3)$.

Ingredients : ① the real structure $\mathcal{H}_{fin,\mathbb{R}}$ defines a canonical polarization $\overline{T_\tau}$

$$T_\tau \oplus \overline{T_\tau} = \mathcal{H}_{fin} \quad \text{(Hodge decomposition)}$$

② The potential $F_g(\tau, \bar\tau)$ w.r.t the polarization $\overline{T_\tau}$ is modular function for $\Gamma_0(3)$ (not holomorphic)

③ $F_g(\tau) \longleftrightarrow F_g(\tau, \bar\tau)$ related by the change of polarizations.