Discontinuity of straightening maps

Hiroyuki Inou

Kyoto University & Université de Toulouse

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Contents

Notations

Let $d \geq 2$.

- $\mathrm{Poly}_d = \{f(z) = z^d + a_{d-2}z^{d-2} + \cdots + a_0\}$.
- $K(f) = \{z;\ \{f^n(z)\}_{n \leq 0} : \text{bdd}\}$: filled Julia set.
- $J(f) = \partial K(f)$: Julia set.
- $\mathcal{C}_d = \{f \in \mathrm{Poly}_d;\ K(f) : \text{connected}\}$: connectedness locus.

Polynomial-like mappings

Definition (Polynomial-like mapping)

A map $f : U' \to U$ is polynomial-like mapping if

- f is proper holomorphic map of degree ≥ 2,
- $U' \Subset U \subset \mathbb{C}$: topological disks.
- $K(f) = \bigcap_{n \geq 0} f^{-n}(U')$: filled Julia set.
- $J(f) = \partial K(f)$: filled Julia set.

Definition (Hybrid equivalence)

Two polynomial-like mappings $f : U' \to U$, $g : V' \to V$ of the same degree are hybrid equivalent if there exists a qc conjugacy ψ defined between neighborhoods of $K(f)$ and $K(g)$ such that $\bar{\partial}\psi \equiv 0$ a.e. on $K(f)$.

Straightening theorem

An example of polynomial-like mapping is a polynomial restricted to a proper sufficiently large domain.

Theorem (Straightening theorem (Douady-Hubbard))

A polynomial-like mapping $f : U' \to U$ is hybrid equivalent to some polynomial g of the same degree. Moreover, if $K(f)$ is connected, then g is unique up to affine conjugacy.

If $K(f)$ is connected, then we call g (more precisely, its affine conjugacy class) the straightening of f and denote by $g = \chi(f)$.

Theorem (Douady-Hubbard)

χ is continuous for any quadratic-like (degree 2 polynomial-like) family, but not continuous when degree ≥ 3 in general.

Quadratic family

Let $Q_c(z) = z^2 + c$.

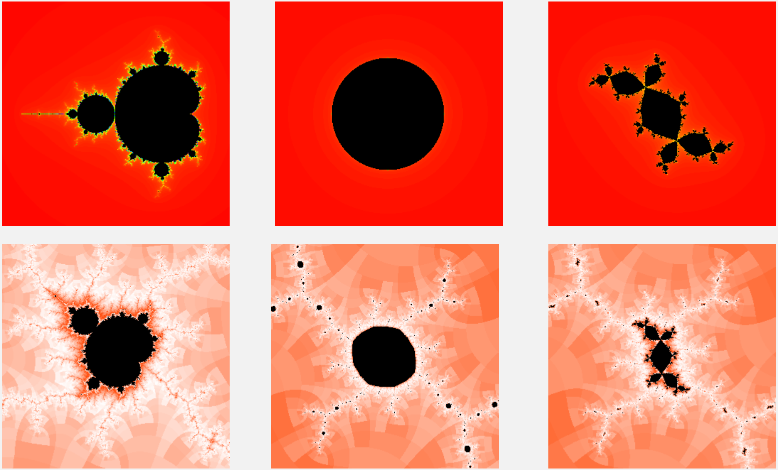
Definition

We say Q_c is *n*-renormalizable if there exist topological disks $0 \in U' \Subset U$ such that $Q_c^n : U' \to U$ is a quadratic-like mapping with connected filled Julia set.

Let c_0 be a parameter such that the critical point 0 is periodic of period n for Q_{c_0} (center of a hyperbolic component). Then there exists a small copy of the Mandelbrot set M_{c_0} such that

- any $c \in M_{c_0}$ is n-renormalizable except at most one point;
- $\chi : M_{c_0} \to M$ is (more precisely, extends to) a homeomorphism s.t. $\chi(c_0) = 0$;
- exceptional point c_1 (if exists) satisfy $\chi(c_1) = 1/4$ (root of M_{c_0}).

Renormalization of quadratic polynomials



Combinatorial renormalization

Definition (Rational lamination)

The rational lamination λ_f of f is the landing relation of external rays of rational angles:
$\theta, \theta' \in \mathbb{Q}/\mathbb{Z}$ are λ_f-equivalent if the external rays $R_f(\theta)$ and $R_f(\theta')$ land at the same point.

Definition (Combinatorial renormalization)

Let f_0 be a center (of a hyperbolic component, i.e., post-critically finite and hyperbolic). We say f is f_0-combinatorially renormalizable if $\lambda_f \supset \lambda_{f_0}$. Let

$$\mathcal{C}(f_0) = \{f : \lambda_f \supset \lambda_{f_0}\}.$$

In the quadratic case, we have

$$\mathcal{C}(Q_{c_0}) = M_{c_0}.$$

Cubic polynomials

Let f_0 be a center, i.e., the critical points ω_1, ω_2 are eventually periodic and lie in the Fatou set.
Such cubic polynomials f_0 are divided into four types:

Adjacent type $\omega_1 = \omega_2$ is periodic. $f_0^n(\omega_1) = \omega_1$.

Bitransitive type $\omega_1 \neq \omega_2$ and they lie in the same periodic orbit.

$f_0^n(\omega_1) = \omega_2$ and $f_0^k(\omega_1) = \omega_2$ for $0 < \exists k < n$.

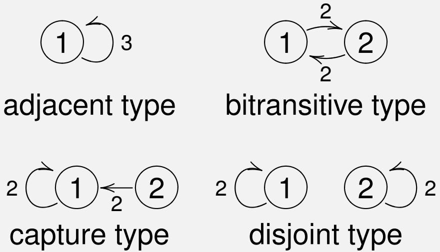
Capture type One of the critical points, say ω_1 is periodic, and ω_2 is not periodic, but preperiodic.

$f_0^n(\omega_1) = \omega_1$ and $f_0^k(\omega_2) = \omega_1$ for $\exists k > 0$.

Disjoint type ω_1 and ω_2 are periodic but lie in different periodic orbits.

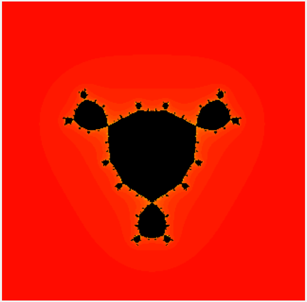
$f^{n_1}(\omega_1) = \omega_1$ and $f^{n_2}(\omega_2) = \omega_2$.

They can be described in terms of "reduced mapping schemata" introduced by Milnor[1]:



[1] Here we do not count the number of critical points(=weights), but we consider the degree of each map(=degree).

A:



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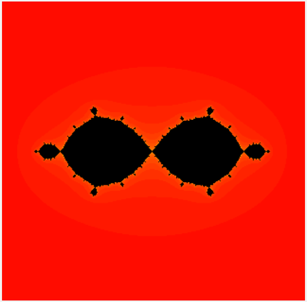

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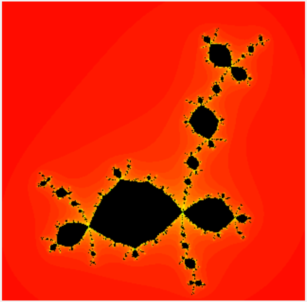

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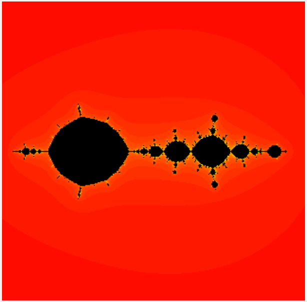

Straightening of renormalizable cubic polynomials

We can consider $\mathcal{C}(f_0)$ and define f_0-renormalizability for $f \in \mathcal{C}(f_0)$.
For a Fatou component Ω of f_0, define $K_f(\Omega)$ as follows:

$$K_f(\Omega) = \bigcap_{\theta_1,\theta_2:\ \lambda_{f_0}\text{-equiv.}} \overline{S_f(\theta_1,\theta_2,\Omega)} \cap K(f)$$

where $S_f(\theta_1,\theta_2,\Omega)$ is the component of $\mathbb{C} \setminus \overline{R_f(\theta_1) \cup R_f(\theta_2)}$ containing $R_f(\theta)$ if and only if $R_{f_0}(\theta)$ and Ω are contained in the same component of $\mathbb{C} \setminus \overline{R_{f_0}(\theta_1) \cup R_{f_0}(\theta_2)}$.

Let Ω_1 and Ω_2 be the Fatou component for f_0 containing the critical points ($\Omega_1 = \Omega_2 \Leftrightarrow f_0$: adjacent).

Adjacent type $\exists f^n : U' \to U$ with $K = K_f(\Omega_1) = K_f(\Omega_2)$, hybrid equivalent to a cubic polynomial g.

Bitransitive type $\exists f^n : U' \to U$ with $K = K_f(\Omega_1)$, and $f^k(K) = K_f(\Omega_2)$ for some $0 < k < n$. It is hybrid equivalent to a biquadratic polynomial $g = g_1 \circ g_2$ where $g_i = Q_{c_i}$ for some c_i.

In those cases, let us define the straightening map χ_{f_0} by $\chi_{f_0}(f) = g$.

Capture type $\exists f^n : U' \to U$ with $K = K_f(\Omega_1)$, hybrid equivalent to some Q_c, and

$$f^k(\omega_2) \in K(f^n : U' \to U)$$

corresponds to some point $z \in K(Q_c)$ by a hybrid conjugacy[2]. The straightening map is given by

$$\chi_{f_0}(f) = (c, z) \in MK = \{(c, z);\ c \in M,\ z \in K(Q_c)\}$$

Disjoint type $\exists f^{n_1} : U_1' \to U_1$ and $\exists f^{n_2} : U_2' \to U_2$ s.t. $K(f^{n_i} : U_i' \to U_i) = K_f(\Omega_i)$. Define

$$\chi_{f_0}(f) = (c_1, c_2)$$

where $f^{n_i} : U_i' \to U_i$ is hybrid equivalent to Q_{c_i}. Note that χ_{f_0} is continuous in this case.

[2] it does not depend on the choice of a hybrid conjugacy

Similarly, we can define renormalizability and straightening maps using mapping schemata for any degree (but we omit it here).

Statement of result

Definition

We say a polynomial has a non-trivial critical relation if two critical points have the same grand orbit, i.e., $f^m(\omega) = f^n(\omega')$ for some $m, n \geq 0$ and critical points ω, ω'.

Note that existence of a multiple critical point is a critical relation.

Main Theorem (I)

Let f_0 be a post-critically finite hyperbolic polynomial with a non-trivial critical relation. Then the straightening map χ_{f_0} is not continuous on any neighborhood of a Misiurewicz polynomial.

A polynomial f is called Misiurewicz if all the critical points are (strictly) preperiodic.

If f_0 is cubic, then the assumption is equivalent that f_0 is not of disjoint type.

Outline of proof

1. Use parabolic implosion and Lavaurs map to relate continuity of straightening map to moduli of multipliers of repelling periodic orbits.
2. Apply Sullivan-Prado-Przytycki-Urbanski thm to obtain analytic conjugacy between quadratic-like restrictions.
3. Extend the analytic conjugacy to get global conjugacy by a correspondence. Such a global conjugacy rarely exists.
4. Find nice perturbations for Misiurewicz maps which we can apply the above to get a contradiction.

Parabolic implosion & continuous straightening map

Consider a family $\mathbf{f} = (f_\lambda : U'_\lambda \to U_\lambda, x_\lambda, y_\lambda)_{\lambda \in \Lambda}$ on a complex manifold Λ s.t.

- $(f_\lambda : U'_\lambda \to U_\lambda)$: analytic family of polynomial-like mappings of degree $d \geq 2$.
- x_λ, y_λ: marked points (holomorphic on λ).

Let

$$\mathcal{C}(\mathbf{f}) = \{\lambda \in \Lambda;\ K(f_\lambda) :\ \text{connected}, x_\lambda, y_\lambda \in K(f_\lambda)\}.$$
$$\mathcal{C}_{d,2} = \{(P, w, z);\ P \in \mathcal{C}_d,\ w, z \in K(P)\}.$$

Then we can define the straightening map as follows:

$$\begin{array}{cccc} \chi_{\mathbf{f}} : & \mathcal{C}(\mathbf{f}) & \to & \mathcal{C}_{d,2} \\ & \lambda & \mapsto & (P_\lambda, \psi_\lambda(x_\lambda), \psi_\lambda(y_\lambda)) \end{array}$$

where f_λ is hybrid equivalent to P_λ by ψ_λ.

In application, x_λ and y_λ are points in forward orbits of critical points.
A straightening map is continuous except when the filled Julia set moves discontinuously in some measurable sense.
Therefore, if $\chi_{\mathbf{f}}$ is discontinuous at $\lambda \in \Lambda$, then f_λ has either

- a parabolic periodic point,
- a Siegel disk, or
- an invariant line field on its Julia set (?).

We do not know whether the Siegel case is possible or not.
So we consider the case f_{λ_0} has a parabolic periodic point for $\lambda = \lambda_0 \in \Lambda$.

For a periodic point α of period $p > 0$ for f, let us denote its multiplier by

$$\mathrm{mult}_f(\alpha) = (f^p)'(\alpha).$$

Lemma 1

In the above setting, assume

- *0 is a periodic point of period $p > 0$ for any $\lambda \in \Lambda$.*
- *when $\lambda = \lambda_0$, $x_{\lambda_0} = y_{\lambda_0}$ and 0 is a parabolic periodic point such that x_{λ_0} lie in its basin.*
- *α_λ: marked repelling periodic point for f_λ.*
- *$\exists \lambda_{m,n} \to \lambda_n \to \lambda_0$ in $\mathcal{C}(\mathbf{f})$ s.t.*
  - *0 still is parabolic for f_{λ_n}, but $x_{\lambda_n} \neq y_{\lambda_n}$.*
  - *$\exists k_{n,m} \xrightarrow{m\to\infty} \infty$ s.t. $f_{\lambda_{n,m}}^{k_{n,m}} \xrightarrow{m\to\infty} \exists g_n$ (Lavaurs map) on the basin of 0, s.t. $g_n(x_{\lambda_n}) = \alpha_{\lambda_n}$.*
  - *$g_n \xrightarrow{n\to\infty} \exists g$ s.t. $g'(x_{\lambda_0}) \neq 0$.*

Then

$$|\,\mathrm{mult}_{f_{\lambda_0}}(\alpha_{\lambda_0})| = |\,\mathrm{mult}_{P_{\lambda_0}}(\psi_{\lambda_0}(\alpha_{\lambda_0}))|$$

where $\chi_{\mathbf{f}}(\lambda_0) = P_{\lambda_0}$ and ψ_{λ_0} is a hb conj between f_{λ_0} and P_{λ_0}.

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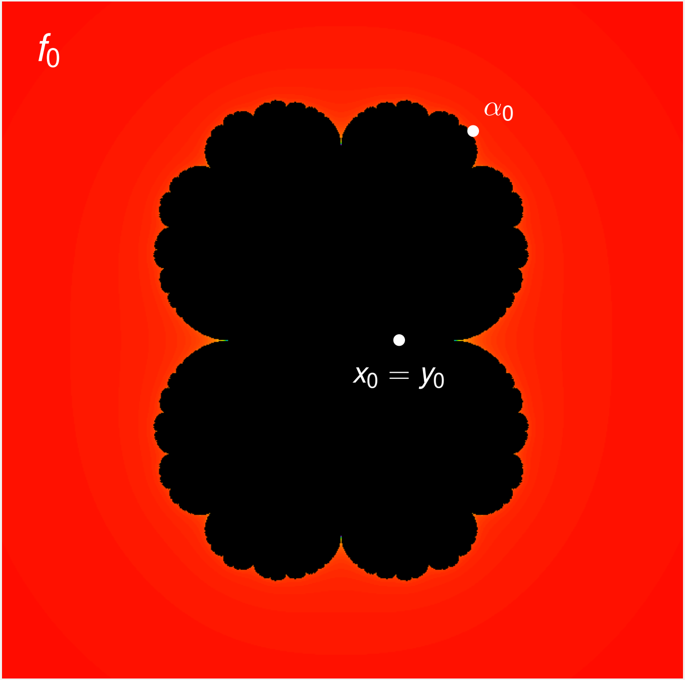
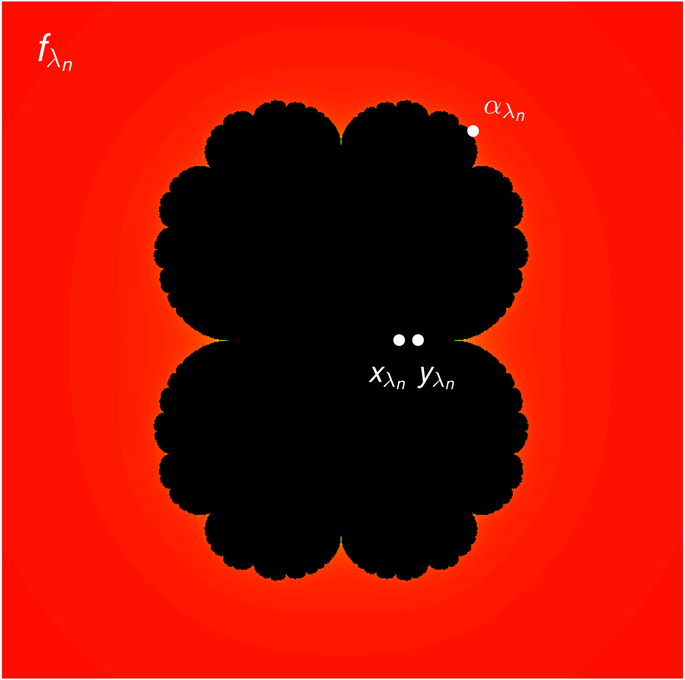
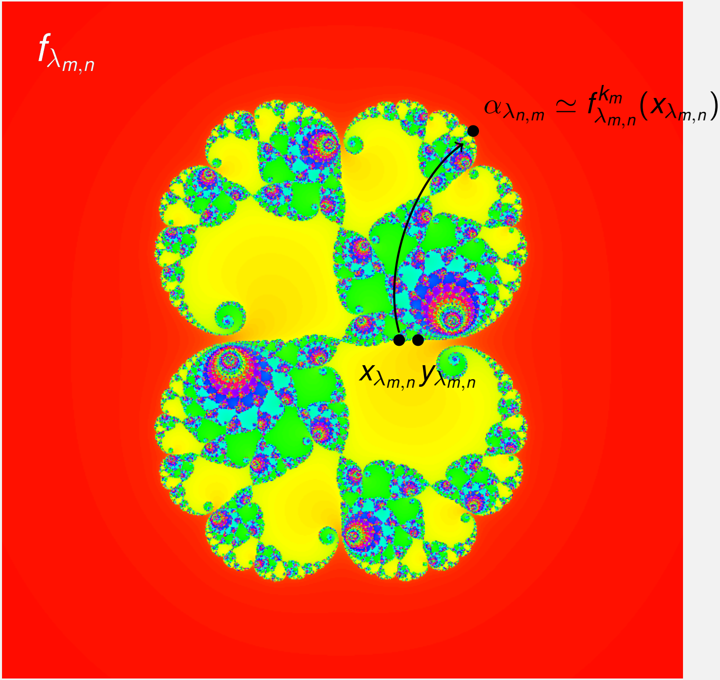



f_{λ_n}
α_{λ_n}
x_{λ_n} y_{λ_n}


$f_{\lambda_{m,n}}$
$\alpha_{\lambda_{n,m}} \simeq f_{\lambda_{m,n}}^{k_m}(x_{\lambda_{m,n}})$
$x_{\lambda_{m,n}}$ $y_{\lambda_{m,n}}$

Multipliers and analytic conjugacy

Theorem 2 (Sullivan-Prado-Przytycki-Urbanski)

If two tame polynomial-like mappings $f : U' \to U$ and $g : V' \to V'$ are hybrid conjugate by a conjugacy ψ, and

$$|\operatorname{mult}_f(\alpha)| = |\operatorname{mult}_g(\psi(\alpha))|$$

for any repelling periodic point α for f, then f and g are analytically conjugate.

Theorem (Urbanski)

Every polynomial-like mapping with no recurrent critical points in its Julia set is tame.

In particular, a quadratic-like mapping hybrid equivalent to $z + z^2$ is tame.

Extending analytic conjugacy

Theorem 3 (I)

If two polynomials f and g have polynomial-like restrictions $f : U' \to U$ and $g : V' \to V$ which are analytically conjugate, then there exist polynomials h, φ_1 and φ_2 such that $f \circ \varphi_1 = \varphi_1 \circ h$ and $g \circ \varphi_2 = \varphi_2 \circ h$:

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{h} & \mathbb{C} \\ \downarrow \varphi_1 & & \downarrow \varphi_1 \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{h} & \mathbb{C} \\ \downarrow \varphi_2 & & \downarrow \varphi_2 \\ \mathbb{C} & \xrightarrow{g} & \mathbb{C}. \end{array}$$

In particular, $\deg f = \deg g = \deg h$.

The conclusion means that f and g are conjugate by an irreducible holomorphic correspondence. We do not know whether it is an equivalence relation or not.

Finding perturbations

Lemma 4

If f_0 is a post-critically finite hyperbolic polynomial with a non-trivial critical relation, then for any Misiurewicz polynomial $f \in \mathcal{C}(f_0)$,

- *f is f_0-renormalizable.*
- *There exists some f_1 arbitrarily close to f s.t.*
  - *$\exists$ a quadratic-like restriction $f_1^m : W' \to W$ hybrid equivalent to $z + z^2$, which is contained in a f_0-renormalization of f_1.*
  - *f_0-renormalization of f_1 satisfies the assumption of the previous lemma except the continuity of the straightening map, with the following:*
    - *$\alpha \in K(f_1^m : W' \to W)$ is an arbitrary repelling periodic point.*
    - *x and y are forward images of critical points.*

Finishing the proof

- Now assume the straightening map χ_{f_0} is continuous in a neighborhood of a Misiurewicz $f \in \mathcal{C}(f_0)$.
- By Lemma 4, we can apply Lemma 1 for any repelling point $\alpha \in K(f_1^m : W' \to W)$.
- By Theorem 2, $P_1 = \chi_{f_0}(f_1)$ has a quadratic-like restriction $P_1^{\tilde{m}} : \tilde{W}' \to \tilde{W}$ analytically conjugate to $f_1^m : W' \to W$.
- Then Theorem 3 contradicts the fact that $\deg P_1 = \deg(f_1^n : U' \to U) < \deg f_1^n$.

The case of rational maps

We do not have such a combinatorial characterization for renormalizable rational maps.
But similar proof can be apply to the case of rational maps, because we know the existence of nice perturbations in the image of the straightening map, which is a family of polynomials:

Theorem 5

Assume a family of rational maps $(f_\lambda)_{\lambda\in\Lambda}$ has polynomial-like restrictions $\mathrm{f} = (f_\lambda : U'_\lambda \to U_\lambda)$ of degree $d \geq 3$. Let $\chi_{\mathrm{f}} : \mathcal{C}(\Lambda) \to \mathcal{C}_d$ be the straightening map, and

- *$\exists$ a Misiurewicz polynomial $P \in \mathcal{C}_d$ and its nbd $\mathcal{U}$ in $\mathcal{C}_d$,*
- *$\exists$ a continuous "section" $t : \mathcal{U} \to \mathcal{C}(\Lambda)$, i.e., a continuous map such that $\chi \circ t = id$,*

then (f_λ) is affinely conjugate to a family of polynomials of degree d.

It follows that there does not exist a natural homeomorphic embedding of $\mathcal{C}_d$ for $d \geq 3$.
We may also consider the case of capture renormalizations.

The case of transcendental entire maps

Theorem 6

Assume a family of transcendental entire maps $(f_\lambda)_{\lambda\in\Lambda}$ has polynomial-like restrictions $\mathrm{f} = (f_\lambda : U'_\lambda \to U_\lambda)$ of degree $d \geq 3$. Let $\chi_{\mathrm{f}} : \mathcal{C}(\Lambda) \to \mathcal{C}_d$ be the straightening map, and

- *$\exists$ a Misiurewicz polynomial $P \in \mathcal{C}_d$ and its nbd $\mathcal{U}$ in $\mathcal{C}_d$,*
- *$\exists$ a continuous "section" $t : \mathcal{U} \to \mathcal{C}(\Lambda)$, i.e., a continuous map such that $\chi \circ t = id$,*

then $\exists P_1 \in t(\mathcal{U})$, Q, φ_1 and a transcendental entire map φ_2 s.t. for $\lambda = t(P_1)$, we have

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{Q} & \mathbb{C} \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_1} \\ \mathbb{C} & \xrightarrow{P_1} & \mathbb{C} \end{array} \qquad \begin{array}{ccc} \mathbb{C} & \xrightarrow{Q} & \mathbb{C} \\ \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_2} \\ \mathbb{C} & \xrightarrow{f_\lambda} & \mathbb{C}. \end{array}$$

Question

Are there exist P, f, φ s.t.

- *P: polynomial of degree ≥ 2,*
- *f, φ: transcendental entire map, and*

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{P} & \mathbb{C} \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \varphi} \\ \mathbb{C} & \xrightarrow{f} & \mathbb{C} \end{array}$$

Remark

If we allow degree 1 polynomials, the Schröder equation for a repelling fixed point of a transcendental entire map gives an example.