Limits of renormalizable polynomials

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Abstract

We give a criterion for closedness of combinatorial properties in the parameter space of polynomials having connected Julia set, and apply it to the renormalizability of polynomials. Consequently, we give two sufficient conditions for the limit of a convergent sequence of renormalizable polynomials to be again renormalizable.

1 Introduction

In the study of dynamics of polynomials in one variable, external rays play an important role. They are used to analyze and describe combinatorially the dynamics of a given polynomial and connectedness loci of parameter spaces. Douady and Hubbard [2] first introduced the notion of external rays and obtain many basic results. One of the most important results is that any rational ray lands at a repelling or parabolic (pre)periodic point and vice versa (see Theorem 2.1).

Therefore, we can consider an equivalence relation on \( \mathbb{Q}/\mathbb{Z} \) called the rational lamination first introduced by Thurston [15]. According to Kiwi [7], the rational lamination \( \lambda_f \) of a polynomial \( f \) with connected Julia set \( J(f) \) is defined that a nontrivial class is the set of landing angles of a repelling or parabolic (pre)periodic point. It is invariant under the action of \( d : \mathbb{Q}/\mathbb{Z} \to \mathbb{Q}/\mathbb{Z} \) where \( d \geq 2 \) is the degree of \( f \). Kiwi discussed the combinatorial characterization of rational laminations of polynomials with connected Julia set.

In this paper, we discuss about closedness of combinatorial properties. Namely, we consider the following question: When a sequence of polynomials having a given combinatorial property converges, then does the limit have the same property? Although we give a general notion to describe combinatorial property, we mainly concerned with the renormalizability of polynomials.

For later use, we introduce the following notation:

**Notation.** For a polynomial \( f \), let us denote by \( K(f) = \{ z \in \mathbb{C} : \{ f^n(z) \}_{n \geq 0} \) is bounded \} the filled Julia set and by \( J(f) = \partial K(f) \) the Julia set of \( f \). Let \( \text{Crit}(f) = \{ c \in \mathbb{C} : f'(c) = 0 \} \) be the set of critical points and \( PC(f) = \bigcup_{n \geq 1} f^n(\text{Crit}(f)) \) be the postcritical set of \( f \). Denote by \( \text{Poly}_d \) the set of monic centered polynomials of degree \( d \) and by \( C_d = \{ f \in \text{Poly}_d : K(f) \) is connected \} the connectedness locus.

In the first half of the paper, we introduce the notion of combinatorial Yoccoz puzzles to describe combinatorial properties precisely and give a criterion for closedness of combinatorial properties. The notion of combinatorial Yoccoz puzzles is a description of Yoccoz puzzles for a polynomial with connected Julia set in terms of rational laminations (Section 4). The notion of Yoccoz puzzles and tableaux is first introduced to study cubic polynomials with one escaping critical point by Branner and Hubbard [1], and local connectivity of the Mandelbrot set at finitely renormalizable parameters by Yoccoz and Hubbard [4]. Figure 1 shows the Julia set of \( z^2 + c \) where \( c \approx 0.123 + 0.745i \) (the Douady rabbit) with Yoccoz puzzles and its rational lamination.

The main result of this part is the following:

**Theorem 1.1** (Criterion for closedness of combinatorial properties). Let \( f_1 \to f_\infty \) be a convergent
sequence of polynomials in the connectedness locus $C_d$ of polynomials of degree $d \geq 2$. Define a $d$-invariant real lamination $\lambda$ by $\lambda = \bigcap_{l>0} \hat{\lambda} f_l$ (as subsets of $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$) where $\hat{\lambda} f_l$ is the real extension of the rational lamination of $f_l$. Assume there exist non-degenerate combinatorial puzzles $\Lambda^j = \{\lambda^j_k\}_{k \geq 0}$ of $\lambda$ for $j = 1, \ldots, d$ such that

1. $\Lambda^j$ is admissible for $f_l$.
2. $\Lambda^j$ separates its postcritical set from $\Lambda^1$ for $j = 2, \ldots, d$.
3. Let $\text{Per}(\Lambda^j)$ be the set of periods of angles in nontrivial classes of $\lambda^j_0$ for $j = 1, \ldots, d$. Then $\text{Per}(\Lambda^1), \ldots, \text{Per}(\Lambda^d)$ are mutually disjoint.

Then $\Lambda^1$ is an admissible puzzle for $f_\infty$ and the landing point of $R_{f_\infty}(\theta)$ for any $\theta \in \text{supp} \lambda^1_0$ is a repelling periodic point.

For definitions and notations, see Section 3 and 4. A proof will be given in Section 5. The point of the theorem is that if we can find as many combinatorial puzzles $\Lambda^2, \Lambda^3, \ldots$ which separate the postcritical set from $\Lambda^1$ as the number of the critical points (or more), then $\Lambda^1$ is admissible in the limit.

Here we give a brief sketch of this theorem. If $\Lambda^1$ is not admissible for $f_\infty$, then some landing point $x$ of a ray whose angle lies in $\Lambda^1$ must be parabolic. Hence some critical orbit of $f_\infty$ must accumulate at $x$. Therefore, if there exist $d - 1$ other admissible combinatorial Yoccoz puzzles $\Lambda^2, \Lambda^3, \ldots, \Lambda^d$ for $f_l$ which “separate” the postcritical set from $x$ (prevent the critical orbits from accumulating at $x$), then to break $\Lambda^1$ in the limit (i.e., if $\Lambda^1$ is not admissible for $f_\infty$), we also have to break $\Lambda^2, \ldots, \Lambda^d$. This implies that the landing point for some external ray of angle in $\Lambda^i$ ($i = 1, \ldots, d$) is a parabolic periodic point for $f_\infty$. Hence, with some technical assumption, $f_\infty$ must have $d$ parabolic periodic orbits. However, at least $d$ critical points are needed for such a phenomenon to occur and it contradicts the fact that $f_\infty$ has at most $d - 1$ critical points.

In the second half of this paper, we apply the above result to renormalizable polynomials and give sufficient conditions that the limit of renormalizable polynomials with a given “combinatorics” (precise statement of is given in Section 7) is again renormalizable with the same combinatorics.

We basically follow the study of combinatorics of renormalizable quadratic polynomials by McMullen [11], and there is also a deep result on combinatorics of renormalizable quadratic polynomials in terms of Yoccoz puzzles by Lyubich [10] to study local connectivity and combinatorial rigidity of quadratic polynomials. In the case of the parameter space of quadratic polynomials, Douady and Hubbard proved that the set of simply renormalizable polynomials having a given combinatorics is homeomorphic to the Mandelbrot set $\mathcal{M} = \mathcal{C}_2$, possibly except the root (the one whose renormalization has a parabolic fixed point). Such a copy of $\mathcal{M}$ is called a baby Mandelbrot set. The exceptional point occurs in the case of
“satellite” copies (copies attached to some hyperbolic components) and not in the “primitive” copies (see [9]). We would like to generalize this results to parameter spaces of higher degree polynomials. Here, we focus on the renormalizability of sequential limits of renormalizable polynomials (or the compactness of the set of renormalizable polynomials) of a given combinatorics.

Combinatorics of “non-crossed” renormalizations can be characterized in Section 6 and 7 in terms of combinatorial Yoccoz puzzles. Therefore, we can apply Theorem 1.1 to obtain the following theorems about renormalizability. We say a renormalization is full if the filled Julia set of the renormalization and its forward images contain all the critical points.

**Theorem 1.2.** Let \( \{f_l\} \) be a sequence in \( C_d \) and assume \( f_l \to f_\infty \). If there exist combinatorial Yoccoz puzzles \( \Lambda^1, \ldots, \Lambda^d \) on increasing period such that for each \( l = 1, 2, \ldots \), \( f_l \) has \( d \) full non-crossed renormalizations characterized by \( \Lambda^1, \ldots, \Lambda^d \), then \( f_\infty \) has a non-crossed renormalization characterized by \( \Lambda^1 \).

Heuristically, this theorem implies that the family of polynomials having \((n + d - 1)\) full non-crossed renormalizations of a given combinatorics forms a relatively compact subset in the family of \( n \) times fully renormalizable polynomials. Note that we only assume \( f_l \) has at least \((not exactly)\) \( d \) renormalizations satisfying the assumption.

For \( d = 2 \), this theorem states that for baby Mandelbrot sets \( M_1 \supset M_2 \), any \( f \in M_2 \) is renormalizable with the combinatorics corresponding to \( M_1 \). This case is trivial because \( M_2 \) does not contain the root of \( M_1 \).

**Theorem 1.3.** Let \( \{f_l\} \) be a sequence in \( C_d \) and assume \( f_l \to f_\infty \). If there exists a combinatorial Yoccoz puzzle \( \Lambda \) such that for each \( l \), \( f_l \) has a full renormalization of disjoint type characterized by \( \Lambda \), then \( f_\infty \) is also renormalizable characterized by \( \Lambda \).

For \( d = 2 \), a combinatorics of renormalizations of disjoint type corresponds to a primitive baby Mandelbrot set. Hence this theorem implies that any polynomial in a primitive baby Mandelbrot set is renormalizable, which is stated above.

Figure 2 shows a cubic polynomial having a period 3 full renormalization.

![Fig. 2 The Julia sets and Yoccoz puzzles (left) for a cubic polynomial \( f(z) \approx z^3 - 2.14z + 0.360 \), which has a period 3 full renormalization of disjoint type hybrid equivalent to \( z^4 \), and its rational lamination (right).](image-url)
Theorem 1.3 implies that the set of polynomials having a full renormalization of disjoint type with a given combinatorics forms a compact set $\mathcal{K}$ in $\mathcal{C}_d$. Hence there exists a neighborhood $\mathcal{U}$ of $\mathcal{K}$ and a holomorphic family of polynomial-like mappings $\{f^*: f^*: U_f \to V_f, f \in \mathcal{U}\}$ such that the connectedness locus of the family is equal to $\mathcal{K}$.

We do not know whether $\mathcal{K}$ is naturally homeomorphic (or even bijective) to the connectedness locus of a family of polynomials, which is true for $d = 2$, where $\mathcal{K}$ is a primitive baby Mandelbrot set.

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## 2 External rays

Let $f \in \mathcal{C}_d$ and let $\varphi_f : (\mathbb{C} \setminus K(f)) \to (\mathbb{C} \setminus \mathbb{D})$ be the Böttcher coordinate of $f$, where $\mathbb{D} = \{|z| < 1\}$ is a unit disk. Namely, a conformal isomorphism tangent to identity at infinity satisfying $\varphi_f \circ f(z) = (\varphi_f(z))^d$.

For $\theta \in \mathbb{R}/\mathbb{Z}$, let us denote the external ray of angle $\theta$ for $f$ by

$$R_f(\theta) = \{\varphi_f^{-1}(\exp(r + 2\pi i \theta)): r > 0\}.$$

And for $r > 0$, we denote the equipotential curve of potential $r$ for $f$ by $E_f(r) = \{\varphi_f^{-1}(\exp(r + 2\pi i \theta)): \theta \in \mathbb{R}/\mathbb{Z}\}$. Let $D_f(r)$ be the region bounded by $E_f(r)$ and let $R_f(\theta, r) = R_f(\theta) \cap D_f(r)$. If the limit $x = \lim_{r \to 0} \varphi_f^{-1}(\exp(r + 2\pi i \theta))$ exists, then we say that $R_f(\theta)$ lands at $x$ and $\theta$ is a landing angle or external angle of $x$. Let $\text{Angle}_f(x)$ (or simply, $\text{Angle}(x)$) denote the set of landing angles of $x$ for $f$.

For a set $A \subset f(J(f), let \text{Angle}(A) = \bigcup_{x \in A} \text{Angle}(x)$. By definition, we have $f(R_f(\theta)) = R_f(d\theta)$ and $f(E_f(r)) = E_f(dr)$. An equipotential curve $E_f(r)$ for some $r > 0$ and external rays $R_f(\theta, r)$ of angle $\theta = 1/7, 2/7, 4/7$ and their inverse images are shown in Figure 1 in the introduction.

**Theorem 2.1.** For any $\theta \in \mathbb{Q}/\mathbb{Z}$, the ray $R_f(\theta)$ lands at a repelling or parabolic (pre)periodic point of $f$.

On the contrary, if $x$ is a repelling or parabolic (pre)periodic point of $f$, then there exists $\theta \in \mathbb{Q}/\mathbb{Z}$ such that $R_f(\theta)$ lands at $x$. Furthermore, when $x$ is periodic and landing angles of $x$ are periodic and have the same period under the map $d : \theta \mapsto d\theta$.

See [2].

**Lemma 2.2.** Let $\alpha$ be a repelling or parabolic (pre)periodic point of $f$. Then the number of the landing angles of $\alpha$ is equal to the number of the components of $K(f) \setminus \{\alpha\}$.

See [11, Corollary 6.9]. Therefore, the rays landing at $\alpha$ divide the complex plane into finite number of components and each component contains exactly one component of $K(f) \setminus \{\alpha\}$. Set $S(\theta \pm \varepsilon)$ for $\theta \in \text{Angle}(\alpha)$ denote such a component which contains $R_f(\theta \pm \varepsilon)$ for small $\varepsilon > 0$. If $\theta$ and $\theta'$ are adjacent in $\text{Angle}(\alpha)$ (with respect to the cyclic order in $\mathbb{R}/\mathbb{Z}$), then we have $S(\theta +) = S(\theta')$ and $S(\theta -) = S(\theta' +)$. Let $S(\theta, \theta')$ denote a domain whose boundary is equal to $\{\alpha\} \cup R_f(\theta) \cup R_f(\theta')$ and contains $R_f(\theta \pm \varepsilon)$ for small $\varepsilon > 0$. We call $S(\theta +)$ and $S(\theta, \theta')$ sectors at $\alpha$.

**Lemma 2.3.** Assume $f_1 \to f_\infty$ in $\mathcal{C}_d$ and $R_{f_\infty}(\theta)$ lands at a repelling (pre)periodic point $x_\infty$ of $f_\infty$. Then for sufficiently large $l$, $R_{f_l}(\theta)$ lands at the repelling (pre)periodic point $x_l$ of $f_l$ which is the continuation of $x_\infty$.

**Proof.** Considering $d^N\theta$ for some $N$ and some iterates of $f_1$ and $f_\infty$, we may assume $R_{f_\infty}(\theta)$ is invariant under $f_\infty$ and lands at a repelling fixed point $x_\infty$ of $f_\infty$.

Take a small neighborhood $U$ of $x_\infty$ such that $U$ is relatively compact in $f_\infty(U)$ and $f_\infty : U \to f_\infty(U)$ is a conformal isomorphism. Then for sufficiently large $l$, $U$ is also relatively compact in $f_l(U)$ and $f_l : U \to f_l(U)$ is a conformal isomorphism. We have $\bigcup_{n}(f_l(U))^{-n}(U) = \{x_l\}$. Take small $\varepsilon > 0$. Since $(z, f) \mapsto \varphi_f(z)$ is continuous, we have $\gamma_l = \{\varphi^{-1}_f(\exp(r + 2\pi i \theta)); \varepsilon \leq r \leq d\varepsilon \} \subset f_l(U)$ for sufficiently large $l$. Thus $\bigcup_{n} (f_l(U))^{-n}(\gamma_l)$ is an arc accumulating at $x_l$. Since $R_{f_\infty}(\theta)$ is forward invariant and connected, $R_{f_\infty}(\theta)$ contains $\bigcup_{n}(f_l(U))^{-n}(\gamma_l)$. Hence $R_{f_l}(\theta)$ lands at $x_l$. \hfill $\Box$
3 Laminations

The notion of rational laminations is very useful to describe combinatorics of polynomial dynamics. See [7] for more details about the facts in this section.

Notation. Let $S^1 = \partial \mathbb{D}$ and denote by

$$e(\theta) = \exp(2\pi i \theta)$$

the standard diffeomorphism from $\mathbb{R}/\mathbb{Z}$ to $S^1$. For an integer $d \geq 2$, let $d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ denote the $d$-fold covering map $\theta \mapsto d\theta$.

Fix an integer $d \geq 2$.

Definition. A lamination $\lambda$ on a set $E \subset \mathbb{R}/\mathbb{Z}$ (usually we treat the case $E \subset \mathbb{Q}/\mathbb{Z}$ or $E = \mathbb{R}/\mathbb{Z}$) is an equivalence relation on $E$ such that

1. $\lambda$ is closed in $E \times E$.
2. Any $\lambda$-equivalence class is a finite set.
3. $\lambda$-equivalence classes are pairwise unlinked.

We say two subsets $A_1, A_2 \subset \mathbb{R}/\mathbb{Z}$ are unlinked if $A_1$ is contained in a component of $\mathbb{R}/\mathbb{Z} \setminus A_2$. It is also equivalent that $A_2$ is contained in a component of $\mathbb{R}/\mathbb{Z} \setminus A_1$, because both conditions are equivalent that the Euclidean convex hulls of $e(A_1)$ and $e(A_2)$ are disjoint. Note that convex hulls in $\mathbb{R}$ under the hyperbolic metric on $\mathbb{D}$ also provide the same definition.

A lamination on $\mathbb{Q}/\mathbb{Z}$ (resp. $\mathbb{R}/\mathbb{Z}$) is called a rational lamination (resp. real lamination). A sublamination $\lambda'$ of a lamination $\lambda$ is a lamination with $\lambda' \subset \lambda$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\math{Z}$. Let us denote by $\text{supp}(\lambda) = \pi(\lambda)$ the support of $\lambda$, where $\pi : \mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ is the projection to the first coordinate.

Let $\lambda_i$ be laminations on $E_i$ for $i = 1, 2$. We denote $d_*, \lambda_1 = \lambda_2$ if $d^{-1}(E_2) = E_1$ and for any $\lambda_1$-equivalence class $A$,

1. $d(A)$ is a $\lambda_2$-class.
2. The map $d_A : A \to d(A)$ is consecutive preserving, that is, if an interval $(\theta_1, \theta_2)$ is a component of $\mathbb{R}/\mathbb{Z} \setminus A$, then $(d\theta_1, d\theta_2)$ is a component of $\mathbb{R}/\mathbb{Z} \setminus d(A)$.

We say a lamination $\lambda$ is $(d,*)$-invariant if $d_\lambda = \lambda$. When $d_* \lambda_1 = \lambda_2$, it is easy to see that for a $\lambda_1$-class $A$, the degree $\deg(d_A : A \to d(A))$ is well-defined. Let us denote this value by $\delta(A) = \deg(d_A)$. We say $A$ is critical if $\delta(A) \geq 2$.

Example 3.1. Let $f \in C_d$. Define an equivalence relation $\lambda_f$ on $\mathbb{Q}/\mathbb{Z}$ such that $\theta_1 \sim_{\lambda_f} \theta_2$ if and only if $R_f(\theta_1)$ and $R_f(\theta_2)$ land at the same point. Then $\lambda_f$ is a $d$-invariant rational lamination. It is called the rational lamination for $f$. Kiwi [7] proved that every invariant rational lamination $\lambda$ can be realized by a polynomial of degree $d$, i.e., $\lambda = \lambda_f$ for some $f \in C_d$ (see also Theorem 3.7). Let $\hat{\lambda}_f$ be the real extension of $\lambda_f$, that is, the smallest equivalence relation which contains the closure of $\lambda_f$ in $\mathbb{R}/\mathbb{Z} \times \mathbb{R}/\mathbb{Z}$. Kiwi also proved that $\hat{\lambda}_f$ is an invariant real lamination, so we can directly refer to the real lamination $\hat{\lambda}_f$ for $f$, avoiding any distinction.

We also consider the $C_d$-extension $\lambda_C$ of a real lamination $\lambda$, which is an equivalence relation on the complex plane defined as follows: We consider $\lambda$ as an equivalence relation on $S^1$ via $e : \mathbb{R}/\mathbb{Z} \to S^1$ and a nontrivial $\lambda_C$-class is defined by the Euclidean convex hull of a nontrivial $\lambda$-class.

Remark 3.2. Although external rays do not always land, we can still consider an invariant real lamination on $\mathbb{R}/\mathbb{Z}$ of $f \in C_d$ defined in terms of prime ends, but we do not use this here.

Lemma 3.3. Assume $d : A \to d(A)$ is consecutive preserving, and let $\theta_1, \theta_2 \in A$. If there exists some $\theta \in (d\theta_1, d\theta_2) \cap d(A)$, there exists $\theta' \in (\theta_1, \theta_2) \cap A$ with $d\theta' = \theta$. 

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Proof. Suppose there exists no such \( \theta' \). Take \( \theta'' \in (\theta_1, \theta_2) \cap d^{-1}(\theta) \). By assumption, \( \theta'' \notin \mathcal{A} \). We may replace \((\theta_1, \theta_2)\) by the component \((s, t)\) of \(\mathbb{R}/\mathbb{Z} \setminus \mathcal{A}\) containing \(\theta''\). Since \(d : A \to d(A)\) is consecutive preserving, \((d\theta_1, d\theta_2)\) must be a component of \(\mathbb{R}/\mathbb{Z} \setminus d(A)\). But it contradicts that \(\theta \in (d\theta_1, d\theta_2) \cap d(A)\).

**Lemma 3.4.** For a \(d\)-invariant lamination \(\lambda\) on \(E\), the restriction \(\lambda|_{E'} = \lambda \cap (E' \times E')\) to a completely invariant subset \(E' \subset E\) is a \(d\)-invariant lamination on \(E'\).

**Proof.** It is clear that \(\lambda' = \lambda|_{E'}\) is a lamination on \(E'\). Let \(A\) be a \(\lambda'\)-class and let \(\tilde{A} \supset A\) be the \(\lambda\)-class containing \(A\). Then we have \(A = \tilde{A} \cap E'\) by definition. By the complete invariance of \(E'\) and surjectivity of \(d : \tilde{A} \to d(\tilde{A})\), \(d(A) = d(\tilde{A} \cap E') = d(\tilde{A}) \cap E'\). Hence \(d(A)\) is a \(\lambda'\)-class.

Let \((\theta_1, \theta_2)\) be a component of \(\mathbb{R}/\mathbb{Z} \setminus \mathcal{A}\) and assume \((d\theta_1, d\theta_2)\) intersects \(d(A)\). By Lemma 3.3, there exists \(\theta \in (\theta_1, \theta_2) \cap \tilde{A}\) such that \(d\theta \in (d\theta_1, d\theta_2) \cap d(A)\). This implies that \(\theta \in A\), that is a contradiction. Therefore, \(d : A \to d(A)\) is consecutive preserving.

It is easy to see the following lemma:

**Lemma 3.5.** A consecutive preserving bijection is cyclic order preserving.

Let \(\lambda\) be a lamination on \(E\). We say \(\theta_1, \theta_2 \in \mathbb{R}/\mathbb{Z}\) are \(\lambda\)-unlinked if \(\theta_1 = \theta_2\), or \(A\) and \(\{\theta_1, \theta_2\}\) are unlinked for any \(\lambda\)-class \(A\).

**Proposition 3.6.** Let \(\lambda\) and \(\lambda'\) be real laminations with \(d_\lambda \lambda = \lambda'\). Then

1. The \(\lambda\)-unlinked relation is an equivalence relation on \(\mathbb{R}/\mathbb{Z}\).
2. For a \(\lambda\)-unlinked class \(L\), there exists a \(\lambda'\)-unlinked class \(L'\) containing \(d(L)\) such that \(L' \setminus (\mathcal{L} \cup (\mathcal{L} \setminus \lambda)) \subset d(\lambda) \setminus d(\lambda')\).
3. For a \(\lambda\)-unlinked class \(L\), any component of \(\mathbb{R}/\mathbb{Z} \setminus L\) is a closed interval \([\theta^-, \theta^+]\) with \(\lambda^- \sim \lambda^+\).
   In particular, \(\mathcal{L}/\lambda\) is homeomorphic to \(S^1\).
4. The map \(d\) induces an orientation-preserving covering map \(\tilde{d} : \mathcal{L}/\lambda \to d(\mathcal{L})/\lambda'\). Hence there exists an integer \(\delta = \delta(L)\) with \(0 < \delta \leq d\) such that \(\tilde{d}\) is topologically conjugate to \(\delta : S^1 \to S^1\).

**Proof.** The proof of [2, Lemma 4.6] can be applied to show (1) and (2). Note that for a \(\lambda\)-class \(A\), its image \(d(A)\) can be a trivial \(\lambda'\)-class (namely, \#\(d(A)\) = 1). In this case, \(d(A)\) is contained in some nontrivial \(\lambda'\)-unlinked class.

Let \(L\) be a \(\lambda\)-unlinked class and \(I\) a component of \(\mathbb{R}/\mathbb{Z} \setminus L\). Denote \(\theta^- = \inf_{\theta \in I} \lambda\) and \(\theta^+ = \sup_{\theta \in I} \lambda\). If \(\theta^- = \theta^+\), then there exist sequences \(\theta^- = \theta^-_n \to \theta^+\). This implies that the \(\lambda\)-equivalence class containing \(\theta\) is equal to \(\{\theta\}\) and \(\theta \in L\), so this is a contradiction. Thus we have \(\theta^- \neq \theta^+\).

Next, assume \(\theta^-\) and \(\theta^+\) are not \(\lambda\)-equivalent. Then for any \(i \in (\theta^-, \theta^+)^\perp\), there exist \(i^-, i^+ \in [\theta^-, \theta^+]\) such that \(i \in (i^-, i^+)\) and \(i^+ \leq i\) and \(i^- \geq i\) are \(\lambda\)-equivalent. For \(i_1, i_2 \in (\theta^-, \theta^+)\), we have one of the following:

- \([i_1, i_2] \cap [i_3, i_4] = \emptyset\).
- \([i_1, i_2] \cap [i_3, i_4] \subseteq [i_1, i_2] \cup [i_3, i_4]\).

Therefore, the interval \(I\) is equal to the union of disjoint closed intervals. But the union of two or more disjoint closed interval cannot be an interval (even when there are infinitely many, the union consists of the complement of the Cantor set and countably many points). Therefore, the union consists of only one interval, which must be equal to \([\theta^-, \theta^+]\), and \(\theta^-\) and \(\theta^+\) are \(\lambda\)-equivalent.

If we collapse each complementary interval of an unlinked class \(L\) to one point, then it is again homeomorphic to \(S^1\) and it is also homeomorphic to \(\mathcal{L}/\lambda\), hence we have proved (3).

Similarly, \(d(\mathcal{L})/\lambda'\) is also homeomorphic to \(S^1\) and \(\tilde{d} : \mathcal{L}/\lambda \to d(\mathcal{L})/\lambda'\) induces a continuous map \(\rho : S^1 \to S^1\). It is monotone non-decreasing and the inverse image of every point by \(\rho\) is finite. Hence \(\rho\) is a covering map of degree \(d > 0\), and we have proved (4).

**Definition.** For real laminations \(\lambda, \lambda'\) with \(d_\lambda \lambda = \lambda\), we say a \(\lambda\)-unlinked class \(L\) is critical if \(\delta(L) \geq 2\).

We say a \(d\)-invariant real lamination \(\lambda\) is postcritically finite if all critical \(\lambda\)-classes and \(\lambda\)-unlinked classes are eventually periodic by \(d\). We say a \(d\)-invariant rational lamination \(\lambda\) is postcritically finite if
its real extension $\tilde{\lambda}$ is so.

It is easy to see when $f$ is postcritically finite (i.e., $PC(f)$ is finite), then its rational lamination $\lambda_f$ is also postcritically finite. Kiwi [7] proved the converse, which plays an important role in Section 9:

**Theorem 3.7.** For a postcritically finite $d$-invariant rational lamination $\lambda$, there exists a postcritically finite polynomial $f$ of degree $d$ such that $\lambda = \lambda_f$.

Now we can introduce the notion of combinatorial Yoccoz puzzles for an invariant lamination. Fix a $d$-invariant real lamination $\lambda$.

**Definition.** A combinatorial puzzle $\Lambda = \{\lambda_k\}_{k \geq 0}$ for $\lambda$ is a sequence of sublaminations of $\lambda$ such that:

1. The support of $\lambda_0$ is a union of finitely many periodic orbits by $d$.
2. Let $A$ be a nontrivial $\lambda_0$-class. Then $d(A)$ is also a $\lambda_0$-class and $d : A \to d(A)$ is consecutive preserving.
3. $d^k\lambda_{k+1} = \lambda_k$ for $k \geq 0$.

This definition implies that $\text{supp} \lambda_k$ is finite and contained in $\mathbb{Q}/\mathbb{Z}$, and $\lambda_k \subset \lambda_{k+1}$ in $\mathbb{Q}/\mathbb{Z} \times \mathbb{Q}/\mathbb{Z}$. Let $\text{supp} \Lambda = \bigcup_{k \geq 0} \text{supp} \lambda_k$ be the support of $\Lambda$. Let $\text{Per}(\Lambda)$ be the set of the periods of orbits contained in $\text{supp}(\lambda_0)$. In most cases, we treat the case that $\text{Per}(\Lambda)$ consists of only one element. We say a combinatorial puzzle $\Lambda$ is non-degenerate if any $\lambda_k$-class satisfies $\delta(A) = 1$. A combinatorial puzzle piece of depth $k \geq 0$ is a $\lambda_k$-unlinked class. Denote by $\mathcal{P}_k(\Lambda)$ the set of combinatorial puzzle pieces of depth $k$. By Lemma 3.6 2, we can define a map $d : \mathcal{P}_{k+1}(\Lambda) \to \mathcal{P}_k(\Lambda)$.

**Example 3.8.** Let $f(z) = z^3 - 3z/4 - \sqrt{7}i/4$. The critical points of $f$ are $\pm 1/2$ and they satisfy $f^2(\pm 1/2) = \pm 1/2$, i.e., they are superattractive periodic points of period two. We can prove that all of the immediate basins of the superattractive periodic orbits have a common point $x_0$ in their boundaries and $x_0$ is a repelling fixed point with $\text{Angle}(x_0) = \{1/4, 5/8, 3/4, 7/8\}$. Let $E_k = \text{Angle}(f^{-k}(x_0))$ and $\lambda_k = \lambda_f|_{E_k}$. Consider the rational lamination $\lambda_f$ of $f$. Then $\Lambda = \{\lambda_k\}$ is a combinatorial Yoccoz puzzle for $\lambda_f$. We have $\bigcup_k \lambda_k = \lambda_f$ and $\text{supp}(\lambda_f) = \text{supp}(\Lambda)$.

![Fig. 3 The Julia set of $f$ (left), its lamination $\lambda_f$ (right, gray regions), and a combinatorial puzzle $\Lambda'$ of $\lambda_f$ (right, black lines). The corresponding Yoccoz puzzles (see Section 4) are also shown in the left figure.](image-url)

Define a sequence of laminations $\Lambda' = \{\lambda'_k\}$ as follows:

- $\lambda'_0$ has only one nontrivial class $\{1/4, 3/4\}$.
- For $k \geq 1$, $\theta, \theta' \in \mathbb{R}/\mathbb{Z}$ are $\lambda'_k$-equivalent if $d^k\theta = 1/4$ and $d^k\theta = 3/4$ (or vice versa) and $\theta, \theta'$ are $\lambda_f$-equivalent.
Then Λ′ is also a combinatorial puzzle for λf. See Figure 3.

**Lemma 3.9.** Let Λ = {λk}k≥0 be a combinatorial puzzle for λ. Then for L ∈ ℙk(Λ), we have

\[ \sum_{L′ ∈ ℙk+1(Λ), L′ ⊆ L} (δ(L′) - 1) + \sum_{A: λ_{k+1} \text{-class}} (δ(A) - 1) = δ(L) - 1. \]

**Proof.** By the property (4) in Proposition 3.6, we need only show the following: Note that λk is a real lamination.

**Claim.** If laminations λ, λ′ satisfies d• λ = λ′, then

\[ \sum_{L: \text{λ-unlinked class}} (δ(L) - 1) + \sum_{A: \text{λ-class}} (δ(A) - 1) = d - 1. \]

To prove this claim, we consider the C-extensions λC and λ′C of λ and λ′, respectively, and construct a branched covering \( h: \mathbb{C}/\mathbb{L}_C \rightarrow \mathbb{C}/\mathbb{L}'_C \) of degree d such that

\[ h([e(\theta)]_{\lambda_C}) = [e(d\theta)]_{\lambda'_C}. \]

By the theorem of Moore [13], the quotient \( \mathbb{C}/\mathbb{L}_C \) is homeomorphic to \( \mathbb{C} \), so h can be regarded as a map \( h: \mathbb{C} \rightarrow \mathbb{C} \). Then for a λ-class A, δ(A) is the local degree of h at hull(Λ)/λC, which is a point in \( \mathbb{C}/\mathbb{L}_C \). For a λ-unlinked class L, δ(L) is the degree of \( h: L/\mathbb{L}_C \rightarrow d(L)/\mathbb{L}'_C \). The required identity is obtained by counting critical points of h with multiplicity. See [15, Proposition 11.4] or [7, Proposition 4.14] for details. (Kiwi proved stronger result for invariant laminations.)

**Definition.** For a non-degenerate combinatorial puzzle Λ = {λk} for λ, let

\[ \mathcal{C}_k(\Lambda) = \{ L ∈ ℙk(Λ); δ(L) ≥ 2 \} \]

be the set of critical combinatorial puzzle pieces of depth k. The postcritical set \( \mathcal{P}C_k(\Lambda) \) of depth k is defined as follows:

\[ \mathcal{P}C_k(\Lambda) = \{ d^n L; n > 0, L ∈ \mathcal{C}_{k+n}(\Lambda) \}. \]

Let Λ = {λk} and Λ′ = {λ′k} be non-degenerate combinatorial puzzles for λ. We say that Λ separates its postcritical set from Λ′ if there exists some k ≥ 0 such that \( \bigcup_{L ∈ \mathcal{P}C_k(\Lambda)} L \) does not intersect any \( λ'_k \)-class. The combinatorial tableau of Λ is a double sequence of the form \{d^n(L_k)\}n,k≥0 where \( L_k ∈ ℙ_k(Λ) \) and \( L_{k+1} ⊆ L_k \). We only consider critical combinatorial tableaus, that is, \( L_k \) is critical for any k ≥ 0. By Lemma 3.9, there are at most d − 1 critical tableaus for a combinatorial puzzle Λ. We say a critical combinatorial tableau is periodic of period s if s ≥ 1 is the smallest integer which satisfies \( d^s L_{k+s} = L_k \) for any k ≥ 0. A periodic critical tableau of period s is full if for all sufficiently large k,

\[ \sum_{0 ≤ m < s} (δ(d^m L_k) - 1) = d - 1. \]

**Example 3.10.** Let \( f(z) = z^2 - 1.14 + 0.241i \) be a quadratic polynomial having a superattracting periodic point of period 6. It has a fixed point \( x_0 \) with Angle(\( x_0 \)) = \{1/3, 2/3\} and a periodic point \( x_1 \) of period two with Angle(\( x_1 \)) = \{11/63, 44/63, 50/63\}. We can construct laminations \( Λ^0 = \{λ_0^f\} \) and \( Λ^2 = \{λ_2^f\} \) from \( x_0 \) and \( \{x_1, f(x_1)\} \) respectively as in Example 3.8. Namely, define \( λ_k^f = f^{-k}(x_1) \). Then \( Λ^0 \) separates the postcritical set from \( Λ^0 \). Figure 4 shows that the condition is fulfilled when k = 2.

The puzzle pieces containing the immediate basins of the superattracting periodic points form the critical tableau (introduced in the next section). We can see that the critical combinatorial tableau corresponds to this critical tableau.

8
4 Yoccoz puzzles

In this section, we introduce the notion of Yoccoz puzzles for a polynomial \( f \in \mathbb{C}_d \). Our definition is slightly different from the usual one; Usually, for \( k \geq 0 \), we divide \( D_f(r/d^k) \) into finitely many domains by landing rays of \( f^{-n}(O) \) for some repelling or parabolic periodic cycle(s) \( O \) and Yoccoz puzzles of depth \( k \) are defined by the closures of components. However, we need to treat “degenerate” puzzles, so we construct a Yoccoz puzzle from a combinatorial Yoccoz puzzle. We say a combinatorial puzzle for \( \lambda_f \) is admissible for \( f \). For an admissible puzzle \( \Lambda = \{\lambda_k\}_{k \geq 0} \) for \( f \), we construct the Yoccoz puzzle for a pair \((f, \Lambda)\) as follows:

Fix \( r > 0 \). For \( L \in \mathcal{P}_k(\Lambda) \), the complement of \( L \) consists of finitely many disjoint closed intervals

\[
\mathbb{R}/\mathbb{Z} \setminus L = \bigcup_{j=1}^{n} [\theta_j, \theta'_j].
\]

By Proposition 3.6, \( \theta_j \) and \( \theta'_j \) are \( \lambda_k \)-equivalent and lie in \( \mathbb{Q}/\mathbb{Z} \). Therefore, \( \theta_j \) and \( \theta'_j \) are \( \lambda_f \)-equivalent and both external rays \( R_f(\theta_j) \) and \( R_f(\theta'_j) \) land at a point \( x_j \).
Assume $\theta'_j$ and $\theta_{j+1}$ are adjacent in $\{\theta_j, \theta'_j; j = 1, \ldots, n\} \subset \mathbb{R}/\mathbb{Z}$ (where $\theta_{n+1} = \theta_1$). Let
\[
\gamma(L) = \bigcup_{j=1}^{n} (R_f(\theta_j, r/d^k) \cup \{x_j\} \cup R_f(\theta'_j, r/d^k) \cup E_j),
\]
where
\[
E_j = \{\varphi_j^{-1}(\exp(r/d^k + 2\pi i\theta)); \theta \in [\theta_j, \theta_{j+1}]\}
\]
is a subarc of the equipotential curve $E_f(\theta/d^k)$. The curve $\gamma(L)$ is a (not necessarily simple) closed curve and $\gamma(L)$ has a non-empty self intersection only when $x_j = x_{j'}$ for some $j \neq j'$. Let $P(L)$ be the complement of the unbounded component of $\mathbb{C} \setminus \gamma(L)$ for $L \in P_k(\Lambda)$. We call $P(L)$ a puzzle piece of depth $k$ for $(f, \Lambda)$. Let $P_k(f, \Lambda)$ be the set of all puzzle pieces of depth $k$ for $(f, \Lambda)$. Let $\Gamma_k = \bigcup_{\theta \in \text{supp.} \Lambda} R_f(\theta, r/d^k) \cup E_f(r/d^k)$.

**Proposition 4.1.** Let $f \in C_d$ and $\Lambda = \{\lambda_k\}_{k \geq 0}$ be an admissible puzzle for $f$. Then for $k \geq 0$ we have the following:

1. For $L \in P_k(\Lambda)$, let $\theta_j$ and $\theta'_j$ be as above and let $U_j = S(\theta_j, \theta'_j) \cap D_f(r/d^k)$. Then
   \[
   P(L) = D_f(r/d^k) \setminus \bigcup_j U_j,
   \]
2. Any $P \in P_k(f, \Lambda)$ is compact, connected, locally connected and full.
3. The interior of $P(L)$ is the union of all bounded components of $\mathbb{C} \setminus \Gamma_k$ which intersect $R_f(\theta)$ for some $\theta \in \Lambda$.
4. $P_k(f, \Lambda)$ is a partition for $D_f(r/d^k)$. More precisely, the interior of the puzzle pieces of depth $k$ are mutually disjoint and
   \[
   D_f(r/d^k) = \bigcup_{P \in P_k(f, \Lambda)} P.
   \]
5. For $P' \in P_{k+1}(f, \Lambda)$, there exists a unique $P \in P_k(f, \Lambda)$ containing $P'$.
6. For $P = P(L) \in P_{k+1}(f, \Lambda)$, we have $f(P) = P(d(L))$ and $f : \text{int } P \to \text{int } f(P)$ is a proper map of degree $\delta(L)$. In particular, $f(P)$ is a puzzle piece of depth $k$.

We say a compact connected set $K \subset \mathbb{C}$ is full if its complement is connected.

**Proof.** We first show (1). Let $\gamma_j$ be a simple closed curve defined by the following:
\[
\gamma_j = R_f(\theta_j, r/d^k) \cup \{x_j\} \cup R_f(\theta'_j, r/d^k) \cup \{\varphi_j^{-1}(e^{r/d^k + 2\pi i\theta}); \theta \in [\theta_j, \theta'_j]\}.
\]
Then $U_j$ is the domain bounded by $\gamma_j$. Since the equipotential part of $\gamma_j$ intersects $\gamma(L)$ only at its end points and $\mathbb{C} \setminus D_f(r/d^k)$ does not intersect $\gamma(L)$, $U_j$ is contained in the unbounded component of $\mathbb{C} \setminus \gamma(L)$. Hence $P(L)$ is contained in $D_f(r/d^k) \setminus \bigcup_j U_j$. Since the boundary of $D_f(r/d^k) \setminus \bigcup_j U_j$ is equal to $\gamma(L)$, the interior of $D_f(r/d^k) \setminus \bigcup_j U_j$ is the unbounded component of $\mathbb{C} \setminus \gamma(L)$. Therefore, we have proved (1).

The boundary $\partial P(L) = \gamma(L)$ is a union of finitely many simple closed curves, which intersect only at finitely many points. Hence $P(L)$ is compact, connected and locally connected. The fullness of $P(L)$ follows from the fact that $U_j$ ($j = 1, \ldots, n$) are connected, full and contained in the unbounded component of $\mathbb{C} \setminus \gamma(L)$. Thus we have proved (2).

To see (3), let $U$ be a component of the interior of $P(L)$. Since $\Gamma_k$ is contained in $\bigcup_j U_j \cup E_f(r/d^k)$, $U$ is a component of $\mathbb{C} \setminus \Gamma_k$. Take $\theta \in \mathbb{R}/\mathbb{Z}$ so that $\varphi_f^{-1}(\exp(r/d^k + 2\pi i\theta))$ lies in the interior of the curve $\partial U \cap E_f(r/d^k)$. Then $R_f(\theta)$ intersects $U$, and $\theta$ lies in $L$ because $\partial U \subset \gamma(L)$. On the other hand, for $\theta \in \mathbb{R}/\mathbb{Z} \setminus I = \bigcup_j (\theta_j, \theta'_j)$, the external ray $R_f(\theta) \subset (\mathbb{C} \setminus D_f(r/2^k)) \cup U_j$ for some $j$. Therefore, we have proved (3).
Take another combinatorial puzzle piece \( L' \in \mathcal{P}_k(\Lambda) \). Then \( L' \) is contained in \((\theta_j, \theta'_j)\) for some \( j \). Thus \( P(L') \subset U_j \) and it follows what the interiors of \( P(L) \) and \( P(L') \) are disjoint. The equation

\[
\mathbb{R}/\mathbb{Z} = \text{supp}(\lambda_k) \cup \bigcup_{L \in \mathcal{P}_k(\Lambda)} L
\]

implies that \( \mathcal{P}_k(f, \Lambda) \) is a partition of \( \overline{D_f(r/2^k)} \) by (3). We have proved (4).

(5) follows from (3), (4) and the fact \( \lambda_k \subset \lambda_{k+1} \).

By definition, we have \( \gamma(d(L)) = f(\gamma(L)) \) and each component of the complement of \( \Gamma_{k+1} \) is properly mapped by \( f \) to a component of the complement of \( \Gamma_k \). Thus \( f(P(L)) = P(d(L)) \) by (3).

Each component of \( \mathbb{C} \setminus \Gamma_{k+1} \) is properly mapped to some component of \( \mathbb{C} \setminus \Gamma_k \). Thus \( f : \text{int} P \rightarrow \text{int} f(P) \) is a proper map. Let \( \theta_1 \in d(L) \) and \( 0 < r_1 < r/d^k \). Then \( z = \varphi_f^{-1}(\exp(r_1 + 2\pi i \theta)) \in f(P) \). The cardinality of \( f^{-1}(z) \cap P \) is it equal to \( \delta(L) \) because

\[
f^{-1}(z) \cap P = \{ \varphi(\exp(r/d + 2\pi i \theta)); \ \theta \in d^{-1}(\theta_1) \cap L \}.
\]

Hence \( f : \text{int} P \rightarrow \text{int} f(P) \) has degree \( \delta(L) \) (although \( \text{int} P \) can be disconnected) and we have proved (6).

**Corollary 4.2.** The map \( P : \mathcal{P}_k(\Lambda) \rightarrow \mathcal{P}_k(f, \Lambda), \ L \mapsto P(L) \) is a bijection and the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{P}_{k+1}(\Lambda) & \xrightarrow{d} & \mathcal{P}_k(\Lambda) \\
\downarrow{P} & & \downarrow{P} \\
\mathcal{P}_{k+1}(f, \Lambda) & \xrightarrow{f} & \mathcal{P}_k(f, \Lambda)
\end{array}
\]

Let \( L = P^{-1} \). Then for \( P \in \mathcal{P}_k(f, \Lambda) \),

\[
L(P) = \{ \theta; \ R_f(\theta) \cap \text{int} P \neq \emptyset \},
\]

by Proposition 4.1 (3). Let \( \text{supp}(f, \lambda_k) \) be the set of landing points of \( R_f(\theta) \) for \( \theta \in \text{supp} \lambda_k \) and let \( \text{supp}(f, \Lambda) = \bigcup_k \text{supp}(f, \lambda_k) \). For \( x \in K(f) \setminus \bigcup_k \text{supp}(f, \lambda_k) \), let \( P_k(x; f, \Lambda) \in \mathcal{P}_k(f, \Lambda) \) be the unique puzzle piece of depth \( k \) for \((f, \Lambda)\) containing \( x \) and let \( L_k(x; f, \Lambda) = L(P_k(x; f, \Lambda)) \) be the corresponding combinatorial puzzle piece for \( \Lambda \). If \( \Lambda \) is non-degenerate and \( c \) is a critical point, then \( \{ L_k(f^n(c); f, \Lambda) \}_{n,k} \) is a critical combinatorial tableau (see the properties below). We call \( T(c; f, \Lambda) = \{ P_k(f^n(c); f, \Lambda) \}_{n,k} \) the critical tableau of \( c \).

By Proposition 4.1, critical tableaux have the following properties:

- \( P_k(f^n(c); f, \Lambda) \supset P_{k+1}(f^{n+1}(c); f, \Lambda) \).
- \( f : P_{k+1}(f^n(c); f, \Lambda) \rightarrow P_k(f^{n+1}(c); f, \Lambda) \) is a proper map of degree \( \delta(L_{k+1}(f^n(c); f, \Lambda)) \).
- If the corresponding combinatorial tableau \( \{ L_k(f^n(c); f, \Lambda) \} \) is periodic of period \( s \), then \( T(c; f, \Lambda) \) is also periodic of the same period \( s \), i.e., \( P_k(f^n(c); f, \Lambda) = P_k(c; f, \Lambda) \) for any \( k \).

When \( c \in \text{supp}(f, \Lambda) \), we may have some choice for \( P_k(f^n(c); f, \Lambda) \). For such a case, as the critical tableau of \( c \), we choose and fix a double sequence of puzzle pieces \( \{ P_k(f^n(c); f, \Lambda) \}_{n,k} \) with \( f^n(c) \in P_k(f^n(c); f, \Lambda) \) which satisfies the above properties.

## 5 Proof of Theorem 1.1

Now we prove Theorem 1.1.

First note that \( \lambda = \bigcap_{j \geq 0} \lambda_{f_j} \) is an equivalence relation defined by \( \theta \sim \theta' \) if and only if they are \( \hat{\lambda}_{f_j} \)-equivalent for all \( j \). Hence \( \lambda \) is a \( d \)-invariant real sublamination of \( \hat{\lambda}_{f_j} \) for any \( j \). This implies that \( \Lambda \) is admissible for \( f_j \).
If $\Lambda^j$ is not admissible for $f_{\infty}$, then there exists some $\theta, \theta'$ such that they are $\lambda_0^j$-equivalent and $R_{f_{\infty}}(\theta)$ and $R_{f_{\infty}}(\theta')$ land at different points. Hence the landing point of either $R_{f_{\infty}}(\theta)$ or $R_{f_{\infty}}(\theta')$ is a parabolic periodic point of $f_{\infty}$ by Lemma 2.3. In other words, if all landing points of $R_{f_{\infty}}(\theta)$ for $\theta \in \supp(\lambda_0^j)$ is repelling, then $\Lambda^j$ is admissible for $f_{\infty}$.

Now assume there exists some $\theta^1 \in \supp \lambda_0^1$ such that $R_{f_{\infty}}(\theta^1)$ lands at a parabolic periodic point $x^1$.

**Claim.** For any $j \geq 2$, there exists some $\theta^j \in \supp \lambda_0^j$ such that $R_{f_{\infty}}(\theta^j)$ lands at a parabolic periodic point $x^j$.

Assume the claim does not hold. Then there exists some $j \geq 2$ such that $R_{f_{\infty}}(\theta)$ lands at a repelling periodic point for any $\theta \in \supp \lambda_0^j$. Since $\Lambda^j$ separates postcritical set from $\Lambda^1$, there exists some $k \geq 0$ such that $P_k(f_{\infty}(c); f_{\infty}, \Lambda^j)$ does not intersect $\supp \lambda_0^1$. However, the immediate basin of $x^1$ must contain at least one critical point, so for any $k \geq 0$, there exists some critical puzzle piece $P \in \mathcal{P}_k(f_{\infty}, \Lambda^j)$ such that $x^1 \in P$, this is a contradiction. Hence we have proved the claim.

Let $x_j$ and $\theta_j$ be as in the claim and let $p_j \in \text{Per}(\Lambda^j)$ be the period of $\theta^j$ by $d$ for $j \geq 2$. By assumption, $p_1, \ldots, p_d$ are distinct. Therefore, the periodic orbits of $x^1, \ldots, x^d$ are mutually disjoint by Theorem 2.1 and parabolic. Namely, $f_{\infty}$ has at least $d$ parabolic periodic orbits. However, the number of parabolic periodic orbit is bounded by the number of the critical points (counted without multiplicity), which is at most $d - 1$. Hence this is a contradiction.

Therefore, $R_{f_{\infty}}(\theta)$ lands at a repelling periodic point for any $\theta \in \supp \lambda_0^1$. This also implies that $\Lambda^1$ is admissible for $f_{\infty}$.

## 6 Renormalizations

In this section, we introduce the notion of renormalization and show the relation between non-crossed renormalizations and periodic critical tableaux.

**Definition.** A map $f : U \to V$ is called a polynomial-like map if it is a proper holomorphic map between topological disks $U, V$ and $U$ is a relatively compact subset of $V$. The filled Julia set $K(f; U, V)$ and the Julia set $J(f; U, V)$ are defined as follows:

$$
K(f; U, V) = \bigcap_{n \geq 0} f^{-n}(U),
$$

$$
J(f; U, V) = \partial K(f; U, V).
$$

We say two polynomial-like maps of the same degree are hybrid equivalent if there exists a quasiconformal conjugacy defined between neighborhoods of the filled Julia sets such that the complex dilatation is zero almost everywhere in the filled Julia set. The following theorem is well-known [3]:

**Theorem 6.1 (Straightening Theorem).** Any polynomial-like map of degree at least two is hybrid equivalent to a polynomial of the same degree. If its filled Julia set is connected, then such a polynomial is unique up to affine conjugacy.

Therefore, we can consider landing angles, rational laminations and puzzles in the case of polynomial-like maps.

**Definition.** A polynomial $f \in \text{Poly}_d$ is renormalizable of period $s$ if there exist topological disks $U, V$ such that

1. $f^s : U \to V$ is a polynomial-like map of degree at least two with connected filled Julia set $K = K(f^s; U, V)$,
2. For any $j, k$ (0 ≤ $j < k < s$), $f^j(U) \cap f^k(U)$ does not contain any critical point of $f$,
3. $s \geq 2$ or $\text{Crit}(f) \not\subset U$.

We call a polynomial-like restriction $f^s : U \to V$ a renormalization of $f$. We do not distinguish renormalizations (whose range and domain differ) if they have the same filled Julia set.
Let

\[ I = I(f^s; U, V) = K \cap \left( \bigcup_{n=1}^{s-1} f^n(K) \right), \]

\[ K_0 = K \setminus \left( \bigcup_{n=1}^{s-1} f^n(K) \right) = K \setminus I(f^s; U, V). \]

Then \( I \) consists of at most one repelling fixed point of \( f^s \) and \( K_0 \) has only finitely many components (see [11] and [5]). We call \( I = I(f^s; U, V) \) the intersecting set for \( f^s : U \to V \). We say a renormalization is simple if \( K_0 \) is connected. A renormalization is called of disjoint type if \( K = K_0 \), or equivalently, \( K \cap f^n(K) \) is empty for all \( 1 \leq n < s \). We say a renormalization is non-crossed if either it is of disjoint type, or \( I = \{ x_0 \} \) and there exist landing angles \( \theta_1, \theta_2 \in \text{Angle}(x_0) \) such that \( K \subset S(\theta_1, \theta_2) \) and that if \( x_0 \in f^n(K) \) (\( 1 \leq n < s \)), then \( S(\theta_1, \theta_2) \cap f^n(K) \) is empty. We call a pair \( \theta_1, \theta_2 \) separating angles for a non-crossed renormalization \( f^s : U \to V \) which is not of disjoint type. If a renormalization is not non-crossed, then it is called crossed. By definition, we have

of disjoint type \( \Rightarrow \) simple \( \Rightarrow \) non-crossed,

but the converses do not hold. However, if \( f \) is unicusrical \( (f(z) = z^d + c) \), then any non-crossed renormalization is simple (see [11, § 7.3]). We say a renormalization is full if

\[ \bigcup_{n=0}^{s-1} f^n(K) \supset \text{Crit}(f). \]

This definition is equivalent to

\[ \bigcup_{n=0}^{s-1} f^n(U) \supset \text{Crit}(f). \]

It is clear that any renormalization of a unicusrical map is full.

**Example 6.2.** Here, we give examples of renormalizations of all of the types above.

1. A renormalization of disjoint type. Let \( f_1(z) = z^2 + c_1 \) with \( c_1 \approx -1.75 \) (the aeroplane), which has a superattracting periodic point of period 3. Hence \( f_1 \) has a full renormalization of
period 3, whose filled Julia set $K_3$ is the closure of the Fatou component containing the origin. It is easy to see that $K_4$, $f(K_1)$ and $f^2(K_1)$ are mutually disjoint and the renormalization is of disjoint type. See Figure 5 (see also Figure 1).

2. **A simple renormalization.** The Douady’s rabbit $f_3(z) = z^2 + c_2$ with $c_2 \approx -0.123 + 0.745i$ (see Figure 5) has a full simple renormalization of period 3. Let $K_2$ be the closure of the Fatou component containing the origin. Take a small neighborhood $V_2$ of $K_2$ and let $U_2 = f_2^{-3}(V_2) \cap V_2$. Then $f_2^3 : U_2 \to V_2$ is a quadratic-like map hybrid equivalent to $z^2$. Its filled Julia set is $K_2$ and $I(f_2^3; U_2, V_2) = \{x_2\}$ where $x_2$ is a repelling fixed point of $f_2$.

3. **A non-crossed renormalization.** Let $f_3(z) = z^3 - 3z/4 - \sqrt{7}i/4$ (the cubic polynomial in Example 3.8). Let $K$ be the closure of the union of the Fatou components containing 1/2 and $f(-1/2) = (1 - \sqrt{7}i)/4$. Let $K_3$ be a component of $\bigcup_{n \geq 0} f_3^{-2n}(K)$ containing $K$ (see Figure 6). Take a small neighborhood $V_3$ of $K_3$ and let $U_3 = f_3^{-2}(V_3) \cap V_3$. Then $f_3^2 : U_3 \to V_3$ is a full renormalization of period two, whose filled Julia set is $K_3$. The intersecting set $I_3 = I(f_3^2; U_3, V_3) = \{x_3\}$, where $x_3$ is a repelling fixed point of $f_3$ with Angle$(x_3) = \{1/4, 5/8, 3/4, 7/8\}$.

Since $K_3 \subset S(3/4, 1/4)$ and $f(K_3) \subset S(1/4, 3/4)$, it is a non-crossed renormalization. However, $R_{f_3^3}(5/8) \cup R_{f_3}(7/8) \cup \{x_3\}$ disconnects $K_3$. This implies that $K_3 \setminus I_3$ is disconnected. Therefore, it is not simple.

It also has non-full (quadratic) renormalizations whose filled Julia sets are the immediate basin of critical points.

4. **A crossed renormalization.** Let $f_4(z) = z^2 + c_4$ with $c_4 \approx 0.282 - 0.530i$, which has a super-attracting periodic orbit of period 4 and full renormalizations of period 2 and 4.

The renormalization $f_4^2 : U_4 \to V_4$ of period 2 is crossed: Its filled Julia set $K_4$ contains a fixed point $x_4 \approx 0.000615 + 0.531i$ with Angle$(x_4) = \{1/15, 2/15, 4/15, 8/15\}$. Furthermore, $K_4$ and $f(K_4)$ intersects at $x_4$ and

\[
K_4 \subset S(8/15, 1/15) \cup S(2/15, 4/15),
\]
\[
f(K_4) \subset S(1/15, 2/15) \cup S(4/15, 8/15).
\]

Therefore, there are no separating angles for $f_4^2 : U_4 \to V_4$. See Figure 6.

![Fig. 6 The Julia sets of $f_3(z) = z^3 - 3z/4 - \sqrt{7}i/4$ (left) and $f_4(z) \approx z^2 + 0.282 - 0.530i$ (right) with their renormalizations and the rays landing at the intersecting set.](image)

**Lemma 6.3.** Let $f^* : U \to V$ be a renormalization of $f$ with filled Julia set $K = K(f^*; U, V)$ and let $x_0 \in K$ be a repelling or parabolic periodic point of $f$. Then different components of $K \setminus \{x_0\}$ lie in different components of $K(f) \setminus \{x_0\}$. 

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For the proof, see [5, Lemma 3.6]. For \( \theta \in \text{Angle}_f(x_0) \), the ray \( R_f(\theta) \) is a path outside \( K(f^*:U,V) \) accumulating at \( x_0 \). By Lindelöf’s theorem, there exists a corresponding angle \( \theta' \in \text{Angle}_{f^*:U,V}(x_0) \) (\( R_{f^*:U,V}(\theta') \) and \( R_f(\theta) \) accumulate at \( x_0 \) from the same direction). Then \( d^s\theta' \in \text{Angle}_{f^*:U,V}(x_0) \) corresponds to \( d^s\theta' \in \text{Angle}_{f^*:U,V}(f^*(x_0)) \), where \( d' = \deg(f^*: U \to V) \). Conversely, for any \( \theta' \in \text{Angle}_{f^*:U,V}(x_0) \), there exists some \( \theta \in \text{Angle}_f(x_0) \) which corresponds to \( \theta' \) by Lemma 6.3.

A parabolic or repelling fixed point of a polynomial (or polynomial-like map) of degree \( d \) is called a \( \alpha \)-fixed point if its landing angles are fixed by \( d \). Otherwise it is called an \( \alpha \)-fixed point.

**Lemma 6.4.** Let \( f^*: U \to V \) be a non-crossed renormalization of \( f \). Let \( K = K(f^*: U, V) \) and assume \( I(f^*: U, V) = \{ x_0 \} \). Then the landing angles of \( x_0 \) for \( f \) have period \( s \). In particular, \( x_0 \) is a \( \beta \)-fixed point for the renormalization.

**Proof.** Let \( \theta_1, \theta_2 \) be separating angles for \( f^*: U \to V \). We can choose \( \theta_1, \theta_2 \) so that the interval \((\theta_1, \theta_2)\) is the smallest. Since \( d^s\theta_1, d^s\theta_2 \) are also separating angles, the interval \((d^s\theta_1, d^s\theta_2)\) contains \((\theta_1, \theta_2)\). Since \( f^* \) preserves the cyclic order of the landing rays for \( x_0 \), this can happen only when \( d^s\theta_1 = \theta_1 \) and \( d^s\theta_2 = \theta_2 \). Therefore, the landing angles of \( x_0 \) for \( f^* \) has period \( s \).

By Lemma 6.3, a landing angle \( \theta_0 \) of \( f \) corresponds to a landing angle \( \theta' \) of \( f^* \). Then an angle \( d\theta' \) for \( f \) corresponds to \( d\theta' \) for \( f^* \). Therefore, \( x_0 \) is a \( \beta \)-fixed point for the renormalization.

**Lemma 6.5.** Let \( f^*: U \to V \) and \( f'^*: U' \to V' \) be full non-crossed renormalizations with \( c \in U \cap U' \cap \text{Crit}(f) \). Assume \( s' \leq s \). Then \( s' \) divides \( s \) and \( K(f^{s'}:U', V') \) contains \( K(f^*:U, V) \).

In particular, \( f^*: U \to V \) can be considered as a full non-crossed renormalization of \( f'^*: U' \to V' \).

For the proof, see [11, Theorem 7.14] and [5, Proposition 3.8].

**Proposition 6.6.** Let \( f \in C_d \) and \( \Lambda = \{ \lambda_k \} \) be a non-degenerate admissible puzzle for \( f \). Take \( c \in \text{Crit}(f) \). If the critical combinatorial tableau \( \{ L_k(f^n(c); f, \Lambda) \} \) is periodic of period \( s \) and full and \( \text{supp}(f, \Lambda_0) \) consists only of repelling periodic orbits, then there exists a non-crossed renormalization \( f^*: U \to V \) with \( c \in U \) and \( K(f^*:U,V) \subset P_0(c; f, \Lambda) \).

**Proof.** First note that \( f^*: P_{k+s}(c; f, \Lambda) \to P_k(c; f, \Lambda) \) is a proper map. We would like to extend it to a polynomial-like map \( f^*: U \to V \) to obtain a desired renormalization.

Consider two puzzle pieces \( P_{k_1}(f^{n_1}(c); f, \Lambda) \) and \( P_{k_2}(f^{n_2}(c); f, \Lambda) \) for \( k_1, k_2 \geq s \), each of which does not contain other. If the intersection \( P_{k_1}(f^{n_1}(c); f, \Lambda) \cap P_{k_2}(f^{n_2}(c); f, \Lambda) \cap \text{K}(f) \subset \text{supp}(f, \Lambda) \) is nonempty, then it consists of only one point (say \( x_0 \)). Since the critical tableau is periodic of period \( s \),

\[
\{ f^n(x_0) \} = f^n(P_{k_1}(f^{n_1}(c); f, \Lambda) \cap P_{k_2}(f^{n_2}(c); f, \Lambda) \cap \text{K}(f)) = P_{k_1-s}(f^{n_1+s}(c); f, \Lambda) \cap P_{k_2-s}(f^{n_2+s}(c); f, \Lambda) \cap \text{K}(f) \cap P_{k_1}(f^{n_1}(c); f, \Lambda) \cap P_{k_2}(f^{n_2}(c); f, \Lambda) \cap \text{K}(f) = \{ x_0 \}.
\]

Therefore, \( x_0 \) is a repelling or parabolic fixed point of \( f^* \). In particular, it is not a critical point. Hence for any critical point \( c' \in \text{Crit}(f) \) and any \( k \geq 0 \), there exists a unique \( n \) such that \( 0 \leq n < s \) and \( c' \in P_k(f^n(c); f, \Lambda) \) because the critical combinatorial tableau is full.

Let \( k \geq 0 \) be sufficiently large so that the puzzle pieces \( P_{k-n}(f^n(c); f, \Lambda) \) \((n = 0, \ldots, s - 1)\) are distinct. Take a sufficiently small neighborhood \( V = V_{k-1,n+1} \cap P_{k-n}(f^{n+1}(c); f, \Lambda) \) so that \( V \setminus P_{k-1}(f^{n+1}(c); f, \Lambda) \) does not contain any critical value. Let \( U = U_{k,n} \) be the component of \( f^{-1}(V) \) containing \( P_{k}(f^n(c); f, \Lambda) \). If \( U \) contains a puzzle piece \( P \in P_k(f^n(c); f, \Lambda) \) with \( P \neq P_k(f^n(c); f, \Lambda) \) and \( f(P) = P_{k-1}(f^{n+1}(c); f, \Lambda) \), then there exists a critical point \( c' \in P \cap P_{k-1}(f^{n+1}(c); f, \Lambda) \). The local degree of \( f \) at \( c \) is strictly greater than that of \( f_{P_k}(f^n(c); f, \Lambda) \). Therefore, there exists some puzzle piece \( P' \in P_k(f^n(c); f, \Lambda) \) such that \( c' \in P' \) and \( \delta(U') > 1 \) for the corresponding combinatorial puzzle \( U' \in P_k(f^n(c); f, \Lambda) \) of \( P' \). Since the critical tableau is full, we have \( P' = P_k(f^n(c); f, \Lambda) \) for some \( n' \) which contradicts the fact \( P_k(f^n(c); f, \Lambda) \) does not contain any critical point. Therefore, \( U \) is a small neighborhood of \( P_k(f^n(c); f, \Lambda) \) and \( f: U \to V \) is a proper map of degree \( \delta(L_k(f^n(c); f, \Lambda)) \).
We may assume \( U_{k,n} \subset V_{k,n} \) and let \( \tilde{U} = f^{-s}(V_{k,s}) \cap U_{k+s,0} \). To see that \( \tilde{U} \) is relatively compact in \( V_{k,0} \), we must construct \( V_{k,0} \) so that its boundary consists of equipotential curve, external rays, and boundaries of small neighborhoods of points in \( \text{supp}(f, \lambda_0) \) and use the assumption that \( \text{supp}(f, \lambda_0) \) consists of repelling periodic orbits. Then \( f^s : \tilde{U} \to V_{k,0} \) is a renormalization and \( K(f^s; \tilde{U}, V_{k,0}) \subset P_{k+s}(c; f, \Lambda) \) (see [12] for more details). It is easy to see that it is full and non-crossed.

Remark 6.7. If \( \text{supp}(f, \lambda_0) \) contains a parabolic periodic orbit, then Proposition 6.6 does not hold. For example, consider a quadratic polynomial \( f(z) = z^2 - 3/4 \), the root of the baby Mandelbrot set \( M_{1/2} \) centered at \(-1\). Since \( M_{1/2} \) is attached to the main cardioid at \(-3/4\), \( f \) is not renormalizable. However, it is easy to see that \( \text{supp}(\lambda_f) = \bigcup_{n \geq 0} f^{-n}(1/3, 2/3) \) and \( \{ \lambda_{f,k} = f^k \} \) is a combinatorial Yoccoz puzzle having a period two critical combinatorial tableau.

7 Combinatorial characterizations for non-crossed renormalizations

In Proposition 6.6, we proved that the existence of a full periodic tableau implies the existence of a full non-crossed renormalization of the same period. In this section, we will prove the converse: if a polynomial has a full non-crossed renormalization of period \( s \), then there exists a combinatorial puzzle having a full periodic tableau of period \( s \) such that the renormalization is the same as the one obtained by Proposition 6.6.

For simplicity, we consider the first non-crossed renormalization, that is, non-crossed renormalization having the smallest period because other cases are similar since the \( n \)-th renormalization can be considered as the first renormalization of the \( (n-1) \)-st renormalization (see Remark 7.9 and Section 8 for details).

Theorem 7.1. Let \( c \in \text{Crit}(f) \). Assume \( f^s : U \to V \) is the non-crossed full renormalization for \( f \) of the smallest period \( s \) with \( c \in U \). Then there exists a non-degenerate admissible puzzle \( \Lambda \) for \( f \) such that the critical tableau of \( c \) for \( f \) is periodic of period \( s \) and full. In particular, the renormalization obtained by Proposition 6.6 is \( f^s : U \to V \).

Moreover, let \( K = K(f^s; U, V) \). Then

- Assume \( f^s : U \to V \) is of disjoint type. If a repelling periodic orbit \( O \) does not intersect \( K \) and each point in \( O \) has two or more landing angles, then we can choose \( \Lambda \) so that \( \text{supp} \lambda_0 = \text{Angle}(O) \).
- Assume \( f^s : U \to V \) is not of disjoint type and let \( \{ x_0 \} = I(f^s; U, V) \). Then \( \text{supp} \lambda_0 \subset \text{Angle}(O(x_0)) \), where \( O(x_0) \) is the forward orbit of \( x_0 \) by \( f \). Furthermore, if it is simple, then we can take \( \lambda_0 = \lambda_f[I_{\text{Angle}(O(x_0))}] \).

Definition. Let \( \Lambda \) be an admissible puzzle \( \Lambda \) for \( f \). We say that \( \Lambda \) characterizes a non-crossed renormalization \( f^s : U \to V \) if \( f^s : U \to V \) is obtained from \( \Lambda \) by Proposition 6.6. Such a puzzle \( \Lambda \) is called a (combinatorial) characterization of \( f^s : U \to V \) for \( f \). The above theorem says that there always exists a characterization.

Remark 7.2. When \( f^s : U \to V \) is of disjoint type, there always exists such a repelling periodic orbit. In fact, we can take \( x \) as a fixed point with non-fixed landing angles. Because there exists no non-repelling periodic point of period less than \( s \geq 2 \), \( f \) has exactly \( d \) repelling fixed points (counted without multiplicity) and since there are only \( d - 1 \) fixed angles by \( d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \), at least one fixed point has non-fixed landing angles. Since \( f^s : U \to V \) is of disjoint type and \( s \geq 2 \), \( x \notin K \).

Later we prove there exist infinitely many such periodic orbits (see Section 9).

Example 7.3. Let us consider the non-crossed renormalizations in Example 6.2.

(1) \( f_1(z) \approx z^2 - 1.75 \) (the aeroplane). We can prove that every real repelling periodic point of \( f_1 \) except the landing point of the fixed external ray \( R_{f_1}(0) \) has exactly two external angles. Therefore, there exists infinitely many periodic orbit satisfying the assumption of Theorem 7.1 (compare Lemma 9.3).

Take a real periodic orbit \( O \) satisfying the assumption of Theorem 7.1. Let \( \lambda_{1,k} \) be a equiva-
lence relation on \( \mathbb{R}/\mathbb{Z} \) whose nontrivial class has the form \( \text{Angle}(z) \) for some \( z \in f_1^{-k}(O) \). Then we can verify that \( \Lambda_1 = \{ \lambda_{1,k} \} \) is non-degenerate admissible combinatorial puzzle for \( f_1 \) and it characterizes the renormalization \( f_1^3 : U_1 \to V_1 \).

(2) \( f_2(z) \approx z^3 - 0.123 + 0.745i \) (the Douady’s rabbit). Since \( I(f_2^3; U_2, V_2) = x_2 \) is a repelling fixed point of \( f_2 \) with \( \text{Angle}(x_2) = \{ 1/7, 2/7, 4/7 \} \), we can construct a non-degenerate admissible combinatorial puzzle \( \Lambda_2 = \{ \lambda_{2,k} \} \) such that every nontrivial \( \lambda_{2,k} \) class has the form \( \text{Angle}(z) \) for some \( z \in f_2^{-k}(x_2) \) (or equivalently, \( \lambda_{2,k} = \lambda_{f_2^{-k}(x_2)} \)). Since \( R_{f_2}(1/7) \cup R_{f_2}(2/7) \cup R_{f_2}(4/7) \cup \{ x_2 \} \) separates \( K_2, f_2(K_2) \) and \( f_2^3(K_2) \), it is easy to see that \( \Lambda_2 \) is a characterization of \( f_2^3 : U_2 \to V_2 \).

(3) \( f_3(z) = z^3 - 3z^2 - \sqrt{7}i/4 \). Since 1/4 and 3/4 are separating angles for the renormalization \( f_3^2 : U_3 \to V_3 \), we can see that \( \Lambda' \) in Example 3.8 is a characterization. See Figure 3.

**Example 7.4.** Consider \( f_5(z) = z^2 + c_5 \) with \( c_5 \approx -1.54 \), such that \( x_5 = f_5^2(0) \approx -0.839 \) is a fixed point. Then it has a renormalization \( f_5^2 : U_5 \to V_5 \) of period two hybrid equivalent to \( z^2 - 2 \). Its (filled) Julia set is the interval \( K_5 = [x_5, f_5^2(0)] \) and it satisfies that \( K_5 \cap f_5^2(K_5) = \{ x_5 \} \).

According to Theorem 7.1, we can construct a characterization \( \Lambda_5 = \{ \lambda_{5,k} \} \) for \( f_5^2 : U_5 \to V_5 \) such that \( \text{supp} \lambda_{5,0} = \text{Angle}(x_5) = \{ 1/3, 2/3 \} \). Since the critical point 0 lies in \( f_5^{-3}(x_5) \), the combinatorial Yoccoz puzzle \( \{ \lambda_{f_5^{-k}(x_5)} \} \) is degenerate (see Figure 7). Hence we should be careful in constructing \( \lambda_{5,3} \).

For \( k = 0, 1, 2 \), let \( \lambda_{5,k} = \lambda_{f_5^{-k}(x_5)} \). Then define \( \lambda_{5,3} \) such that each nontrivial class has one of the following forms:

- \( \text{Angle}(z) \) for some \( z \in f_5^{-3}(x_5) \setminus \{ 0 \} \).
- \( \text{Angle}(0) \cup (0, 1/2) = [5/24, 7/24] \).
- \( \text{Angle}(0) \cup (1/2, 1) = [17/24, 19/24] \).

And for \( k \geq 4 \), define \( \lambda_{5,k} \) such that every nontrivial class is either nontrivial \( \lambda_{5,k-1} \)-class or of the form \( \text{Angle}(z) \cap 2^{-1}(A) \) where \( A \) is a \( \lambda_{5,k-1} \)-class and \( z \in f_5^{-k}(x_5) \setminus f_5^{-k-1}(x_5) \) (note that \( z : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z} \) is considered as a covering map).

We can verify that every nontrivial \( \lambda_{5,k} \) class has the form \( \text{Angle}(z) \) for \( z \in f_5^{-k}(\bar{x}_5) \) where \( f_5(z) = z^2 - 1 \) and \( \bar{x}_5 = \frac{1 - \sqrt{5}}{2} \) is a fixed point of \( f_5 \). Therefore, \( \Lambda_5 \) is non-degenerate.

Since \( P_k(f^{2n}(0); f, \Lambda_5) \) contains \( K_5 \) for any \( n, k \geq 0 \), \( \Lambda_5 \) is a characterization of \( f_5^2 : U_5 \to V_5 \).

Now let us give a proof of Theorem 7.1. Let \( K = K(f^s; U, V) \) be the filled Julia set of the renormalization and let \( K = \bigcup_{n=0}^{-1} f^K_n(K) \).

**Proof of Theorem 7.1 for renormalizations of disjoint type.** Assume the first non-crossed renormalization \( f^s : U \to V \) is of disjoint type.

Let \( O \) be a repelling periodic orbit as described. Let \( E_0 = \text{Angle}(O) \), \( E_k = d^{-k}(E_0) \) and \( \lambda_k = \lambda_k(E_k) \). Then \( d_k \lambda_k = \lambda_{k-1} \) and \( \Lambda = \{ \lambda_k \} \) is an admissible puzzle for \( f \). If \( K \) is not contained in any puzzle piece of depth \( k \) for some \( k \geq 0 \), then \( f^k(K) \) must intersect with \( O \) and that is a contradiction. Therefore, \( K \) is contained in a unique puzzle piece \( P_k(c; f, \Lambda) \) for any \( k \geq 0 \). Since \( f^s(K) = K \), \( P_k(f^s(c); f, \Lambda) = f^s(P_k(c); f, \Lambda) \) contains \( K \). Hence \( P_k(c; f, \Lambda) = P_k(f^s(c); f, \Lambda) \) and the critical tableau of \( f \) is periodic of period \( s \). By assumption, \( \bigcup_{n=0}^{-1} f^K_n(K) \subset \bigcup_{n} P_k(f^{n}(c); f, \Lambda) \) contains \( \text{Crit}(f) \). Therefore, the critical tableau of \( f \) is full. This also implies that \( \text{supp}(f, \Lambda) = \bigcup_{k} f^{-k}(O) \) does not intersect \( \text{Crit}(f) \), so \( \Lambda \) is non-degenerate.

By Proposition 6.6, there exists a full simple renormalization \( f^s' : U' \to V' \) with \( s' \leq s \). Since \( f^s : U \to V \) is the first full simple renormalization, we have \( s' = s \) and these two renormalizations are the same.

Now we assume \( f^s : U \to V \) is not of disjoint type. Then \( I(f^s; U, V) = \{ x_0 \} \) where \( x_0 \) is a repelling
Fig. 7 The Julia sets and puzzles for \( f^5(z) \approx z^2 - 1.54 \) (left) and \( \tilde{f}_5(z) = z^2 - 1 \) (right), and the combinatorial puzzle \( \Lambda_5 \) and the rational lamination \( \lambda_{f_5} \) (bottom).

fixed point of \( f^s \) and let \( \theta_0, \theta_0' \) be separating angles for \( f^s: U \to V \). Let

\[
E_0 = \{ d^n\theta_0, d^n\theta_0'; n \geq 0 \}, \\
E_k = f^{-k}(E_0), \\
E_\infty = \bigcup_k E_k.
\]

Then \( E_k \subset E_{k+1} \) and \( E_k \subset \text{Angle}(f^{-k}(O(x_0))) \). For \( z \in f^{-k}(O(x_0)) \), let \( E_k(z) = E_k \cap \text{Angle}(z) \). By Lemma 3.4, \( \lambda' = \lambda_f|_{E_\infty} \) is an invariant lamination on \( E_\infty \). Let \( \lambda_0 = \lambda'_{|E_0} = \lambda_f|_{E_0} \). We construct \( \lambda_k \) \((k > 0) \) by induction. Assume we have defined \( \lambda_{k-1} \). First, define \( \lambda_k = \lambda_{k-1} \) on \( E_{k-1} \). Take \( z \in f^{-k}(O(x_0)) \setminus f^{-k+1}(O(x_0)) \). If \( z \) is not a critical point, then \( \lambda_k \) on \( E_k(z) \) is defined by the pullback of \( \lambda_{k-1} \). Namely, \( \theta_1, \theta_2 \in E_k(z) \) are \( \lambda_k \)-equivalent if and only if \( d\theta_1 \) and \( d\theta_2 \) are \( \lambda_{k-1} \)-equivalent.

If \( z \) is a critical point, then \( z \in f^n(K) \) for a unique \( 0 < n < s \), because \( f^s: U \to V \) is full and \( z \not\in O(x_0) \). We define \( \lambda_k \) on \( E_k(z) \) as follows: \( \theta_1, \theta_2 \in E_k(z) \) are \( \lambda_k \)-equivalent if \( d\theta_1 \) and \( d\theta_2 \) are \( \lambda_{k-1} \)-equivalent and

\[
S(\theta_1, \theta_2) \cap f^n(K) = \emptyset \quad \text{or} \quad S(\theta_2, \theta_1) \cap f^n(K) = \emptyset.
\]

(1)

To see that \( \Lambda = \{ \lambda_k \}_{k \geq 0} \) is an admissible puzzle for \( f \), we need only show the following claim:
Claim. $d_n \lambda_k = \lambda_{k-1}$.

For $z \in f^{-k}(O(x_0)) \setminus f^{-k+1}(O(x_0))$, take a small neighborhood $U$ of $z$ where $f$ is a proper mapping into its image without any critical point other than possibly $z$. By coordinate change, $f(z)$ behaves like $w \mapsto w^\delta$ where $\delta = \delta(\text{Angle}(z))$. If $\delta = 1$ (that is, if $z$ is not a critical point), then $d : \text{Angle}(z) \to \text{Angle}(f(z))$ is an orientation preserving bijection and it is easy to see the claim. Assume $\delta \geq 2$, i.e., $z \in \text{Crit}(f)$.

Then the renormalization is full, there exists some $n \geq 0$ such that $z \in f^n(K)$. If $f^n(K)$ intersects a sector $S = S(\theta_1, \theta_2)$ for $\theta_1, \theta_2 \in E_k(z)$, then $f^n(K)$ intersects $S \cap U$, because $f^n(K) \subset K(f)$ is connected and does not intersect any external ray.

**Lemma 7.6.** The equivalence relation defined by (1) and $\lambda_k$ are equal on $E_k(z)$.

**Proof.** We prove the lemma by induction. Note that when $k = 0$ and $z \in O(x_0)$, the lemma holds even though there are several choices of $n$. Indeed, a $\lambda_0$-class on $E_0(z)$ is $E_0(z)$ itself. Each $f^n(K)$ is contained in a unique puzzle piece of depth 0. Hence the condition (1) is always true.

Let $k \geq 1$ and $\theta \in E_k(z)$. By construction, the $\lambda_k$-class of $\theta$ is contained in a $(1)$-equivalence class of $\theta$. Conversely, assume $\theta_1$ and $\theta_2$ satisfy (1). We may assume $S(\theta_1, \theta_2) \cap f^n(K)$ is empty. Then $f(S(\theta_1, \theta_2) \cap U)$ does not intersect $f^{n+1}(K)$. This implies that $S(\theta_1, \theta_2) \cap U \setminus S(d\theta_1, d\theta_2)$ also does not intersect $f^{n+1}(K)$. Hence, $d : A \to d(A)$ is a bijection. Since $A = \theta \in E_k(z)$ and $d : E_k(z) \to E_{k-1}(f(z))$ is a consecutive preserving bijection, $d(A) = [d\theta_1, d\theta_2] \subset E_k(z)$ by Lemma 3.3. Therefore, $d : A \to d(A)$ is a bijection.

On the contrary, assume $S(\theta_1, \theta_2)$ does not intersect $f^{n+1}(K)$ and $d : A \to [d\theta_1, d\theta_2] \subset E_{k-1}(f(z))$ is a consecutive preserving bijection. Take a component $(\theta, \theta')$ of $[\theta_1, \theta_2]$ \setminus $A$. Then $(\theta, \theta')$ is also a component of $\mathbb{R}/\mathbb{Z} \setminus E_k(z)$ and $(d\theta, d\theta')$ is a connected component of $\mathbb{R}/\mathbb{Z} \setminus E_{k-1}(f(z))$. If $f(S(\theta, \theta') \cap U)$ is not contained in $S(\theta', \theta'') \cap U$, then $f(S(\theta', \theta'') \cap U)$ is a neighborhood of $f(z)$. This implies that $d((\theta, \theta') \cap E_k(z)) = E_{k-1}(f(z))$.

By Lemma 3.3. However, it contradicts that $(\theta, \theta') \cap E_k(z)$ is empty. Thus $f(S(\theta, \theta') \cap U) \subset S(d\theta, d\theta')$. Taking a union, we have $f(S(\theta_1, \theta_2) \cap U) = S(d\theta_1, d\theta_2) \subset f(U)$. Therefore, if $S(\theta_1, \theta_2)$ intersects $f^n(K)$, then $f(S(\theta_1, \theta_2) \cap U) \subset S(d\theta_1, d\theta_2)$ also intersects $f^{n+1}(K)$. This proves the lemma.

**Lemma 7.7.** For every $z \in E_k \setminus E_{k-1}$ and any $\lambda_k$-class $A \subset E_k(z)$, $d(A)$ is an $\lambda_{k-1}$-class and $d : A \to d(A)$ is a consecutive preserving bijection.

**Proof.** By Lemma 7.5 and Lemma 7.6, any $\lambda_k$-class $A$ can be written as $[\theta_1, \theta_2] \cap E_k(z)$ where $\theta_1, \theta_2 \in E_k(z)$ such that $S(\theta_1)$ and $S(\theta_2)$ intersects $f^n(K)$ and $S(\theta_1, \theta_2)$ does not intersect $f^n(K)$. Thus by Lemma 7.5, $S(d\theta_1, d\theta_2)$ does not intersect $f^{n+1}(K)$, $d : A \to d(A)$ is a consecutive preserving bijection, and $d(A) = [d\theta_1, d\theta_2] \cap E_{k-1}(f(z))$ is a $\lambda_{k-1}$-class by Lemma 7.6. Therefore, we have proved the claim. We have also proved that $\Lambda$ is non-degenerate. It remains to prove that the critical tableau of $c$ for $f$ is periodic of period $s$ and full.

**Lemma 7.8.** For any $k \geq 0$ and $0 \leq n < s$, $f^n(K)$ is contained in a unique puzzle piece of depth $k$. 

Hence we have $p$ is the period of landing angles of $f^n(K)$. By Proposition 4.1 (1), $P$ can be written as

$$P = D_f(r/d^k) \setminus \bigcup_j U_j,$$

where $U_j = S(\theta_j, \theta'_j) \cap D_f(r/d^k)$ and $\theta_j$ and $\theta'_j$ are $\lambda_0$-equivalent. By definition, $\theta_j$ and $\theta'_j$ satisfy the condition (1) and since $S(\theta'_j, \theta_j)$ intersect $f^n(K)$, $S(\theta_j, \theta'_j)$ does not intersect $f^n(K)$. Therefore, $P$ contains $f^n(K)$. \hfill \square

By the lemma above, there exists a unique puzzle piece $P_k(f^n(c); f, \Lambda)$ containing $f^n(c)$ such that $f^n(K) \subset P_k(f^n(c); f, \Lambda)$. Since $f^n(K) = K$, we have $K \subset P_k(f^n(c); f, \Lambda)$. Therefore, $P_k(f^n(c); f, \Lambda) = P_k(c; f, \Lambda)$ and $P_k(f^n(c); f, \Lambda)$ is a periodic critical tableau of period $s$. The fullness of $f^s : U \to V$ implies that $\text{Crit}(f) \subset \bigcup_n f^n(K) \subset \bigcup_n P_k(f^n(c); f, \Lambda)$. Therefore, the critical tableau $\{P_k(f^n(c); f, \Lambda)\}$ of $c$ is full. As in the proof of renormalization of disjoint type, $\Lambda$ is a characterization of $f^s : U \to V$. Therefore, we have proved the Theorem 7.1.

Remark 7.9. In the case of non-disjoint type, we can show without the assumption that the renormalization has the smallest period. Indeed, assume $f^n(K)$ and $f^m(K)$ (0 $\leq$ $n$ $\neq$ $m$ $<$ $s$) are contained in the same puzzle piece of depth $k$ for $(f, \Lambda)$ and any $k$ $\geq$ 0. If $f^n(K) \cap f^m(K)$ is nonempty, then it is equal to $(f^n(x_0))$ and $f^n(K)$ and $f^m(K)$ must lie in the different puzzle piece of depth zero. Thus $f^n(K) \cap f^m(K)$ is empty and we have $f^n(x_0) \neq f^m(x_0)$. Let $p(< s)$ be the period of $x_0$. Then $f^{n+p}(K)$ and $f^{m+p}(K)$ must lie in the different puzzle piece of depth zero because there exists only one puzzle piece $P \in P_0(f, \Lambda)$ containing both $f^n(x_0)$ and $f^m(x_0)$, but $f^n(K) \neq f^{n+p}(K)$ and $f^m(K) \cap f^{n+p}(K) = \{f^n(x_0)\}$, so $f^{n+p}(K)$ is not contained in $P$. Therefore, the critical tableau has period $s$ and the renormalization obtained by Proposition 6.6 is equal to $f^s : U \to V$.

In the last part of this section, we show a relation for the period of angles in $E_0$ and the period $s$ of the renormalization.

Proposition 7.10. In Theorem 7.1, we can take $\Lambda$ so that the period $p$ of landing angles in $E_0 = \text{supp}(\lambda_0)$ satisfies that

$$1 < p \leq s.$$

Proof. First we show the proposition in the case of renormalizations of disjoint type. We can take $x_0$ in the proof of Theorem 7.1 as a fixed point having non-fixed landing angles (see Remark 7.2). Thus we have $p > 1$. The map $\delta = d_{\text{Angle}(x_0)} : \text{Angle}(x_0) \to \text{Angle}(x_0)$ is a cyclic order preserving bijection. The landing rays for $x_0$ divide the plane into at least $p$ component. Assume there exists a component $S = S(\theta_1, \theta_2)$ such that $S$ does not intersect $f^n(K)$ for any $0 \leq n < s$. Since $f^s : U \to V$ is full, $S$ does not have any critical value. Therefore, $S_1 = S(\delta^{-1}\theta_1, \delta^{-1}\theta_2)$ does not intersect $f^n(K)$ for any $n \geq 0$. Repeating this argument, we can show that $S_k = S(\delta^{-k}\theta_1, \delta^{-k}\theta_2)$ does not intersect $f^n(K)$ for any $k$ and $n$. Since $S_p = S$, this implies $\{f^n\}_{n \geq 0}$ forms a normal family on $S$, that is a contradiction. Therefore, we have $s \geq p$.

Now we prove the proposition in the case of renormalizations not of disjoint type. By the construction, $p$ is the period of landing angles of $x_0$. We have $p = s$ by Lemma 6.4. Furthermore, the period of the landing angles of $x_0$ is greater than one. (otherwise, we have $s = 1$ and $f^s : U \to V$ cannot be full). Hence we have $p > 1$. \hfill \square

8 Proof of Theorem 1.2

First we make a remark about a characterization for a renormalization not of the smallest period. Although Theorem 7.1 treats only the first renormalization, there exists a characterization for any non-crossed renormalization. Indeed, any full non-crossed renormalization $f^s : U \to V$ can be considered as the first full non-crossed renormalization of some full non-crossed renormalization (polynomial-like map) $f^s : U' \to V'$ (or $f$ itself) by Lemma 6.5. By Theorem 7.1, there exists a characterization $\Lambda'$ of $f^s : U \to V$ for $f^s : U' \to V'$. Let $K = K(f^s; U, V)$ and $K' = K(f^s; U', V')$. 

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By Lemma 6.3, an external angle for \( f^s : U' \to V' \) corresponds to some external angle for \( f \) (although not unique). By using this, we can obtain a sequence of laminations for \( f \) and as in the proof of Theorem 7.1, we can “extend” it to a characterization of \( f^s : U \to V \) for \( f \).

This also proves that we can construct a characterization \( \Lambda = \{ \lambda_k \} \) of \( f^s : U \to V \) for \( f \) satisfying that \( \text{supp}(f, \lambda_0) \subset \bigcup_n f^n(K') \). Furthermore, if \( f^s : U \to V \) is of disjoint type, we can take \( \text{supp}(\lambda_0) = \text{Ang}(O(x_0)) \) for any repelling periodic point \( x_0 \in K' \) with \( O(x_0) \cap K = \emptyset \). In particular, we can take \( x_0 \) as a repelling fixed point of \( f^s : U' \to V' \), whose landing angles for \( f \) are not fixed by \( d' \).

Now we claim that \( s' < \text{Per}(\Lambda) \leq s \). If we have \( x_0 \notin \bigcup_{n \geq 0} f^n(I(f^s; U, V)) \), then the period of \( f^s \) is equal to \( s' \) by Lemma 6.4 (see also Remark 7.9). Therefore, the landing angles of \( y \) for \( f^s : U' \to V' \) are fixed. By Proposition 7.10, the period of landing angles of \( x_0 \) for \( f^s : U' \to V' \) is strictly greater than one. Therefore, \( x_0 \) does not lie in the periodic orbit of \( y \).

Now we have proved the following:

**Lemma 8.1.** Assume a polynomial \( f \in \mathcal{C}_d \) has \( J \) full non-crossed renormalizations \( f^s_j : U_j^j \to V_j^j \) for \( j = 1, \ldots, J \) and assume \( s_1 < s_2 < \cdots < s_J \). Then there exists a characterization \( \Lambda = \{ \lambda_k \} \) for \( f^s_j : U_j^j \to V_j^j \) for \( j = 1, \ldots, J \) such that \( \text{Per}(\Lambda) = \{ p_j \} \) and \( s_j < p_j \leq s_{j+1} \). In particular, we have \( p_j < p_{j+1} \).

Let \( f^s : U \to V \) be a full non-crossed renormalization. Applying the lemma above, for a maximal sequence of full non-crossed renormalizations with \( s_j = s \), \( U^j = U \) and \( V^j = V \), we can obtain characterizations with some nice properties in a natural way.

**Definition.** For a full non-crossed renormalization \( f^s : U \to V \) for \( f \in \mathcal{C}_d \), a characterization \( \Lambda \) is called nice if \( \text{Per}(\Lambda) = \{ p \} \), \( s < p \) and \( p < s' \) for any full non-crossed renormalization \( f^{s'} : U' \to V' \) with \( s' > s \).

Because there may be several choices for a fixed point \( x_0 \) to construct \( \Lambda \) when a renormalization is of disjoint type, a nice characterization is not unique in general.

Now we can give the detailed statement and a proof of the Theorem 1.2.

**Theorem 1.2.** Let \( f_t \to f_\infty \) be a convergent sequence in \( \mathcal{C}_d \). Assume \( f_j \) has \( d \) full non-crossed renormalizations \( f_j^l : U_j^l \to V_j^l \), whose nice characterizations \( \Lambda^l \) are independent of \( l \) for any \( j = 1, \ldots, d \), and assume \( s_1 < s_2 < \cdots < s_d \).

Then the limit \( f_\infty \) has a full non-crossed renormalization which is characterized by \( \Lambda^1 \).

**Proof.** Let \( \Lambda^j = \{ \lambda_k^j \}_{k \geq 0} \) and let \( \text{Per}(\Lambda^j) = \{ p_j \} \) (\( p_1 < p_2 < \cdots < p_d \)). For \( j \geq 2 \), if \( \bigcap_{k > 0} \mathcal{P}_k(f^l_j(c); f_1, \Lambda^j) = K(f^s_j; f_1, \Lambda^j) \) contain some periodic point of period less than \( p_j \), then it is a \( \beta \)-fixed point of \( f^s_j \). In particular, \( \mathcal{P}_k(f^j_l; f_j, \Lambda^j) \) does not intersect \( \text{supp}(f, \lambda_0^j) \) for sufficiently large \( k \). This implies that the corresponding combinatorial puzzle piece \( L_k(f^j_l(c); f_j, \Lambda^j) \) does not intersect \( \text{supp}(\lambda_0^j) \) for sufficiently large \( k \). Since the critical combinatorial table \( \{ L_k(f^j_l(c); f_j, \Lambda^j) \} \) is periodic and full, it contains the postcritical set \( \mathcal{P}(\Lambda^j) \). Therefore, \( \Lambda^j \) separates the postcritical set from \( \Lambda^1 \).

By Theorem 1.1, \( \Lambda^1 \) is an admissible puzzle for \( f_\infty \) and \( \text{supp}(f_\infty, \lambda_0) \) consists of repelling periodic points, hence by Proposition 6.6, \( f_\infty \) has a renormalization for which \( \Lambda^1 \) is a nice characterization.

**Corollary 8.2.** Let \( f_t \to f_\infty \) be a convergent sequence in \( \mathcal{C}_d \). Assume there exists a sequence of combinatorial Yoccoz puzzles \( (\Lambda^j)_{j \geq 1} \) such that \( f_t \) has \( l \) full non-crossed renormalizations \( f^l_j : U_j^l \to V_j^l \) characterized by \( \Lambda^j \) for \( j = 1, \ldots, l \) and assume \( s_j < s_{j+1} \) for all \( j \). Then \( f_\infty \) has infinitely many renormalizations characterized by \( \Lambda^1, \Lambda^2, \ldots \).

This corollary immediately implies the following:

**Corollary 8.3.** The family of infinitely fully renormalizable polynomials of a given combinatorics forms a compact subset of the connectedness locus.

**Remark 8.4.** As noted after Theorem 1.2 in the Introduction, we do not require that \( f_l \) has exactly \( l \) full
non-crossed renormalizations. Therefore, for any (not necessarily convergent) sequence \( \{f_l\} \subset \mathcal{C}_d \) such that \( f_l \) has \( l \) full non-crossed renormalizations characterized by \( \Lambda^1, \ldots, \Lambda^l \), we can take a subsequence satisfying the assumption of Corollary 8.2 as follows:

Since \( (\Lambda^i)_{i \geq 1} \) satisfies that \( \lim_{i \to \infty} \text{Per}(\Lambda^i) = \infty \), we can take a subsequence \( (\Lambda^{i_k})_{k \geq 1} \) such that \( (\text{Per}(\Lambda^{i_k}))_{k \geq 1} \) is an increasing sequence. Since \( \mathcal{C}_d \) is compact, we can take a convergent subsequence \( \{f_{i_k}\}_{k \geq 1} \). Then \( f_{i_k} \) has \( j_k \) renormalizations characterized by \( \Lambda^{i_1}, \ldots, \Lambda^{i_k} \). In particular, \( f_{i_k} \) has \( k \) renormalizations characterized by \( \Lambda^{i_1}, \ldots, \Lambda^{i_k} \) (note that \( k \leq j_k \)).

**Proof.** Apply Theorem 1.2 to \( \{f_{i_k}\}_{k \geq 1} \to f_\infty \) and combinatorial Yoccoz puzzles \( \Lambda^0, \ldots, \Lambda^{a+d} \) for each \( l_0 \geq 1 \).

\[\square\]

### 9 Proof of Theorem 1.3

The detailed statement of Theorem 1.3 is the following:

**Theorem 1.3.** Let \( f_l \to f_\infty \) be a convergent sequence in \( \mathcal{C}_d \). Assume for any \( l \), \( f_l \) has a full renormalization \( f^*_l : U_l \to V_l \) of disjoint type, whose characterization \( \Lambda \) is independent of \( l \).

Then the limit \( f_\infty \) has a full renormalization of disjoint type characterized by \( \Lambda \).

This theorem is a corollary of the following theorem. Let \( K_l = K(f^*_l; U_l, V_l) \) be the filled Julia set of the renormalization for \( f_l \) and let \( K_\infty = \bigcup_{n \geq 0} f^n(K_l) \).

**Theorem 9.1.** Under the same assumptions as Theorem 1.3, there exist infinitely many admissible puzzles \( \Lambda' = \{\lambda^l_k\}_{k \geq 0} \) for \( f_l \) characterizing \( f^*_l : U_l \to V_l \) such that \( \Lambda' \) is independent of \( l \), the ray of any angle \( \theta \in \text{supp}(\lambda^l_k) \) for \( f_l \) lands at a repelling periodic point outside \( K_l \), and \( \text{supp}(f_l, \lambda^l_0) \) consists of only one periodic cycle.

First, we assume this theorem and prove Theorem 1.3.

The proof of Theorem 1.3. Let \( \Lambda = \{\lambda_k\} \) and let \( x \in \text{supp}(f_\infty, \lambda_0) \) be a repelling periodic point of period \( p \). Let \( j \geq 1 \). Since \( \Lambda^j \) characterizes \( f^*_l : U_l \to V_l \), it follows that

\[
\bigcap_{k} \bigcup_{n} P_k(f^n(c); f_l, \Lambda^j) = K_l \supset \text{PC}(f).
\]

Therefore, \( x \) does not lie in the puzzle piece of the form \( P_k(f^n(c); f_l, \Lambda^j) \) for sufficiently large \( k \). Hence \( \Lambda^j \) separates the postcritical set from \( \Lambda^l \) for any \( j \neq j' \).

In particular, \( \Lambda^j \) separates the postcritical set from \( \Lambda \) for any \( j \geq 1 \). By Theorem 1.1, \( \Lambda \) is admissible for \( f_\infty \) and \( \text{supp}(f_\infty, \Lambda) \) consists of repelling periodic points. Hence Proposition 6.6 implies that \( f_\infty \) has a renormalization characterized by \( \Lambda \).

The proof of Theorem 9.1 is divided into two parts.

**Lemma 9.2.** Assume there exists a repelling periodic point \( x \) for \( f_1 \) which satisfy \# \( \text{Angle}_{f_1}(x) \geq 2 \) and \( x \notin K_1 \). Then for any \( l \), there exists a repelling periodic point \( x_l \notin K_1 \) for \( f_l \) such that \( \text{Angle}_{f_1}(x_l) = \text{Angle}_{f_1}(x) \).

In particular, there exists a combinatorial puzzle \( \Lambda' = \{\lambda^l_k\} \) such that \( \Lambda' \) is admissible for \( f_l \), \( \text{supp}(f_l, \lambda^l_0) \) consists of the periodic cycle of \( x_l \) and \( \Lambda' \) characterizes \( f^*_l : U_l \to V_l \) for any \( l \).

**Lemma 9.3.** There exist infinitely many repelling periodic points \( x_j \) (\( j = 1, 2, \ldots \)) for \( f_1 \) such that \# \( \text{Angle}_{f_1}(x_j) \geq 2 \) and \( x_j \notin K_1 \).

Clearly, the two lemmas above imply Theorem 9.1.

**Proof of Lemma 9.2.** For a repelling periodic point \( x \notin K_1 \) of period \( p \) for \( f_1 \), let us consider puzzle pieces \( P_k(x; f_1, \Lambda) \) (\( k = 0, 1, \ldots \)) containing \( x \). They are nested and \( f^p : P_{k+p}(x; f_1, \Lambda) \to P_k(x; f_1, \Lambda) \)
has degree one for sufficiently large $k$. Since $P_{k+n}(x; f, \Lambda) = (f^n|_{P_{k+n}(x; f, \Lambda)})^{-1}(P_{k}(x; f, \Lambda))$, we have
\[
\bigcap_{k \geq 0} P_{k}(x; f, \Lambda) = \{x\},
\]
by the Schwarz lemma. Hence we have
\[
\bigcap_{k \geq 0} L_k(x; f, \Lambda) = \text{Angle}_{f_1}(x).
\]
For $l \geq 2$, let $P_{k,l} \in \mathcal{P}_{f_1}(f, \Lambda)$ be the corresponding puzzle piece of $L_k(x; f, \Lambda)$ for $(f, \Lambda)$. Then $f^n_l : P_{k+n,l} \to P_{k,l}$ also has degree one. Hence we have $\bigcap_{k \geq 0} P_{k,l} = \{x_l\}$ for some $x_l$. Since critical orbits are contained in the critical tableau, $x_l$ is a repelling periodic point of period $p$ for $f_l$ and
\[
\text{Angle}_{f_1}(x_l) = \bigcap_k L_k(x_l; f_1, \Lambda) = \bigcap_k L_k(x; f_1, \Lambda) = \text{Angle}_{f_1}(x).
\]
Similarly, all backward images of $x_l$ by iterates of $f_l$ and their landing angles are determined only in terms of $\Lambda$. Therefore, the combinatorial puzzle constructed from $x$ by Theorem 7.1 is admissible for $f_l$. \hfill \Box

To prove Lemma 9.3, we use the result of Poirier [14] on Hubbard trees, which was first introduced by Douady and Hubbard [2].

**Definition.** For a postcritically finite polynomial $f$, a regulated arc is a Jordan arc $\gamma$ in $K(f)$ such that the intersection with the closure of a bounded Fatou component consists of (at most two) segments of internal rays. For any points $z, z' \in K(f)$, there is a unique regulated arc joining them, which we denote by $[z, z']_{K(f)}$.

We say a subset $X \subset K(f)$ is regulated connected if for any $z, z' \in X$, we have $[z, z']_{K(f)} \subset X$. The regulated hull $[X]_{K(f)}$ of $X \subset K(f)$ is the smallest closed regulated connected subset of $K(f)$ containing $X$.

The Hubbard tree $H = H_f$ of $f$ is the regulated hull of $\text{Crit}(f) \cup \text{PC}(f)$. For $z \in H$, the incidence number $\nu_H(z)$ is the number of components of $H \setminus \{z\}$. We say $z$ is a branching point if $\nu_H(z) > 2$ and an end if $\nu_H(z) = 1$.

Note that for a finite set $X$, the regulated hull $[X]_{K(f)}$ is a finite topological tree [14, Proposition 1.6]. Since $\text{Crit}(f) \cup \text{PC}(f)$ is forward invariant and contains all critical points, $f(H) \subset H$ [14, Lemma 1.10]. Therefore, we always have $\nu_H(f(z)) \geq \nu_H(z)/\deg_f(z)$ where $\deg_f(x)$ is the local degree of $f$ at $z$. In particular, let
\[
S_0 = \text{Crit}(f) \cup \text{PC}(f) \cup \{\text{branching points}\} \subset H.
\]
Then $S_0$ is forward invariant by $f$ because for a branching point $z \in H$, either $f(z)$ is also a branching point or $z$ is a critical point. For a forward invariant set $S \subset H$ containing $S_0$, we can consider $S$ as the set of vertices for the tree $H$. (Note that all ends of $H$ is contained in $\text{Crit}(f) \cup \text{PC}(f)$. ) Define the distance $d_S(v, v')$ for $v, v' \in S$ by the number of edges between $v$ and $v'$. Then $f : H \to H$ is “expanding” in the following sense.

**Proposition 9.4.** Let $S \subset H$ be a set of vertices. Then for any $v, v' \in S \cap J(f)$ with $d(v, v') = 1$, there is an $n \geq 1$ such that $d(f^n(v), f^n(v')) > 1$.

See [14, Proposition 1.21]. One of the main results of [14] is the following:

**Theorem 9.5.** For a eventually periodic point $z \in H \cap J(f)$, the number of rays which land at $z$ is equal to $\nu_H(z)$.

**Proof of Lemma 9.3.** First we construct a postcritically finite polynomial $f$ with a renormalization characterized by $\Lambda$. Then by Lemma 9.2, it is enough to show the lemma for $f$ instead of $f_1$. 23
Claim 1. There exists a postcritically finite $d$-invariant rational lamination $\lambda$ such that $\Lambda$ is a combinatorial Yoccoz puzzle for $\lambda$.

Define an equivalence relation $\lambda$ on $\mathbb{Q}/\mathbb{Z}$ such that $\theta_1$ and $\theta_2$ are $\lambda$-equivalent if they are $\lambda_{f_i}$-equivalent and

$$K_1 \subset \overline{S(d^n\theta_1, d^n\theta_2)} \text{ or } \overline{S(d^n\theta_2, d^n\theta_1)}$$

for any $n \geq 0$. It is easy to see that $\lambda_k \subset \lambda$.

Since $\lambda \subset \lambda_{f_i}$, any $\lambda$-equivalence class is finite and it is easy to see that $\lambda$-equivalence classes are pairwise unlinked. Assume $\theta_{i,j} \rightarrow \theta_i$ (j $\rightarrow$ $\infty$) for $i = 1, 2$ and $\theta_{1,j}, \theta_{2,j}$ are $\lambda$-equivalent for any $j$. Fix $n \geq 0$ and we may assume $K_1 \subset \overline{S(d^n\theta_{1,j}, d^n\theta_{2,j})}$ for any $j$. Then $K_1 \subset \bigcap_j \overline{S(d^n\theta_{1,j}, d^n\theta_{2,j})} \subset \overline{S(d^n\theta_1, d^n\theta_2)}$ (note that $\theta_1$ and $\theta_2$ are $\lambda_{f_i}$-equivalent). Therefore, $\lambda$ is closed and $\lambda$ is a rational lamination. Moreover, we can prove that $\lambda$ is $d$-invariant in the same way as the claim in the proof of Theorem 7.1.

By definition, $K_1$ is contained in $P(L)$ (the definition is the same as in the case of Yoccoz puzzle pieces) for some $\lambda$-unlinked class $L$ with $d^k(L) = L$. Since the renormalization $f^+_i : U_1 \rightarrow V_1$ is full, all critical $\lambda$-unlinked class lies in the periodic orbit of $L$ and there exists no critical $\lambda$-class. In particular, $\lambda$ is postcritically finite and we have proved the claim.

By Theorem 3.7, there exists a postcritically finite polynomial $f$ of degree $d$ such that $\lambda = \lambda_f$. Since $\lambda_k \subset \lambda$ for any $k$, $\Lambda$ is admissible for $f$ and since $f$ has no parabolic periodic point, there exists a renormalization $f^* : U \rightarrow V$ characterized by $\Lambda$ by Proposition 6.6. Let $K = K(f^*; U, V)$ and $\mathcal{K} = \bigcup_{n \geq 0} f^n(K)$.

It remains to prove the following:

Claim 2. $f$ has infinitely many repelling periodic orbits outside $\mathcal{K}$.

Consider the Hubbard tree $H$ of $f$. We will find a horseshoe structure in the dynamics on $H$ to prove the claim.

Since $f^i : U \rightarrow V$ is of disjoint type, $f^i(K) \cap f^j(K) = \emptyset$ for any $0 \leq i \neq j < s$. This implies that $H \setminus \mathcal{K}$ is nonempty. Let

$$S = S_0 \cup (\mathcal{K} \cap H \setminus \mathcal{K}).$$

Since $S$ is a finite set, we consider $S$ as the vertices of $H$. The following lemma and the forward invariance of $S_0$ asserts that $S$ is forward invariant by $f$.

Lemma 9.6. Let $v \in \mathcal{K} \cap H \setminus \mathcal{K}$ and consider an edge $e = [v, v']_{K(f)}$ of $H$. Then $e \subset H \setminus \mathcal{K}$ if and only if there exists some neighborhood $U$ of $v$ in $e$ such that $f(U) \subset H \setminus \mathcal{K}$.

Proof. First note that $f$ is injective on $U$ and if $U$ is sufficiently small, then either $f(U) \subset H \setminus \mathcal{K}$ or $f(U) \subset \mathcal{K}$.

If $U \subset \mathcal{K}$, then $f(U) \subset \mathcal{K}$ because $\mathcal{K}$ is forward invariant. So suppose $U \subset H \setminus \mathcal{K}$. There exists an edge $e' = [v, v']_{K(f)}$ such that $e' \subset \mathcal{K}$. We may assume $e' \subset f^i(K)$ for some $0 \leq i < s$. If $f(U) \subset \mathcal{K}$, then $f(U) \subset f^i(K)$ for some $j$. Since $v$ is not a critical point, we must have $i \neq j$. But this implies that $f^* : U \rightarrow V$ is not of disjoint type and it is a contradiction. Therefore, $f(U) \subset H \setminus \mathcal{K}$. \qed

Let $d_{S, (\cdot, \cdot)}$ be the distance on $S$ explained before. Every edge of $H$ is contained in either $\mathcal{K}$ or $H \setminus \mathcal{K}$. Let $E = \{e_1, \ldots, e_N\}$ be the set of edges which are contained in $H \setminus \mathcal{K}$. Since $f^* : U \rightarrow V$ is of disjoint type, $E$ is nonempty. Furthermore, the ends of $e_n$ lie in the Julia set for any $n$. In fact, let $v \in S$ be an end of $e_n$. Then either $v \in \mathcal{K} \cap H \setminus \mathcal{K} \subset \partial \mathcal{K} \subset J(f)$ or $v$ is a branching point. Hence we may assume $v$ is a branching point. Since $f^* : U \rightarrow V$ is full, $\text{Crit}(f)$ is contained in the interior of $K$, so $v$ is not a critical point. This implies that $f^k(v)$ is also a branching point for any $k \geq 0$. Since the Fatou set of $f$ consists of the basin of infinity and the backward images of the interior of $\mathcal{K}$, we have $v \in J(f)$.

Lemma 9.7. Denote $e_n = [v_n, v'_n]_{K(f)}$. Then there exists some $k \geq 1$ such that $f^k(e_n)$ contains two or more edges in $E$.
Proof. By Proposition 9.4, there exists some $k \geq 1$ such that $d_S(f^k(v), f^k(v')) > 1$. Hence there exists a point $z, z' \in [v, v']_K(f)$ such that $e = f^k([v, z])$ and $e' = f^k([v', z'])$ are different edges of $H$. By Lemma 9.6, $e$ and $e'$ are contained in $H \setminus K$, i.e., $e, e' \in E$. 

Consider an $n \times n$ matrix $A = (a_{ij})$ where

$$a_{ij} = \begin{cases} 1 & \text{if } e_n \subset f(e_i), \\ 0 & \text{otherwise}, \end{cases}$$

and let us denote $A^k = (a_{ij}^k)$.

**Corollary 9.8.** For each $j$, there exists some $k = k(j) \geq 1$ such that $\sum_{i=1}^n a_{ij}^k \geq 2$.

Consider an irreducible submatrix $B$ of $A$, i.e., $B = (a_{ij})_{i,j \in I}$ for some $I \subset \{1, 2, \ldots, n\}$ such that $B$ cannot write as

$$P^{-1}BP = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}$$

by a permutation matrix $P$, square matrices $B_1$, $B_3$ and a matrix $B_2$. We may assume $B$ is maximal, i.e., for some permutation matrix $\tilde{P}$, we have

$$\tilde{P}^{-1}A\tilde{P} = \begin{pmatrix} B_1 & B_2 \\ 0 & B_3 \end{pmatrix}.$$ 

Let us denote $B = (b_{ij})_{i,j \in I}$ and $B^k = (b_{ij})_{i,j \in I}$. Let $k_0 = \max_{j \in I} k(j)$ where $k(j)$ is as in Corollary 9.8. Then for any $j \in I$, we have $\sum_{i \in I} b_{ij}^{k_0} \geq 2$. Hence

$$\sum_{i=1}^n b_{ij}^{k_0} \geq \#I + 1$$

for sufficiently large $k_1$ (more precisely, if $k_1 \geq k_0 \log(\#I+1)/\log 2$). In particular, $b_{ij}^{k_1} \geq 2$ for some $i \in I$. Furthermore, the irreducibility of $B$ implies that there exists some $k_2 > 0$ such that $b_{ij}^{k_2} > 0$. Let $k = k_1 + k_2$. Then

$$b_{ij}^k \geq b_{ij}^{k_1} b_{ij}^{k_2} \geq 2.$$ 

This implies that there exist regulated arcs $e_{j1} = [z_1, z_2']_K(f)$, $e_{j2} = [z_2, z_2']_K(f) \subset e_j$ such that $f^k$ maps $e_{j1}$ and $e_{j2}$ homeomorphically to $e_j$. In particular, $f^k$ has infinitely many repelling periodic points in the interior of $e_j$, which does not intersect $K$. Therefore, we have proved the claim and Lemma 9.3. 

**10 Conclusion**

We conclude with some remarks for non-full renormalizations. Our main theorems (Theorem 1.2 and Theorem 1.3) do not hold when we do not assume renormalizations are full. We present the following two types of counterexamples here. We notice that any renormalization of period one is of disjoint type.

**Example 10.1.** (1) The case that a periodic point in the filled Julia set of a renormalization becomes parabolic in the limit and it attracts a critical point outside the renormalization. Consider

$$f_a(z) = -\frac{1}{a}z^2(z - (1 + a)).$$

Then the origin is a superattracting fixed point of $f_a$ and the other fixed points are 1 and $a$. When $|f_a''(a)| = |a - 2| < 1$, then $a$ is also attracting. Thus $f_a$ has a polynomial-like restriction $f_a : U \to V$ near 0 which is hybrid equivalent to $z \mapsto z^2$. But if $a = 1$, then the fixed point 1 is parabolic and lies in the immediate basin of 0. So $f_a$ do not have quadratic-like restriction around the origin. See Figure 8.
(2) The case that a critical point outside the renormalization approaches in the limit a preperiodic point inside the filled Julia set of a renormalization. Consider

\[ g_c(z) = \frac{z^2}{2c - 1}(3z^2 - 4(1 + c)z + 6c). \]

Then \( \text{Crit}(g_c) = \{0, 1, c\} \) and \( g_c \) has two superattracting fixed points 0 and 1. For

\[ c = c_0 = \frac{1}{6}(3 - 2\sqrt{3} - \sqrt{21}) \approx -0.841, \]

\( g_{c_0}(c_0) \) is a fixed point and \( 0 < g_{c_0}(c_0) < 1 \). The immediate basin of 0 contains an interval \((c_0, g_{c_0}(c_0))\) and the boundary of the immediate basin of 1 contains \( g_{c_0}(c_0) \). Thus it is a repelling fixed point. Since

\[ \frac{d}{dc}(g^2_c(c) - g_c(c)) \approx -1.11 < 0, \]

\( g_c(c) \) lies in the immediate basin of 1 for \( c < c_0 \) and \( g_c(c) \) lies in the immediate basin of 0 for \( c > c_0 \). Therefore, for \( c < c_0 \), \( g_c \) has a quadratic-like restriction near 0 hybrid equivalent to \( z \mapsto z^2 \), but for \( c = c_0 \), the boundary of the immediate basin of 0 contains a critical point \( c \). Thus \( g_{c_0} \) does not have a quadratic-like restriction near 0. See Figure 9.

Although the examples above treat only polynomials having only one renormalization of period one, we may construct examples having two or more renormalizations using quasiconformal surgery [6].

However, if we know that all the critical points outside the renormalization behaves “nicely” (e.g., attracted to some attracting periodic point, preperiodic,...), then we can still apply Theorem 1.1 to show that the limit is also renormalizable.

We do not know whether there exist counterexamples other than two types above. If there does not exist such counterexamples, then we will need only two renormalizations to obtain a renormalizable polynomial in the limit:

**Conjecture 10.2.** Let \( f_l \to f_{\infty} \) be a convergent sequence in \( \mathcal{C}_d \). If \( f_l \) has two full non-crossed renormalizations \( f^{s_j} : U_j \to V_j \) for \( j = 1, 2 \) with \( s_1 < s_2 \), and if a nice characterization \( \Lambda^j \) of \( f^{s_j} : U^j \to V^j \) is independent of \( l \) for \( j = 1, 2 \), then the limit \( f_{\infty} \) has a full non-crossed renormalization which is characterized by \( \Lambda^j \).
Fig. 9 The Julia sets for $g_{c_0 - 0.01}$, $g_{c_0}$ and $g_{c_0 + 0.01}$.

References


