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2024 BICMR-IBSCGP, Conference on Gromov-Witten Theory and Related Topics

Quantum cohomology of blowups

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(based on arXiv:2307.13555)

Related joint work:

[I-Koto] *Quantum cohomology of projective bundles*, arXiv:2307.03696

[I-Sanda] in preparation

Talk Plan:

1. Decomposition of quantum cohomology from birational geometry
   — decomposition associated with an extremal ray
2. Decomposition theorem for blowups
3. Teleman's conjecture (D-module version)
4. Proof via Fourier analysis of equivariant quantum cohomology

Usage of Colours

Blue: keywords

Red: important

Magenta: to be explained on the whiteboard

Yellow: notes for myself

§1. Decomposition of quantum cohomology from birational geometry

(See [Galkin-I-Hu-Ke-Li-Su, §6] *Counter-examples to Gamma conjecture I, arXiv:2405.16979*)

Small quantum cohomology

$$\mathrm{QH}(X) = (H^*(X) \otimes \mathbb{C}[\![Q]\!], \star)$$

sometimes decomposes as a ring (after an extension of the Novikov ring $\mathbb{C}[\![Q]\!]$).

Example:

$$\mathrm{QH}(\mathbb{P}^{r-1}) \cong \mathrm{QH}(\mathrm{pt})^{\oplus r}$$

Mori cone: $\overline{\mathrm{NE}}(X) \subset H^2(X, \mathbb{R})$

Extremal ray is a 1-dimensional face (edge) $R \subset \overline{\mathrm{NE}}(X)$ whose generator $d_0 \in R$ satisfies $c_1(X) \cdot d_0 > 0$ (we choose d_0 to be primitive integral).

$\exists$ extremal contraction $f\colon X \to Y$ such that $C \subset X$ is contracted to a point $\iff [C] \in R$.

Exceptional quantum product $\star_{\mathrm{exc}}$ is defined by

$$(\alpha \star_{\mathrm{exc}} \beta, \gamma) = \sum_{n \geq 0} \langle \alpha, \beta, \gamma \rangle_{0,3,nd_0}^{X} Q^{nd_0}$$

— the sum in the right-hand side is finite.

The exceptional quantum cohomology $\mathrm{QH}_{\mathrm{exc}}(X) = (H^*(X) \otimes \mathbb{C}[q], \star_{\mathrm{exc}})$ can be defined over the polynomial ring $\mathbb{C}[q]$, where $q = Q^{d_0}$.

$$\mathrm{QH}(X) \underset{\text{deformation}}{\overset{\text{specialization}}{\rightleftarrows}} \mathrm{QH}_{\mathrm{exc}}(X)$$

decomp of $\mathrm{QH}_{\mathrm{exc}}(X)|_{q=1}$ (finite dimensional $\mathbb{C}$-algebra)
$\rightsquigarrow$ induces a decomp of $\mathrm{QH}(X)$.

Question: What is a geometric meaning of each summand of the decomposition of $\mathrm{QH}(X)$ (associated with an extremal ray)?

Example 1 [Gonzalez-Woodward, Lee-Lin-Wang, I]. Let $X \dashrightarrow X^+$ be a toric flip. $\mathrm{QH}(X)$ contains big-$\mathrm{QH}(X^+)$ as a direct summand.

Example 2 [I-Koto] Let $V \to Y$ be a vector bundle of rank r. The projective bundle $\mathbb{P}(V) \to Y$ is an extremal contraction and we have $\mathrm{QH}(\mathbb{P}(V))_\tau \cong \bigoplus_{i=1}^{r} \mathrm{QH}(Y)_{\sigma_i(\tau)}$.

§2. Decomposition theorem for blowups.

X: smooth projective variety
$Z \subset X$: subvariety of codimension r
$\widetilde{X} = \mathrm{Bl}_Z(X)$: blowup of X along Z

Theorem:

$$\mathrm{QH}(\widetilde{X})_{\tilde{\tau}} \cong \mathrm{QH}(X)_{\tau(\tilde{\tau})} \oplus \bigoplus_{i=1}^{r-1} \mathrm{QH}(Z)_{\sigma_i(\tilde{\tau})}$$

This decomposition lifts to (formal) quantum D-modules. (F-manifold decomp, Euler eigenvalues).

Problems/Applications

- Relative $\widehat{\Gamma}$-conjecture: relate this result to an SOD of derived categories such as:

$$D^b(\widetilde{X}) \cong \langle D^b(X), D^b(Z), \ldots, D^b(Z) \rangle$$

- announced by [Katzarkov-Kontsevich-Pantev-Yu]
  - Application to rationality question: irrationality of generic cubic fourfolds
  - Birational Calabi-Yaus have the same (quantum) cohomology (Batyrev, McLean)

§3. Teleman's conjecture (D-module version)

Let W be a smooth proj variety with T-action.

$$\mathrm{QH}_T(W) \quad \underset{\text{Fourier}}{\Longleftrightarrow} \quad \mathrm{QH}(W/\!\!/_t T)$$

equiv para λ	quantum conn $z\nabla_{q\partial_q}$
shift op $\mathbb{S}$	Kähler para $q = e^{-t}$
$[\lambda, \mathbb{S}] = z\mathbb{S}$	$[z\nabla_{q\partial_q}, q] = zq$

- $\mathbb{S}$hift op $=$ Seidel rep $=$ Stability para
- We apply this to $W = \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{P}^1)$

Shift operator:

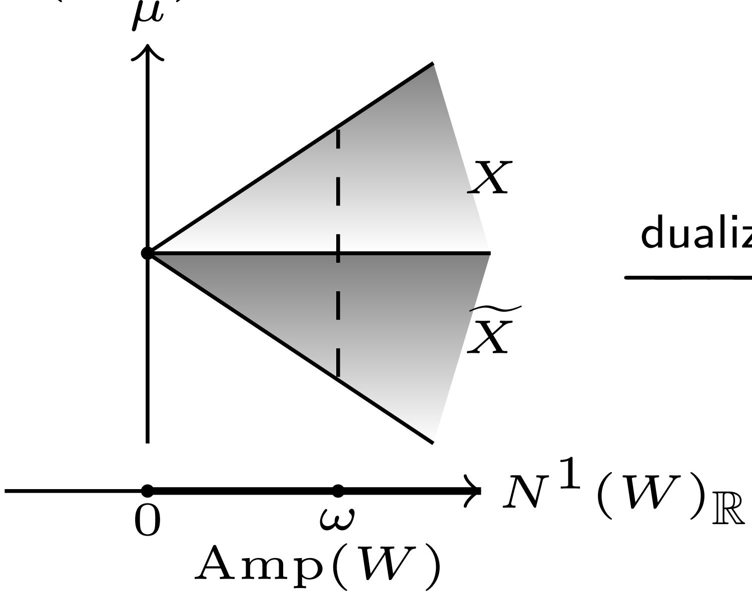
- The action $\mathrm{Hom}(\mathbb{C}^\times, T) \curvearrowright \mathcal{L}W$: free loop space
  $\gamma(e^{\mathtt{i}\theta}) \mapsto k(e^{\mathtt{i}\theta})\gamma(e^{\mathtt{i}\theta})$ ($k \in \mathrm{Hom}(\mathbb{C}^\times, T)$, $\gamma \in \mathcal{L}W$)
  induces $\mathrm{Hom}(\mathbb{C}^\times, T) \curvearrowright \mathrm{QH}_T(W)[Q^{-1}]$ (shift op)
- More precisely, we have the action
  $H_2^T(W, \mathbb{Z}) \curvearrowright \mathrm{QH}_T(W)[Q^{-1}]$:

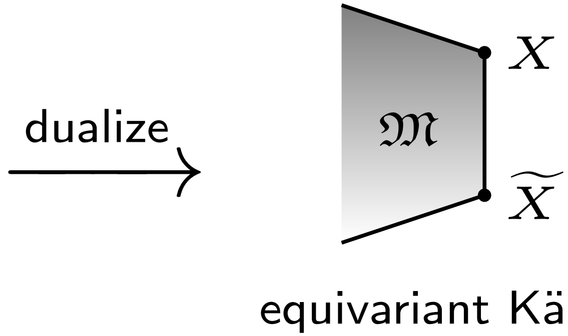
$$
\begin{array}{ccccccccc}
0 \to & H_2(W,\mathbb{Z}) & \to & H_2^T(W,\mathbb{Z}) & \to & H_2^T(\mathrm{pt},\mathbb{Z}) & \to 0 \\
 & \cup & & \cup & & \| & \\
 & \mathrm{NE}_{\mathbb{N}}(W) & \longrightarrow & \mathrm{NE}^T_{\mathbb{N}}(W) & \longrightarrow & \mathrm{Hom}(\mathbb{C}^\times, T) &
\end{array}
$$

Mori — equiv Mori — cocharacters

“Equivairant” Kähler moduli space $\mathfrak{M}$:

- Equivariant Mori cone $\overline{\mathrm{NE}}^T(W)$ is dual to the T-ample cone [Dolgachev-Hu, Thaddeus]
  $C_T(W) = \{\omega \in N^1_T(W) : W_{\mathsf{st}}(\omega) \neq \emptyset\}$
- $C_T(W)$ has the wall and chamber structure





$\mathrm{QH}_T(W)$ is a sheaf over $\mathfrak{M}$

§4. Proof via Fourier analysis of equivariant quantum cohomology

Construct Fourier transformations:

$$
\begin{array}{ccccc}
 & & \mathrm{QH}_T(W) & & \\
 & \swarrow^{F_{\widetilde{X}}} & \downarrow{\scriptstyle F_Z^i} & \searrow^{F_X} & \\
\mathrm{QH}(\widetilde{X}) & & \mathrm{QH}(Z) & & \mathrm{QH}(X)
\end{array}
$$

$(i = 1, \dots, r-1)$ and show that

- $F_{\widetilde{X}}$ is an isomorphism
- $F_X \oplus F_Z^1 \oplus \cdots \oplus F_Z^{r-1}$ is an isomorphism

Discrete Fourier Transformation for GIT quotients

Reduction conjecture [I-Sanda]: The discrete Fourier transformation I of the equivariant J-function J_W of W lies in the Givental cone of the smooth GIT quotient $W /\!\!/_t T$.

$$I := \sum_{k \in \mathrm{Hom}(\mathbb{C}^\times, T)} \kappa(\mathcal{S}^{-k}(J_W))q^k$$

Example: $\mathbb{P}^{r-1} = \mathbb{C}^r /\!\!/ \mathbb{C}^\times$, $\mathrm{pt} = \mathbb{P}^1 /\!\!/ \mathbb{C}^\times$.

Continuous Fourier transformations for fixed components

<u>Prop</u> (follows from [Coates-Givental]). Let $F \subset W$ be a T-fixed component and set $G_F := \prod_\varrho \frac{1}{\sqrt{-2\pi z}}(-z)^{-\varrho/z}\Gamma(-\varrho/z) \in H^*(F)$ (where ϱ ranges over Chern roots of $\mathcal{N}_{F/W}$). The formal stationary phase asymptotics $\mathscr{I}$ (as $z \to 0$)

$$\int J_W|_F \cdot G_F\, e^{\lambda \log q/z} d\lambda \sim \sqrt{2\pi z}\, e^{u/z} \mathscr{I}$$

lies in the Givental cone of F.

Example: $W = \mathbb{C}^r$ with scalar $T = \mathbb{C}^\times$-action. $J_W = 1$. The continuous Fourier transformation associated with the fixed point $0 \in \mathbb{C}^r$ is the Mellin-Barnes integral

$$\text{const.} \int 1 \cdot (-z)^{-r\lambda/z} \Gamma(-\lambda/z)^r e^{\lambda \log q/z} d\lambda$$

This has r many asymptotic expansions corresponding to r many critical points.

$$\rightsquigarrow \mathrm{QH}(\mathbb{P}^{r-1}) \cong \mathrm{QH}_T(\mathbb{C}^r) \cong \mathrm{QH}(\mathrm{pt})^{\oplus r}.$$

Comparison between discrete and continuous Fourier transformations:

Example: $\mathbb{C}^r /\!\!/ \mathbb{C}^\times$. By residue calculations,

$$\frac{1}{-2\pi \mathtt{i} z} \underbrace{\int (-z)^{-r\lambda/z} \Gamma(-\lambda/z)^r e^{\lambda \log q/z} d\lambda}_{\text{continuous FT}}$$

$$= \left(\widehat{\Gamma}_{\mathbb{P}^{r-1}}, (-z)^{c_1} (-z)^{\frac{\deg}{2}} \underbrace{J_{\mathbb{P}^{r-1}}(q,z)}_{\text{discrete FT}} \right)$$

$$\left(\underset{\text{mirror}}{=} \int e^{W(x)/z} \frac{dx_1 \cdots dx_{r-1}}{x_1 \cdots x_{r-1}} \right)$$

Thank you for your attention!