SATELLITE TUNING VIA INTERTWINING SURGERY

HIROYUKI INOU AND YIMIN WANG*

Dedicated to Professor Mitsuhiro Shishikura on the occasion of his sixtieth birthday

Abstract. In this paper, we show that for any critically periodic unicritical polynomial of degree d can be tuned by an arbitrary polynomial of degree d with connected Julia set and without parabolic fixed points. This partially answers a question raised by Douady [2].

1. Introduction

Renormalization is an important and powerful tool in dynamical systems. Roughly speaking, renormalization is a rescaling of a first return map. *Tuning* is an inverse procedure of renormalizations for dynamics of complex polynomials in one variable. Let $\mathcal{M}$ denote the celebrated Mandelbrot set, that is, the set of parameters c for which $Q_c = z^2+c$ has a connected Julia set. Douady-Hubbard [3] and Haissinsky[6] proved that any critically periodic quadratic polynomial f_0 can be tuned by any Q_c for $c \in \mathcal{M} \setminus \{1/4\}$. More precisely, there exists an embedding

$$\phi = \phi_{f_0} : \mathcal{M} \setminus \{1/4\} \to \mathcal{M}$$

such that $Q^n_{\phi(c)}$ has a quadratic-like restriction that is hybrid equivalent to Q_c for all $c \in \mathcal{M}\setminus\{1/4\}$, where n is the period of 0 under f_0. Such an embedding is called a *tuning* and the inverse operator of ϕ is called a *straightening*. Douady raised a general problem on tuning for post-critically finite polynomials with superattracting cycles [2]. For every integer $d \geq 2$, let $\mathcal{C}_d$ denote the set of monic and centered polynomials of degree d with connected Julia set. In this paper, we consider the tuning problem for unicritical polynomials. More precisely, we consider that for a given critically periodic unicritical polynomial $f_0 \in \mathcal{C}_d$, whether f_0 can be tuned by any $g \in \mathcal{C}_d$ without parabolic fixed points or not. Note that the maps in $\mathcal{C}_d$ may be multi-critical, so this problem is more complicated than Douady-Hubbard's result.

In [8], the first author and Kiwi mostly generalized Douady-Hubbard's result to polynomials of an arbitrary degree. Fix an integer $d \geq 2$, for any $f \in \mathcal{C}_d$, the *rational lamination* λ_f of f is an equivalent relation in $\mathbb{Q}/\mathbb{Z}$ that identifies s and t if and only if the external rays for f with angle s and t land at a common point. For any critically periodic unicritical polynomial $f_0 \in \mathcal{C}_d$, let $\mathcal{C}(\lambda_{f_0}) = \{f \in \mathcal{C}_d \mid \lambda_f \supset \lambda_{f_0}\}$. They define a subset $\mathcal{R}(\lambda_{f_0})$ of $\mathcal{C}(\lambda_{f_0})$ consisting of renormalizable maps in some

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certain sense. Moreover, they construct a map

$$\chi = \chi_{f_0} : \mathcal{R}(\lambda_{f_0}) \to \mathcal{C}_d,$$

which we call *straightening map*, and they showed such a map is injective and onto the set of hyperbolic maps in $\mathcal{C}_d$. In [16], Shen and the second author proved χ is a bijection and $\mathcal{R}(\lambda_{f_0})$ is connected when f_0 is *primitive*, that is, the closures of any two Fatou component of f_0 are disjoint.

In this paper, we mainly deal with the case when f_0 is not primitive. Then by combining with the surjectivity for primitive case in [16], we show that for any $g \in \mathcal{C}_d$ without parabolic fixed points, $\chi^{-1}(g)$ is non-empty.

Main Theorem. *Let $f_0 \in \mathcal{C}_d$ be a critically periodic uncritical polynomial of degree d for some integer $d \geq 2$. The straightening map*

$$\chi = \chi_{f_0} : \mathcal{R}(\lambda_{f_0}) \to \mathcal{C}_d$$

is almost surjective, that is, for any $g \in \mathcal{C}_d$ without parabolic fixed points,

$$\chi^{-1}(g) \neq \emptyset.$$

This partially answers Douady's question. The proof is based on so-called intertwining surgery [4]. Although straightening maps are normally discontinuous [7], the main theorem explains some self-similarity of $\mathcal{C}_d$.

This paper is organized as follows. In section 2 and 3, we recall some results about generalized polynomial-like maps and straightening maps. In section 4, we prove a criterion on existence of rotatory intertwining, which is a main tool for us to deal with the case when f_0 is immediately renormalizable. In section 5, we show that rotatory intertwnings are unique up to affine conjugacies. The main theorem will be proved in section 6.

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2. N-PL MAPS

In this section, we summarize some results on N-PL maps, which are special generalized polynomial-like maps introduced in [15, 8].

2.1. N-polynomial maps.

Definition 2.1. Let N be a positive integer. An *N-polynomial map* is a skew product $F : \mathbb{Z}_N \times \mathbb{C} \to \mathbb{Z}_N \times \mathbb{C}$ such that

$$F(k, z) = (k + 1 \mod N, F_k(z))$$

where F_k is a monic centered polynomial.

We also denote an N-polynomial map $F : \mathbb{Z}_N \times \mathbb{C} \to \mathbb{Z}_N \times \mathbb{C}$ simply by $F = (F_k)_{k \in \mathbb{Z}_N}$.

Observe that

$$\deg F^N = \prod_k \deg F_k.$$

We call it the *degree of return* of F. The *total degree* of F is defined as follows:

$$\text{t-deg}\, F = \sum_{k \in \mathbb{Z}_N} (\deg F_k - 1) + 1.$$

In the following, we only consider non-affine N-polynomials; i.e., we always assume the degree of return $\deg F^N$ is greater than one (or equivalently, there exists some $k \in \mathbb{Z}_N$ such that $\deg F_k \geq 2$). The *filled Julia set* $K(F)$ of F is the set of all points whose forward orbits by F are bounded. The *Julia set* $J(F)$ of F is the boundary of $K(F)$. The *k-th small filled Julia set* is defined by $K_k(F) = \{ z \,|\, (k,z) \in K(F)\}$ and the *k-th small Julia set* $J_k(F) = \partial K_k(F)$. We say the (filled) Julia is *fiberwise connected* if k-th small (filled) Julia set is connected for some (hence any) k.

Let $C(F) = \{ (k,z) \,|\, F_k'(z) = 0\}$ be the set of critical points of F and define the *postcritical set* of F as

$$P(F) = \overline{\bigcup_{n>0} F^n(C(F))}.$$

Let $F = (F_k)_{k\in\mathbb{Z}_N}$ be an N-polynomial map with a fiberwise connected Julia set. Similarly as in the case of a single polynomial, we can define external rays and Green functions for F. Indeed, for each $k \in \mathbb{Z}_N$ there exists a unique conformal $\varphi_k : \mathbb{C} \setminus K_k(F) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ such that $\varphi_k(z)/z \to 1$ as $z \to \infty$ and these maps satisfy

$$\varphi_{k+1}(F_k(z)) = \varphi_k(z)^{\deg F_k} \text{ for all } z \in \mathbb{C} \setminus K_k(F).$$

The *Green function* of F is defined as

$$G_F(k,z) = \begin{cases} \log|\varphi_k(z)|, & \text{if } z \notin K_k(F); \\ 0, & \text{otherwise.} \end{cases}$$

For $t \in \mathbb{R}/\mathbb{Z}$, the *k-th external ray* $\mathcal{R}_k(F;t)$ of angle t is defined as

$$\mathcal{R}_k(F;t) = \varphi_k^{-1}(\{re^{2\pi i t} \mid r > 1\}).$$

If the limit

$$x = \lim_{r\to 1} \varphi_k^{-1}(r\exp(2\pi i\theta))$$

exists, then we say $\mathcal{R}_k(F;\theta)$ *lands at* x and θ is the *landing angle for* (k,x).

Let $R > 1$ and $\epsilon > 0$. We also define

$$\mathcal{R}_k(F;\theta,R) = \left\{ \varphi_k^{-1}(r\exp(2\pi i\theta)) \;\middle|\; 1 < r < R \right\},$$
$$\mathcal{R}_k(F;\theta,R,\epsilon) = \left\{ \varphi_k^{-1}(r\exp(2\pi i\eta(\theta,\varepsilon,r))) \;\middle|\; 1 < r < R \right\},$$

where $\eta(\theta,\varepsilon,r) = \theta + \epsilon\log r$. If $\mathcal{R}(F;\theta)$ lands at x, by Lindelöf's theorem, then $\mathcal{R}(F;\theta,R,\epsilon)$ also converges to x. It is easy to check that

$$F(\mathcal{R}_k(F;\theta)) = \mathcal{R}_{k+1}(F;\deg(F_k)\cdot\theta),$$
$$F(\mathcal{R}_k(F;\theta,R)) = \mathcal{R}_{k+1}(F;\deg(F_k)\cdot\theta,\, R^{\deg(F_k)}),$$
$$F(\mathcal{R}_k(F;\theta,R,\epsilon)) = \mathcal{R}_{k+1}(F;\deg(F_k)\cdot\theta,\, R^{\deg(F_k)},\, \epsilon).$$

We say the ray is *periodic* if $F^n(\mathcal{R}_k(F;\theta)) = \mathcal{R}_k(F;\theta)$ for some $n > 0$. The least such n is called the *period* of this ray. Clearly, the period of every periodic ray is divisible by N.

Let $x = (k,z)$ be a periodic point of period n for F. If x is repelling or parabolic, then there are finite number of rays landing at x and they have the same period (see [10, Theorem 2.1] for example). Assume that the common period of these rays is $q \in \mathbb{Z}_+$. Choose $\theta \in \mathbb{Q}/\mathbb{Z}$ so that the external ray with angle θ lands at x. Let $\theta_0, \dots \theta_{q-1}$ be the angles of the external rays $\{F^{jn}(\mathcal{R}_k(F;\theta)) \mid j \in \mathbb{N}\}$ ordered counterclockwise. Since F^n permutes the rays landing at x and it preserves the cyclic order of them, there exists an integer p such that $F^n(\mathcal{R}_k(F;\theta_j)) =$

$F^n(\mathcal{R}_k(F;\theta_{j+p}))$ for every $j \in \mathbb{Z}_q$. We say that the (*combinatorial*) *rotation number* of the periodic point x is p/q. It does not depend on the choice of θ.

2.2. N**-PL maps.** Let $N \in \mathbb{Z}_+$. We say $\mathcal{U}$ is a *topological N-disk* in $\mathbb{Z}_N \times \mathbb{C}$ if there exist topological disks $\{U_k\}_{k\in\mathbb{Z}_N}$ such that $\mathcal{U} \cap (\{k\} \times \mathbb{C}) = \{k\} \times U_k$ for all $k \in \mathbb{Z}_N$.

Definition 2.2. Let N be a positive integer. An *N-polynomial-like map* (*N-PL map* for simplicity) is a skew product

$$F : \mathcal{U} \to \mathcal{V}, \quad (k, z) \mapsto (k + 1, F_k(z))$$

with the following properties:

- $\mathcal{U}$ and $\mathcal{V}$ are topological N-disks and $U_k \Subset V_k$ for all $k \in \mathbb{Z}_N$;
- $F_k : U_k \to V_{k+1}$ is a holomorphic proper map for each k.

We also denote $F = (F_k : U_k \to V_{k+1})_{k\in\mathbb{Z}_N}$ for an N-PL map $F : \mathcal{U} \to \mathcal{V}$.

The *degree of return* of F is defined by $\deg F^N = \prod_k \deg F_k$ and the *total degree* of F is defined by $\text{t-deg}\, F = \sum_k(\deg F_k - 1) + 1$. As in the case of N-polynomials, we assume every N-PL map satisfies $\deg F^N \geq 2$.

The *filled Julia set* $K(F)$ of F is the set of all points whose forward orbits by F lie in $\mathcal{U}$. The *Julia set* $J(F)$ of F is the boundary of $K(F)$.

The *k-th small filled Julia set* $K_k(F)$ is defined by

$$K_k(F) = \left\{ z \in U_k \;\middle|\; \{F^n(k, z)\}_{n\in\mathbb{N}} \subset \mathcal{U} \right\}$$

and the *k-th small Julia set* $J_k(F)$ is defined by the boundary of $K_k(F)$.

We say the (filled) Julia set is *fiberwise connected* if k-th small (filled) Julia set is connected for some (hence any) k.

Let $F = (F_k : U_k \to V_{k+1})$ and $G = (G_k : U'_k \to V'_{k+1})$ be N-PL maps. We say F and G are *hybrid equivalent* if there exist quasiconformal homeomorphisms ϕ_k ($k \in \mathbb{Z}_N$) between some neighborhoods of $K_k(F)$ and $K_k(G)$ such that $G_k \circ \phi_k = \phi_{k+1} \circ F_k$ and $\bar\partial\phi_k \equiv 0$ on $K_k(F)$.

The Douady-Hubbard straightening theorem [3] extends in a straightforward way.

Theorem 2.1 (The straightening theorem for N-polynomial-like maps)**.** *For any N-polynomial-like map F, there exists an N-polynomial map G of the same degree as F hybrid equivalent to F.*

Furthermore, if the Julia set of F is fiberwise connected, then G is unique up to affine conjugacy.

Proof. See [8, Theorem A] for a proof. □

3. Straightening maps

In this section, we recall the definition of straightening maps [8]. Let us fix a critically periodic unicritical polynomial $f_0 \in \mathcal{C}_d$ throughout this section. Let U_0 denote the Fatou component of f_0 that contains the unique critical point 0. It is well known there exists a homeomorphism $h : \partial U_0 \to \mathbb{R}/\mathbb{Z}$ such that

$$h \circ f_0^p = m_d \circ h,$$

where p is the period of U_0 and m_d is the multiplication map by d on $\mathbb{R}/\mathbb{Z}$, that is,

$$m_d : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}, \quad x \mapsto dx \mod 1.$$

Such a homeomorphism can be required to be quasisymmetric since any two expanding circle covering maps of the same degree are quasisymmetrically conjugate, see [11, The proof of Lemma 3.8] for example. We call such a conjugacy h *an internal angle system* of f_0. Notice that there exists a unique cycle of external rays (with period p) of f_0 landing at $h^{-1}(0)$. Let θ_0 be an angle for which $\mathcal{R}(f_0;\theta_0)$ lands at $h^{-1}(0)$.

Let $F : U \to V$ be a polynomial-like map with connected filled Julia set. We say a smooth arc $\gamma : (0,1] \to \mathbb{C} \setminus K(F)$ is *an external marking for* F if $F(\gamma \cap U) = \gamma$. Clearly, every external marking γ for F *lands* on $K(F)$, that is,

$$\lim_{t \to 0} \gamma(t) \in K(F).$$

Let $F_i : U_i \to V_i$ be a polynomial-like map and let γ_i be an external marking for F_i ($i = 1, 2$). We say (F_1, γ_1) is *hybrid equivalent to* (F_2, γ_2) if there exists a hybrid conjugacy between F_1 and F_2 that sends γ_1 to γ_2.

Let $P \in \mathcal{C}_d$, we call the external ray $\mathcal{R}(P; 0)$ with angle 0 *the standard external marking for* P and denote it by γ_P. [8, Theorem A] asserts the following.

Theorem 3.1. *Let F be a polynomial-like map with connected filled Julia set. Fix an external marking γ for F. There exists a unique monic and centered polynomial P such that (F, γ) is hybrid equivalent to (P, γ_P)*

For any $f \in \mathcal{C}(\lambda_{f_0})$, let

$$\hat{K}_f := \bigcap_{\theta \sim_{\lambda_{f_0}} \theta'} \overline{S_f(\theta, \theta') \cap K(f)},$$

where $S_f(\theta, \theta')$ is the component of $\mathbb{C} \setminus \overline{\mathcal{R}(f;\theta) \cup \mathcal{R}(f;\theta')}$ with the following property. For any $t \in \mathbb{R}/\mathbb{Z}$, the external ray $\mathcal{R}(f;t)$ lies in $S_f(\theta, \theta')$ if and only if the external ray $\mathcal{R}(f_0;t)$ lies in the component of $\mathbb{C} \setminus \overline{\mathcal{R}(f_0;\theta) \cup \mathcal{R}(f_0;\theta')}$ that contains U_0. We say f is λ_{f_0}*-renormalizable* if $f^p|_{\hat{K}_f}$ extends to a polynomial-like map with filled Julia set $\hat{K}_f$. Let $\mathcal{R}(\lambda_{f_0})$ denote the set of all the λ_{f_0}-renormalizable maps.

We proceed to define the straightening map $\chi = \chi_{f_0} : \mathcal{R}(\lambda_{f_0}) \to \mathcal{C}_d$. Recall that $\mathcal{R}(f_0;\theta_0)$ is a periodic ray of period p under f_0. For any $f \in \mathcal{R}(\lambda_{f_0})$, let $\Gamma(f) = f^p(\mathcal{R}(f;\theta_0))$. Clearly, it is an external marking for any polynomial-like extension of $f^p|_{\hat{K}_f}$. Fix a polynomial-like extension F of $f^p|_{\hat{K}_f}$, by Theorem 3.1, there exists a unique polynomial $P \in \mathcal{C}_d$ such that $(F, \Gamma(f))$ is hybrid equivalent to (P, γ_P). We call such a polynomial P the *straightening of* f and denote it by $\chi(f)$.

4. Renormalizations and intertwinings

Throughout this section, we fix an N-polynomial map F with fiberwise connected Julia set and assume that $\mathcal{O} = \{\,(k, x_k)\,|\, k \in \mathbb{Z}_N\}$ is a repelling periodic orbit of period N with rotation number p_0/q_0. In this section, we will define a rotatory intertwining of $(F, \mathcal{O})$. The aim of this section is to give a combinatorial criterion for the existence of such a rotatory intertwining.

We say a polynomial f is *renormalizable* for period N if there exist topological disks U_k and V_k ($k \in \mathbb{Z}_N$) such that:

- $G = (f : U_k \to V_{k+1})_{k \in \mathbb{Z}_N}$ is an N-PL map with fiberwise connected Julia set.

- For every critical point c of f, there exists at most one $k \in \mathbb{Z}_N$ such that $c \in U_k$.
- When $N = 1$, U_0 does not contain all the critical points of f.

We call G an *N-PL renormalization* of f.

Definition 4.1. We say a monic centered polynomial with a marked fixed point (g, x) is a *p-rotatory intertwining* of $(F, \mathcal{O})$ if:

- g has an N-PL renormalization which is hybrid equivalent to F.
- x corresponds to $\mathcal{O}$ by the hybrid conjugacy above.
- x has a rotation number $p/(Nq_0)$.
- $\deg g = \text{t-deg}\, F$. (Or equivalently, all critical points of g lie in the filled Julia set of the renormalization above.)

The last property implies that the filled Julia set of an intertwining is connected.

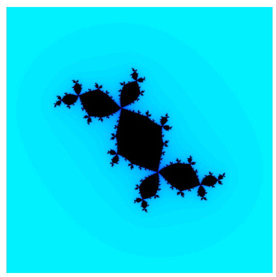

FIGURE 1. (g, x) with $g(z) = z^2 - 0.1225611\ldots + i0.7448617\ldots$ and $x = 0.2762983\ldots + i0.47979213\ldots$ is a 1-rotatory of $(F, \mathcal{O})$, where $F = (z^2, z, z)$ and $\mathcal{O}$=(1,1,1)

Definition 4.2. A four-tuple of integers (N, p_0, q_0, p) is *admissible* if p and N are relatively prime and $p \equiv p_0 \mod q_0$.

Note that the above definition also makes sense when N and q_0 are integers, $p_0 \in \mathbb{Z}_{q_0}$ and $p \in \mathbb{Z}_{Nq_0}$. It is easy to show the following proposition.

Proposition 4.1. *If a p-rotatory intertwining of $(F, \mathcal{O})$ exists, then (N, p_0, q_0, p) is admissible.*

Our main purpose is to show the converse:

Theorem 4.1. *Let F be an N-polynomial map with fiberwise connected Julia set and $\mathcal{O} = \{(k, x_k)\}$ be a repelling periodic orbit of period N with rotation number p_0/q_0.*

If an integer p satisfies that (N, p_0, q_0, p) is admissible, then there exists a p-rotatory intertwining (g, x) of $(F, \mathcal{O})$ and it is unique up to affine conjugacy.

Note that we assume a rotatory intertwining is a monic and centered polynomial. Hence it is unique up to conjugation by rotation by $(\deg g - 1)$-th root of unity.

Remark 4.1. More generally, we allow the case where several cycles land at $\mathcal{O}$, in which case we choose one cycle and ignore the others.

We may even use several (or all) of the ray cycles to intertwine under an appropriate combinatorial assumption, but for simplicity we only consider one ray cycle.

Example 4.1. Figure 1 shows the quadratic polynomial called the *rabbit*, which is of the form

$$g(z) = z^2 - 0.1225611\ldots + 0.7448617\ldots i$$

and have a period 3 superattracting cycle. Then $g(z)$ has a repelling fixed point at $x = 0.2762983\ldots + i0.47979213\ldots$ of rotation number $1/3$ and (g, x) is a 1-rotatory of $(F, \mathcal{O})$, where $F = (z^2, z, z)$ and $\mathcal{O} = (1, 1, 1)$.

Example 4.2. Consider a cubic polynomial

$$g(z) = z^3 - \frac{3}{4}z - \frac{\sqrt{7}}{4}i.$$

The critical points $\pm\frac{1}{2}$ are periodic of period 2 and the unique fixed point on the imaginary axis

$$x = i1.481997\ldots$$

is a repelling fixed point of rotation number $1/2$. There are four rays landing at x, of angles $1/4$, $5/8$, $3/4$ and $7/8$.

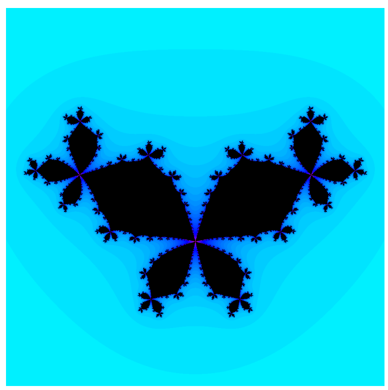

FIGURE 2. The Julia set of $z^3 - \frac{3}{4}z - \frac{\sqrt{7}}{4}i$. (the *butterfly*).

Hence there are two ways to consider $g(z)$ as a 1-rotatory intertwining; the up–down renormalization and the left–right renormalization.

- **Up–down renormalization:** Consider $5/8$- and $7/8$-rays landing at x and ignore the rays of angle $1/4$ and $3/4$. Then we have a 2-PL renormalization hybrid equivalent to $F_1 = (z^3 + \frac{3}{2}z, z)$.
- **Left–right renormalization:** If we consider only the rays of angle $1/4$ and $3/4$, then we have another 2-PL renormalization hybrid equivalent to $F_2 = (z^2 - 1, z^2 - 1)$.

To obtain such a PL renormalization, we construct Yoccoz puzzles from the given two rays in each case, and apply the standard thickening technique.

Note that both F_1 and F_2 have two superattracting cycles, and $\text{t-deg}\, F_1 = \text{t-deg}\, F_2 = 3$.

We leave the uniqueness part of Theorem 4.1 to the next section. The proof for the existence is based on the intertwining surgery [4], [5].

Fix $R > 0$ and let

$$V_k = \left\{ (k, z) \;\middle|\; |\varphi_k(z)| < R \right\} \cup K_k(F)$$

and $U_k = F_k^{-1}(V_{k+1})$. Let $\mathcal{V} = \bigsqcup V_k$ and $\mathcal{U} = \bigsqcup U_k$. Then

$$(F_k : U_k \to V_{k+1})$$

(or equivalently, $F|_{\mathcal{U}} : \mathcal{U} \to \mathcal{V}$) is an N-polynomial-like map.

Fix $\theta \in \mathbb{Q}/\mathbb{Z}$ such that $\mathcal{R}_0(F;\theta)$ lands at $(0, x_0)$. Let $\theta_0, \dots, \theta_{q_0-1}$ be all the external angles of $\{F^{Ni}(\mathcal{R}_0(F;\theta))\}$ ordered counterclockwise.

Let $\epsilon > 0$ small and let $0 < \delta < \epsilon/(2N)$. For $0 \le k < N$ and $l \in \mathbb{Z}_{q_0}$, consider arcs

$$\gamma_0(k+Nl) = \mathcal{R}_0\left(F;\theta_l, R, \left(\frac{k}{N} - \frac{1}{2}\right)\epsilon\right),$$
$$\gamma_0^{\pm}(k+Nl) = \mathcal{R}_0\left(F;\theta_l, R, \left(\frac{k}{N} - \frac{1}{2}\right)\epsilon \pm \delta\right).$$

All these arcs are perturbations of the truncated external ray $\mathcal{R}_0(F;\theta_l, R)$ and they are periodic under F with the same period as $\mathcal{R}_0(F;\theta_l,R)$. If ϵ is sufficiently small, then these arcs are mutually disjoint. For $j \in \mathbb{Z}_{Nq_0}$, let

$$\begin{aligned} \gamma_k(j) &= F_k(\gamma_{k-1}(j-p) \cap U_{k-1}), \\ \gamma_k^{\pm}(j) &= F_k(\gamma_{k-1}^{\pm}(j-p) \cap U_{k-1}), \end{aligned} \tag{4.1}$$

for $k = 1, \dots, N-1$. Let $S_k(j)$ (*resp.* $L_k(j)$) be the open sector in V_k between $\gamma_k(j-1)$ and $\gamma_k(j)$ (*resp.* $\gamma_k^+(j-1)$ and $\gamma_k^-(j)$).

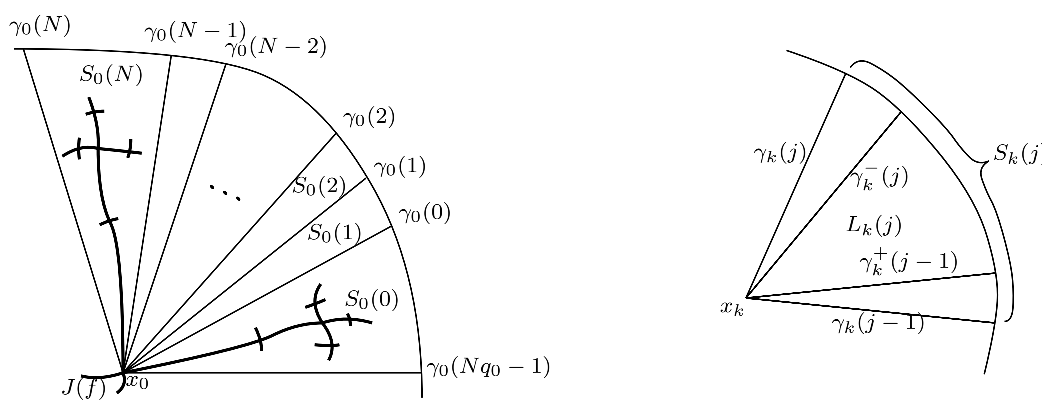

Figure 3. Sectors.

Note that $F_0^N(\gamma_0(j) \cap F_0^{-N}(V_0)) = \gamma_0(j+Np_0)$ since the rotation number of x_0 for F_0^N is p_0/q_0. Therefore, by the assumption that (N, p_0, q_0, p) is admissible,

$$\begin{aligned} F_{N-1}(\gamma_{N-1}(j-p) \cap U_{N-1}) &= F^N(\gamma_0(j-Np) \cap F^{-(N-1)}(U_{N-1})) \\ &= \gamma_0(j - Np + Np_0) \\ &= \gamma_0(j). \end{aligned}$$

This equation also holds for $\gamma_k^{\pm}$ instead of γ_k. Therefore, the equations (4.1) hold for any $k \in \mathbb{Z}_N$.

Since $\mathcal{O}$ is repelling, F is linearizable at $\mathcal{O}$. Namely, there are a neighborhood O_k of x_k and a holomorphic embedding $\psi_k : O_k \to \mathbb{C}$ for each k such that $\psi_k(x_k) = 0$

and $\psi_{k+1} \circ F_k(z) = \lambda_k \cdot \psi_k(z)$ on O'_k, where $\lambda_k = F'_k(x_k)$ and O'_k is the component of $F_k^{-1}(O_{k+1})$ containing x_k.

For each $j \in \mathbb{Z}_{Nq_0}$, the quotient space $(L_k(j) \cap O_k)/F^{Nq_0}$ is an annulus of finite modulus. So we denote the modulus of this quotient annulus by $\operatorname{mod} L_k(j)$. Since F maps $L_k(j) \cap O'_k$ univalently to $L_{k+1}(j+p) \cap O_{k+1}$, we have

$$\operatorname{mod} L_k(j) = \operatorname{mod} L_{k+1}(j+p).$$

To prove Theorem 4.1, we are going to identify N disks $V_0, \dots, V_{N-1}$ quasiconformally and define a quasiregular map on it. Then we can use the measurable Riemann mapping theorem to construct a desired rotatory intertwining. For some technical reasons, before doing this, we deform the N-polynomial-like map $F : \mathcal{U} \to \mathcal{V}$ by some hybrid conjugacy.

Lemma 4.1. *There exists an N-polynomial-like map $\hat{F} = (\hat{F}_k : \hat{U}_k \to \hat{V}_{k+1})_{k \in \mathbb{Z}_N}$ hybrid equivalent to F such that the sector $\hat{L}_k(j)$ which corresponds to $L_k(j)$ satisfies that*

$$\operatorname{mod} \hat{L}_k(j) = \operatorname{mod} \hat{L}_{k'}(j)$$

for any $k, k' \in \mathbb{Z}_N$ and $j \in \mathbb{Z}_{Nq_0}$.

Proof. For $l \in \mathbb{Z}_{q_0}$, let A_l be an annulus with $\operatorname{mod} A_l = m_l = \operatorname{mod}\ L_0(lN)$. For $k = 1, \dots, N-1$, take a quasiconformal homeomorphism

$$\rho_{l,k} : (L_0(lN + pk) \cap O_0)/F^{Nq_0} \to A_l.$$

Since F^N induces a conformal isomorphism between $L_0(lN+pk)/F^{Nq_0}$ and $L_0((l+p)N + pk)/F^{Nq_0}$, we can choose $\rho_{l,k}$ so that

$$\rho_{l+p,k} \circ F^N = \rho_{l,k}. \tag{4.2}$$

Define a complex structure σ on $L_0(lN + pk) \cap O_0$ by pulling back $\rho_{l,k}$ and lifting the standard complex structure σ_0 on A_l. Note that, since (N, p_0, q_0, p) is admissible, $L_0(lN + pk)$ does not intersect the filled Julia set and we can push forward a complex structure on $(L_0(lN + pk)) \cap O_0$ to $L_0(lN + pk)$ by F^{Nq_0}. By (4.2), σ is F^N-invariant. The modulus $\operatorname{mod}(L_0(lN + pk), \sigma)$ with respect to the complex structure σ is equal to m_l.

Now we define an F-invariant complex structure $\hat{\sigma}$ on $\mathbb{Z}_N \times \mathbb{C}$ as follows: let

$$\hat{\sigma} = \begin{cases} (F^n)^*\sigma & \text{on } F^{-n}(L_0(lN + pk)) \text{ for some } k \in \{1, \dots, N-1\} \text{ and } n > -N. \\ \sigma_0 & \text{otherwise.} \end{cases}$$

Since

$$\bigcup_{l \in \mathbb{Z}_{q_0}, k=1,\dots,N-1} L_0(lN + pk)$$

is forward invariant by F^N and σ is F^N-invariant, $\hat{\sigma}$ is well-defined and F-invariant.

By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism

$$\phi \colon \mathbb{Z}_N \times \mathbb{C} \to \mathbb{Z}_N \times \mathbb{C}, \quad \phi(\{k\} \times \mathbb{C}) = \{k\} \times \mathbb{C}$$

such that $\phi^*\sigma_0 = \sigma$. Let

$$\hat{V}_k = \phi(V_k), \qquad \hat{U}_k = \phi(U_k), \qquad \hat{L}_k(j) = \phi(L_k(j)),$$

and $\hat{F} = \phi \circ F \circ \phi^{-1}$ on $\hat{U}_k$. Then $\hat{F} \colon \bigsqcup \hat{U}_k \to \bigsqcup \hat{V}_k$ is holomorphic and by the equation $m_l = m_{l+p}$ and (4.2), we have

$$\operatorname{mod} \hat{L}_0(j) = \operatorname{mod} \hat{L}_0(j+p)$$

and

$$\begin{aligned}\operatorname{mod} \hat{L}_k(j) &= \operatorname{mod} \hat{L}_0(j-pk)\\ &= \operatorname{mod} \hat{L}_0(j).\end{aligned}$$

□

To simplify the notations, we replace F by $\hat{F}$ and so on and assume that $F \colon \bigsqcup U_k \to \bigsqcup V_k$ is an N-PL map such that

$$\operatorname{mod} L_k(j) = \operatorname{mod} L_{k'}(j)$$

for any $k, k' \in \mathbb{Z}_N$ and $j \in \mathbb{Z}_{Nq_0}$.

Remark 4.2. By the proof of the above theorem, we can see that the quasiconformal map ϕ is conformal in a small sector containing $\gamma_k(j)$ for all $k \in \mathbb{Z}_N$ and $0 \leq j < Nq_0 - 1$. Indeed, for any $k \in \mathbb{Z}_N$ and $0 \leq j < Nq_0 - 1$, there is a invariant sector $S_{k,j}$ containing $\gamma_k(j)$ such that

$$F^n(S_{k,j}) \cap F^{-n}(L_0(lN + pk')) = \emptyset$$

for all $k' \in \{1, 2, \cdots, N-1\}$ and $l \in \mathbb{Z}_{q_0}$. Thus the complex structure $\hat{\sigma} = 0$ on $S_{k,j}$. Hence, $\phi(\gamma_k(j))$'s are smooth.

Now we construct quasiconformal maps $\tau_k : V_0 \to V_k$ ($k \in \mathbb{Z}_N$) to identify $V_0, \ldots, V_{N-1}$ together.

Lemma 4.2. *There exist quasiconformal maps $\tau_k : V_0 \to V_k$ ($k \in \mathbb{Z}_N$) with the following properties. For any $0 \leq j < q_0 N$,*

- *$\tau_k|_{\gamma_0(j)}$ is a diffeomorphism onto $\gamma_k(j)$ such that $F^N \circ \tau_k|_{\gamma_0(j)} = \tau_k|_{\gamma_0(j)} \circ F^N$ when both sides are defined;*
- *$\tau_k(S_0(j)) = S_k(j)$ and $\tau_k|_{L_0(j)}$ is a conformal map onto $L_k(j)$.*

We postpone the proof of this lemma to the end of this section and utilize it to show the existence part of Theorem 4.1.

Proof of Theorem 4.1 (Existence part). Let $V = V_0$ and

$$U = \bigcup_{j \in \mathbb{Z}_{q_0}, k=0,\ldots,N-1} \tau_k^{-1}\left(\overline{S_k(jN+kp)} \cap U_k\right).$$

Note that since (N, p_0, q_0, p) is admissible and (4.6), $jN + kp$ runs over all elements of $\mathbb{Z}_{Nq_0}$ and U is a Jordan disk. Define a quasiregular map $\tilde{g} : U \to V$ as follows. When $z \in S_0(jN + kp) \cap U$ for some $j \in \mathbb{Z}_{q_0}$, let

$$\tilde{g}(z) = \tau_{k+1}^{-1} \circ F_k \circ \tau_k(z).$$

By (4.4), $\tilde{g}$ extends continuously on U.

We define an almost complex structure on V as follows. For any $j \in \mathbb{Z}_{q_0}$ and $k \in \mathbb{Z}_N$, let us denote $S_0(jN + kp) \setminus L_0(jN + kp)$ by $Y_{j,k}$ and define

$$\sigma|_{Y_{j,k}} = \tau_k^*(\sigma_0|_{S_k(jN+kp) \setminus L_k(jN+kp)}),$$

where σ_0 is the standard complex structure on V_k. Let $Y = \bigcup Y_{j,k}$. Note that $\tilde{g}$ is holomorphic on $Y \cap U$ with respect to σ. In other words, σ is $\tilde{g}$-invariant, that is, $\tilde{g}^*\sigma = \sigma$ on $Y \cap U$. Indeed, for any $j \in \mathbb{Z}_q$ and $k \in \mathbb{Z}_N$, F maps each sector $\tau_k(Y_{j,k} \cap U)$ onto the sector $\tau_{k+1}(Y_{j,k+1})$ and thus

$$\begin{aligned}\tilde{g}^*\sigma &= \tau_k^* F^* (\tau_{k+1}^{-1})^* \sigma \\ &= \tau_k^* F^* \sigma_0 \\ &= \tau_k^* \sigma_0 = \sigma\end{aligned}$$

Then we define σ on $\bigcup \tilde{g}^{-n}(Y)$ by pulling back $\sigma|_Y$ via g and let $\sigma = \sigma_0$ everywhere else. Note that σ is a well-defined complex structure on V and the maximal dilatation of σ equals to that of $\sigma|_Y$ since $\tilde{g}^*\sigma = \sigma$ on Y and $\tilde{g}$ is conformal outside Y.

By the measurable Riemann mapping theorem, there exists a quasiconformal mapping $h : V \to \mathbb{C}$ such that $h^*\sigma_0 = \sigma$. Then $\hat{g} = h \circ \tilde{g} \circ h^{-1}$ is a polynomial-like map, so there exists a polynomial g hybrid equivalent to $\hat{g}$. By taking an affine conjugacy, we may assume g is monic and centered, and it is easy to check this g is a p-rotatory intertwining of F by using the thickening technique. □

Now we finish the proof of Lemma 4.2 to complete this section. For the proof, we first construct C^1 diffeomorphisms

$$\tau_k : \bigcup_j \gamma_0(j) \to \bigcup_j \gamma_k(j)$$

such that

- τ_k maps $\gamma_0(j)$ diffeomorphically onto $\gamma_k(j)$ for all $k \in \mathbb{Z}_N$ and $0 \le j < Nq_0 - 1$;
- $\tau_k \circ F^N = F^N \circ \tau_k$ for all $k \in \mathbb{Z}_N$.

To this end, we construct C^1 diffeomorphisms

$$\tilde{\tau}_k : \bigcup_j \gamma_k(j) \to \bigcup_j \gamma_{k+1}(j)$$

which maps $\gamma_k(j)$ onto $\gamma_{k+1}(j)$ for any j and $F \circ \tilde{\tau}_k = \tilde{\tau}_{k+1} \circ F$ for all $k \in \mathbb{Z}_N$. The construction is based on induction.

Lemma 4.3. *There exists a diffeomorphism*

$$\tilde{\tau}_0 : \bigcup_j \gamma_0(j) \to \bigcup_j \gamma_1(j)$$

such that

- $\tilde{\tau}_0 \circ F^N = F^N \circ \tilde{\tau}_0$.
- $(\tilde{\tau}_0^{-1} \circ F)^N = F^N$.

Proof. Let $y_k^0(j)$ be the end point of $\gamma_k(j)$ other than x_k. For $n > 0$, let $y_k^n(j)$ be the point in $\gamma_k(j)$ which satisfies that $F^n(y_k^n(j)) = y_{k+n}^0(j + pn)$.

All rays $\gamma_k(j)$ are periodic of the same period q_0N under F. The quotient space $(\bigcup_j \gamma_k(j))/F^{q_0N}$ is diffeomorphic to the disjoint union of circles and each component is of the form $\eta_k(j) = \gamma_k(j)/F^{q_0N}$. The points $\{y_k^n(j)\}$ corresponds to the q_0N points $\{[y_k^n(j)]\}_{k=0,\ldots,q_0N-1}$ (the equivalent class $[y_k^n(j)] = [y_k^{n+q_0N}(j)]$ in $\eta_k(j)$). F induces a diffeomorphism

$$\alpha : \eta_k(j) \to \eta_{k+1}(j + p)$$

(for simplicity, we omit indices k and j for α). Then $\alpha([y_k^n(j)]) = [y_{k+1}^{n-1}(j+p)]$ and

$$\alpha^N : \eta_k(j) \to \eta_k(j+pN)$$

is the identity map on $(\bigcup_j \gamma_k(j))/F^N$.

Furthermore, we identify each $\eta_0(j)$ with $\mathbb{R}/(q_0 N\mathbb{Z})$ diffeomorphically so that $\{[y_0^n(j)]\}$ corresponds to $\{[n]\}$. Define $R : \mathbb{R}/(q_0N\mathbb{Z}) \to \mathbb{R}/(q_0N\mathbb{Z})$ by $R(x) = x-1$. Since $R^{q_0N} = \alpha^{q_0N} = \mathrm{id}$, we may assume that the following diagram commutes:

$$\begin{tikzcd}
\eta_0(j) \arrow[r, "\alpha^N"] \arrow[d, phantom, "\cong\,\|"] & \eta_0(j+pN) \arrow[d, phantom, "\cong\,\|"] \\
\mathbb{R}/(q_0N\mathbb{Z}) \arrow[r, "R^N"'] & \mathbb{R}/(q_0N\mathbb{Z}).
\end{tikzcd}$$

Let $\hat{\tau} : \eta_0(j) \to \eta_1(j)$ be the diffeomorphism defined by:

$$\eta_0(j) \cong \mathbb{R}/(q_0N\mathbb{Z}) \xrightarrow{R^{-1}} \mathbb{R}/(q_0N\mathbb{Z}) \cong \eta_0(j-p) \xrightarrow{\alpha} \eta_1(j).$$

Then the following diagram commutes:

$$\begin{tikzcd}
\mathbb{R}/(q_0N\mathbb{Z}) \arrow[r, "R"] \arrow[d, phantom, "\cong\,\|"] & \mathbb{R}/(q_0N\mathbb{Z}) \arrow[r, "R"] \arrow[d, phantom, "\cong\,\|"] & {} & \cdots & {} \arrow[r, "R"] & \mathbb{R}/(q_0N\mathbb{Z}) \arrow[d, phantom, "\cong\,\|"] \\
\eta_0(j) \arrow[d, "\hat{\tau}"] \arrow[dr, "\alpha"] & \eta_0(j+p) \arrow[d, "\hat{\tau}"] \arrow[dr, "\alpha"] & & \cdots & {} \arrow[dr, "\alpha"] & \eta_0(j+pN) \arrow[d, "\hat{\tau}"] \\
\eta_1(j) & \eta_1(j+p) & {} & \cdots & & \eta_1(j+pN),
\end{tikzcd}$$

so,

$$(\hat{\tau}^{-1} \circ \alpha)^N = R^N = \alpha^N. \tag{4.3}$$

Let $\tilde{\tau}_0 : \gamma_0(j) \to \gamma_1(j)$ be the diffeomorphism which is a lift of $\hat{\tau}$. Then $\tilde{\tau}_0 \circ F^N = F^N \circ \tilde{\tau}_0$. Furthermore, since F is a lift of α, (4.3) implies $({\tilde{\tau}_0}^{-1} \circ F)^N = F^N$. $\square$

Now define $\tilde{\tau}_k$ for $k = 1, \ldots, N-1$ inductively by the equation

$$F_k \circ \tilde{\tau}_{k-1} = \tilde{\tau}_k \circ F_{k-1}. \tag{4.4}$$

Then this equation is also valid for $k = 0$. Indeed,

$$\begin{aligned}
\tilde{\tau}_0 \circ F_{N-1} \circ F^{N-1} &= \tilde{\tau}_0 \circ F^N = F^N \circ \tilde{\tau}_0 \\
&= F_0 \circ F_{N-1} \circ \cdots \circ F_1 \circ \tilde{\tau}_0 \\
&= F_0 \circ F_{N-1} \circ \cdots \circ \tilde{\tau}_1 \circ F_0 \\
&\cdots \\
&= F_0 \circ \tilde{\tau}_{N-1} \circ F^{N-1}.
\end{aligned}$$

Since F^{N-1} maps the subarc of $\gamma_0(j-(N-1)p)$ from x_k to $y_0^{N-1}(j-(N-1)p))$ diffeomorphically onto $\gamma_{N-1}(j)$, we have $F_0 \circ \tilde{\tau}_{N-1} = \tilde{\tau}_0 \circ F_{N-1}$.

Similarly, we can also show that

$$\begin{aligned}
F^N &= (\tilde{\tau}_0^{-1} \circ F)^N \\
&= \tilde{\tau}_{N-1} \circ \cdots \circ \tilde{\tau}_0 \circ F^N,
\end{aligned}$$

so

$$\tilde{\tau}_{N-1} \circ \cdots \circ \tilde{\tau}_0 = \mathrm{id}. \tag{4.5}$$

And it is easy to see that

$$\tilde{\tau}_k(y_k^n(j)) = y_{k+1}^n(j) \tag{4.6}$$

Now let $\tau_k = \tilde{\tau}_{k-1} \circ \cdots \circ \tilde{\tau}_0$ on $\bigcup \gamma_0(j)$. Clearly, τ_k is our desired diffeomorphism.

Proof of Lemma 4.2. The proof is due to Bielefeld [1, Lemma 6.4, 6.5].

Let

$$\tau_k : \bigcup_j \gamma_0(j) \to \bigcup_j \gamma_k(j)$$

be the diffeomorphisms we constructed as above and let $\tau_k|_{L_0(j)} : L_0(j) \to L_k(j)$ be the conformal isomorphism which sends x_0 to x_k, $\gamma_0^+(j-1)$ to $\gamma_k^+(j-1)$, and $\gamma_0^-(j)$ to $\gamma_k^-(j)$. Taking $L_k(j)$ smaller (that is, taking δ greater) if necessary, we may assume that $\tau_k|_{L_0(j)}$ extends smoothly on $\gamma_0^+(j-1)$ and $\gamma_0^-(j)$.

It remains to extend τ_k on $\bigcup(S_k(j) \setminus \overline{L_k(j)})$ quasiconformally. For $k \in \mathbb{Z}_N$ and $j \in \mathbb{Z}_{Nq_0}$, let $S_k^-(j)$ (*resp.* $S_k^+(j)$) be the open sector between $\gamma_k^-(j)$ and $\gamma_k(j)$ (*resp.* $\gamma_k(j)$ and $\gamma_k^+(j)$). Then $\bigcup(S_k(j) \setminus \overline{L_k(j)}) = \bigcup(S_k^+(j) \cup S_k^-(j))$. So we need to extend τ_k quasiconformally on $S_0^-(j)$, which maps to $S_k^-(j)$ (the case for $S_0^+(j)$ is quite similar). Furthermore, since τ_k is smooth on $\gamma_0^\pm(j)$, and we need only show the extendability of τ_k on O_0, where F is linearizable. So we consider $S_0^-(j) \cap O_0$ instead of $S_0^-(j)$.

To this end, we will use the log-Koenigs coordinate. Recall that the Koenigs coordinate is a univalent map $\psi_k : O_k \to \mathbb{C}$ such that $\psi_k(F_k^{Nq_0}(z)) = \tilde{\lambda}^{q_0}\psi_k(z)$, where $\tilde{\lambda}$ is the multiplier of the orbit $\mathcal{O}$. Set $\lambda = \tilde{\lambda}^{q_0}$. Let $h_k(z) = \log \psi_k(z)$ on $(\overline{S_k^-(j) \cup L_k(j)}) \cap O_k$. Then we have

$$h_k(F_k^{Nq_0}(z)) = h_k(z) + \log \lambda. \tag{4.7}$$

Let $T_k^-(j) = h_k(S_0^-(j) \cap O_0)$, $M_k(j) = h_k(L_k(j) \cap O_0)$ and

$$\chi_k = h_k \circ \tau_k \circ h_0^{-1} : \partial^\pm T_0^-(j) \cup M_0(j) \to \partial^\pm T_k^-(j) \cup M_k(j).$$

where $\partial^+ T_k^-(j) = h_k(\gamma_k(j))$ and $\partial^- T_k^-(j) = h_k(\gamma_k^-(j))$ are the upper and lower boundaries of the strip $T_k^-(j)$. Then, by (4.7), for $z \in \partial^+ T_k^-(j)$,

$$\begin{aligned} \chi_k(z + \log\lambda) &= h_k \circ \tau_k \circ h_0^{-1}(z + \log\lambda) \\ &= h_k \circ \tau_k \circ F^{Nq_0}(h_0^{-1}(z)) \\ &= h_k \circ F_k^{Nq_0} \circ \tau_k \circ h_0^{-1}(z) \\ &= h_k(\tau_k \circ h_0^{-1}(z)) + \log\lambda \\ &= \chi_k(z) + \log\lambda. \end{aligned}$$

We call such a diffeomorphism on curves in $\mathbb{C}$ a *near translation*. More precisely, we say a diffeomorphism on a curve in $\mathbb{C}$ onto its image in $\mathbb{C}$ is a near translation if it is of the form $z + O(1)$ and its derivative is uniformly bounded away from zero and infinity.

Claim. $\chi_k|_{\partial^- T_0^-(j)}$ is a near translation.

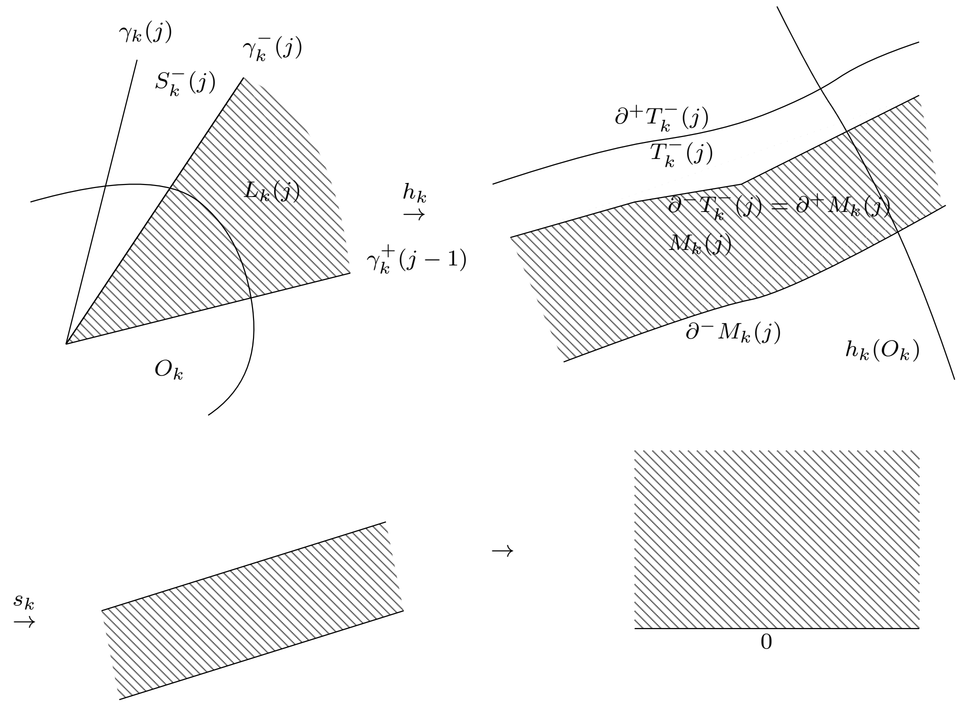

FIGURE 4. Conjugacy to translations and linear expansion.

Since $\tau_k|_{L_k(j)}$ is conformal, $\chi_k|_{M_0(j)} : M_0(j) \to M_k(j)$ is conformal and it maps the upper boundary $\partial^+ M_0(j)$ $(= \partial^- T_0(j))$ to $\partial^+ M_k(j)$ $(= \partial^- T_k(j))$ diffeomorphically. Let $m = \operatorname{mod}(M_k(j)/(z \mapsto z+\log\lambda)) = \operatorname{mod} L_k(j)$. Then by the assumption, m is independent of k.

Let $H_\nu = \{\, z\log\lambda \,|\, 0 < \operatorname{Im} z < \nu\}$. Then there exists some $\nu > 0$ such that for any $k \in \mathbb{Z}_N$, there is a conformal map s_k from $M_k(j)$ into H_ν which maps the upper and lower boundary to the upper and lower boundary respectively, and which gives a conjugacy from $z \mapsto z + \log\lambda$ to itself. (ν is given by the equation $\operatorname{mod}(H_\nu/(z \mapsto z + \log\lambda)) = m$.) Since $s_k(z + \log\lambda) = s_k(z) + \log\lambda$, $s_k|_{\partial^+ M_0(j)}$ is a near translation. Let

$$\hat{\chi}_k = e \circ s_k \circ \chi_k \circ s_0^{-1} \circ e^{-1}$$
$$\text{where } e(z) = \exp\left(\frac{\pi}{\nu\log\lambda} z\right).$$

Then $\hat{\chi}_k$ can be extended to some neighborhood of 0 by the reflection principle. Hence it is of the form $rz + O(z^2)$. Thus

$$\begin{aligned} s_k \circ \chi_k \circ s_0^{-1}(z) &= e^{-1} \circ \hat{\chi}_k \circ e(z) \\ &= z + O(1), \end{aligned}$$

so it is a near translation. Since the composition of near translations is also a near translation, χ_k is also a near translation.

Just as in the case of $M_k(j)$, let $\hat{t}_k$ be a conformal map from $T_k^-(j)$ into H_{ν_k} which gives a conjugacy from $z \mapsto z + \log\lambda$ to itself. (Note that in this case, ν_k may depends on k.) Let $t_k = \hat{t}_k/(\nu_0 \log\lambda) : T_k^-(j) \to \{0 < \operatorname{Im} z < \nu_k/\nu_1\}$. Then $\tilde{\chi}_k^\pm = t_k \circ \chi_k \circ t_0^{-1}$ restricted to the upper and lower boundary respectively are both near translations. We define $\tilde{\chi}_k : t_0(T_0^-(j)) \to t_k(T_k^-(j))$ as follows:

$$\tilde{\chi}_k(x+iy) = y(\tilde{\chi}_k^+(x+i) - i) + (1-y)\tilde{\chi}_k^-(x) + i\frac{\nu_k}{\nu_0}y.$$

(Although it may not be mapped into $t_k(T_k^-(j))$, it makes no problem because we only need to construct this map near the left infinity.) It is easy to check $\tilde{\chi}_k$ is a quasiconformal diffeomorphism. Therefore, $\tau_k = h_k^{-1} \circ t_k^{-1} \circ \tilde{\chi}_k \circ t_0 \circ h_0$ on $S_0^-(j) \cap O_0$ is a quasiconformal extension. $\square$

5. Uniqueness for the rotatory intertwning

In this section, we show that rotatory intertwnings are unique up to rotational conjugacies. It suffices to show that two p-rotatory intertwinings (g, x) and (g', x') of $(F, \mathcal{O})$ are affinely conjugate. Uniqueness of intertwinings are first proved by Epstein and Yampolsky for intertwinings (not rotatory) of two quadratic polynomials [4]. Their proof is based on the properness of the straightening map in the paremeter space, which is true only for quadratic renormalizations. Here, we prove uniqueness of rotatory intertwinings for any degree $d \geq 2$.

Let (g, x) be a p-rotatory intertwining of an N-polynomial map $(F, \mathcal{O})$ with marked periodic point of period N. Let $\varphi : \mathbb{C} \setminus K(g) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ be the Böttcher coordinate of g. Set $D_0 := \{z \mid |\varphi(z)| < 1\} \cup K(g)$. There are Nq_0 external rays $\mathcal{R}_0, \mathcal{R}_1, \cdots, \mathcal{R}_{Nq_0-1}$ landing at the repelling fixed point x. They are ordered by $g(\mathcal{R}_i) = \mathcal{R}_{i+1}$ for all $i = 0, 1, \cdots, Nq_0 - 1$. We are going to use the thickening technique as follows. For each external ray $\mathcal{R}_i$, set a sector S_i contained in D_0 with the following properties:

- $S_i \supset (\mathcal{R}_i \cap D_0)$,
- $f(S_i \cap f^{-1}(D_0)) = S_{i+1}$,
- the components of $\partial S_i \setminus (\{x\} \cup \partial D_0)$ are smooth.

By the thickening procedure, we can obtain a N-PL map

$$G = (g : U_k \to V_{k+1})_{k \in \mathbb{Z}_N}$$

with the following properties:

- The N-PL map G is hybrid equivalent to $(F, \mathcal{O})$.
- For every $k \in \mathbb{Z}_N$, V_k is a Jordan disk with smooth boundary that is contained in D_0.
- The components of $V_k \setminus (\bigcup_i \overline{S_i})$ are mutually disjoint Jordan disks.

Let h be a hybrid conjugacy between G and $(F, \mathcal{O})$. Denote by $\mathcal{K}$ the filled Julia set of the N-PL map G.

Let (g', x') be another p-rotatory intertwining of an N-polynomial map $(F, \mathcal{O})$ with marked periodic point of period N. For g', we attach a prime to each notation (*e.g.*, $D_0', G', \mathcal{K}'$). Our aim is to show (g, x) and (g', x') are affinely conjugate. Since the Julia sets of g and g' are connected, it suffices to show that g and g' are hybrid equivalent.

Theorem 5.1. *There exists a qc map $\phi : \mathbb{C} \to \mathbb{C}$ such that*

- $\phi \circ g = g' \circ \phi$ *on* $\mathbb{C}$,
- $\bar{\partial}\phi = 0$ *a.e. on* $K(g)$.

Proof. Let W_k denote the union of the components of $V_k \setminus (\bigcup_{i\in\mathbb{Z}_N} \overline{S_i})$ that intersects K_k for each $k \in \mathbb{Z}_N$. Set $\Phi = h'^{-1} \circ h$ on $\bigcup_{k\in\mathbb{Z}_N} W_k$. Clearly, Φ can be extended continuously to $\bigcup_{k\in\mathbb{Z}_N} \overline{W_k}$ and the corresponding boundary map satisfies $\Phi\circ g = g'\circ\Phi$. Since x is repelling and each component of $\partial S_i \setminus x$ is smooth, it is well known that Φ can be extended to a qc map Φ form D_0 onto D'_0. See [9, Lemma 5.3] for example. Consequently, we can get a qc homeomorphism $\Phi : \mathbb{C} \to \mathbb{C}$ such that $\Phi(\mathcal{K}) = \mathcal{K}'$ and $\Phi \circ g = g' \circ \Phi$ on $\mathcal{K}$. Since the post-critical set $P(g)$ of g lies in $\mathcal{K}$, we can lift Φ through branched coverings g and g' to get a qc homeomorphism $\Psi : \mathbb{C} \to \mathbb{C}$ such that Ψ is homotopic to Φ rel $\mathcal{K} \supset P(g)$. By the proof of [14, Theorem A.1], there is a qc conjugacy ϕ between g and g' such that $\phi = \Phi$ on $\mathcal{K}$. It follows that $\bar{\partial}\phi = 0$ on $\bigcup_{n=0}^{\infty} g^{-n}(\mathcal{K})$ since $\bar{\partial}\phi = \bar{\partial}\Phi = 0$ on $\mathcal{K}$. Note that $K(g) \setminus \bigcup_{n=0}^{\infty} g^{-n}(\mathcal{K})$ has zero Lebesgue measure (see [8, Key Lemma 6.1] for example). Thus ϕ is actually a hybrid conjugacy between g and g'. □

6. Tuning for unicritical polynomials

The aim of this section is to prove the Main Theorem. More precisely, fix an integer $d \geq 2$ and a critically periodic unicritical polynomial $f_0 \in \mathcal{C}_d$, we show that f_0 can be tuned by any polynomial in $\mathcal{C}_d$ that does not have parabolic fixed points.

Throughout this section, we fix such an $f_0 \neq z^d$ with an internal angle system h. Note that f_0 has d distinct repelling fixed point. Among these fixed point, there exists a unique *dividing fixed point* $\alpha(f_0)$, that is, $K(f_0) \setminus \{\alpha(f_0)\}$ is disconnected. Furthermore, there exists a unique cycle of external rays landing at $\alpha(f_0)$. Assume that the period of the critical point 0 is q'. Let q denote the number of external rays landing at $\alpha(f_0)$.

6.1. **Immediate renormalization.**
Let us begin a naive but essential case.

If $q' = q$, then we call f_0 *immediately renormalizable.*

Theorem 6.1. *If f_0 is immediately renormalizable, then for any $g \in \mathcal{C}_d$ without parabolic fixed points, $\chi_{f_0}^{-1}(g) \neq \emptyset$.*

Proof. Assume that the rotation number of $\alpha(f_0)$ is p/q. Let

$$G = (G_k)_{k\in\mathbb{Z}_q} : \mathbb{Z}_q \times \mathbb{C} \to \mathbb{Z}_q \times \mathbb{C}$$

be a q-polynomial such that $G_0 = g$ and $G_k = \mathrm{id}$ for $k = 1, \dots, q-1$. Let β be the fixed point of g at which the external ray for g with angle 0 lands. Set $\mathcal{O} = \{(k, \beta) \mid k \in \mathbb{Z}_q\}$. It follows from Theorem 4.1 that there exists a p-rotatory intertwining (f, x) of $(G, \mathcal{O})$. Here, we apply Theorem 4.1 with $p_0 = 0$, $q_0 = 1$, $N = q$. By a conjugacy via a rotation, we may assume that $\mathcal{R}(f; \theta)$ lands at x whenever $\mathcal{R}(f_0; \theta)$ lands at $\alpha(f_0)$. By [8, Corollary 4.11], we have $\lambda_f \supset \lambda_{f_0}$. Finally it is easy to check that g is a λ_{f_0}-renormalization for f by the definition of p-rotatory intertwining. Hence $f \in \chi_{f_0}^{-1}(g)$. □

6.2. General cases.

Lemma 6.1. *Let $f = z^d + c$ be a critically periodic polynomial. Let $\mathcal{G}$ be the grand orbit of the external rays landing at $\alpha(f)$ under f and let*

$$\Theta = \{(s,t) \mid \mathcal{R}(f;s) \text{ and } \mathcal{R}(f;t) \in \mathcal{G} \text{ land at a common point}\}.$$

Assume that λ_f coincides with the smallest rational lamination that contains Θ. Then either f is immediately renormalizable or primitive.

Proof. Assume that f is not immediately renormalizable. We will show that f is primitive. For otherwise, there exist two periodic Fatou component U, U' and a periodic point x such that

$$x \in \overline{U} \cap \overline{U'}.$$

Note that there are two external angles $a \neq b$ such that $\mathcal{R}(f;a)$ and $\mathcal{R}(f;b)$ lands at x.

We first show that $(a,b) \notin \Theta$. This is a consequence of the following claim.

Claim. *For any $(s,t) \in \Theta$, the corresponding landing point z_* of $\mathcal{R}(f;t)$ is a buried point. In other words, z_* does not lie on the boundary of any Fatou component.*

For otherwise, there exists a periodic Fatou component V such that $\alpha(f) \in \partial V$. It follows the period of V is the ray period of $\alpha(f)$. See the proof of [16, Lemma 2.1]. Hence, f is immediately renormalizable.

Since λ_f is the smallest rational lamination that contains Θ, there exist a sequence $\{(a_n,b_n)\} \subset \Theta$ such that $\lim_{n\to\infty} |a_n - a| = \lim_{n\to\infty} |b_n - b| = 0$. Let W_n denote the simply connected domain that is bounded by $\overline{\mathcal{R}(f;a) \cup \mathcal{R}(f;b)}$ and $\overline{\mathcal{R}(f;a_n) \cup \mathcal{R}(f;b_n)}$. Then either $U \subset W_n$ or $U' \subset W_n$. Without loss of generality, we may assume that $U \subset W_n$ for all $n \in \mathbb{N}$. Let θ be an external angle such that $\mathcal{R}(f;\theta)$ lands on ∂U. Then either $|\theta - a| \leq |a_n - a|$ or $|\theta - b| \leq |b_n - b|$ for all $n \in \mathbb{N}$. This implies that θ must equal to a or b. So only $\mathcal{R}(f;a)$ and $\mathcal{R}(f;b)$ can land on ∂U. This is impossible since U is a Jordan disk. □

Lemma 6.2. *There exists a finite sequence $(F_i)_{i=0}^n \subset \mathcal{C}_d$ of critically periodic unicritical polynomial with internal angle system H_i satisfying the following.*

- *For each $0 \leq i \leq n$, F_n is either primitive or immediately renormalizable.*
- *$\chi_{f_0} = \chi_n \circ \cdots \circ \chi_0$, where χ_i is the straightening map induced by F_i and H_i.*

Proof. Recall that h is the internal angle system for f_0 and the period of the critical point 0 under f_0 is q'. Let $\mathcal{G}_0$ be the grand orbit of the external rays landing at $\alpha(f_0)$ under f_0 and let

$$\Theta_0 = \{(s,t) \mid \mathcal{R}(f_0;s) \text{ and } \mathcal{R}(f_0;t) \in \mathcal{G}_0 \text{ land at a common point}\}.$$

By [8, Theorems 5.17,5.18], there exists a unique critically periodic unicritical polynomial $F_0 \in \mathcal{C}_d$ such that λ_{F_0} is the smallest rational lamination that contains Θ_0. If F_0 is not immediately renormalizable, then it follows that F_0 is primitive from Lemma 6.1. Assume that the period of the critical point 0 under F_0 is p_0. If $p_0 = q'$, then we choose the internal angle system $H_0 = h$ and let $n = 0$. Otherwise, we choose an arbitrary internal angle system H_0 of F_0.

Now we proceed to construct (F_k) inductively. Assume that $F_0, \cdots, F_k$ and $H_0, \cdots, H_{k-1}$ has been constructed. Let p_i be the period of the critical point 0 under F_i $(0 \leq i \leq k)$.

Case 1. If $q' > \Pi_{i=0}^{k} p_i$, then we choose an arbitrary internal angle system H_k of F_k and we let $f_{k+1} = \chi_k \circ \cdots \circ \chi_0(f_0)$. Let α_{k+1} be the unique dividing fixed point of f_{k+1} and let $\mathcal{G}_{k+1}$ be the grand orbit of the external rays landing at α_{k+1} under f_{k+1}. Let

$$\Theta_{k+1} = \{(s,t) \mid \mathcal{R}(f_{k+1}; s) \text{ and } \mathcal{R}(f_{k+1}; t) \in \mathcal{G}_{k+1} \text{ land at a common point}\}.$$

Again by [8, Theorem 5.17,5.18], there exists a unique critically periodic unicritical polynomial $F_{k+1} \in \mathcal{C}_d$ such that $\lambda_{F_{k+1}}$ is the smallest rational lamination that contains Θ_{k+1}. By Lemma 6.1, either F_{k+1} is immediately renormalizable or primitive.

Case 2. If $q' = \Pi_{i=0}^{k} p_i$, we let $n = k$ and choose H_k such that $H_k^{-1}(0)$ corresponds to $h^{-1}(0)$ in the following sense. Make a convention that $\chi_{-1} = \mathrm{id}$. Indeed, since $q' > \Pi_{i=0}^{k-1} p_i$, $F_k = f_k = \chi_{k-1} \circ \cdots \circ \chi_0(f_0)$ by the construction. Let ϕ_i be a hybrid conjugacy between $f_i^{p_i} = (\chi_{i-1} \circ \cdots \circ \chi_0(f_0))^{p_i}$ and $f_{i+1} = \chi_i \circ \cdots \circ \chi_0(f_0)$. Let $\gamma_k = \phi_{k-1} \circ \cdots \circ \phi_0(\mathcal{R}(f_0; \theta_0))$. Note that γ_k is $F_k^{p_k}$-invariant and so it lands at a periodic point $z_k \in \partial U_k$ of period p_k under F_k, where U_k is the Fatou component of F_k containing 0. Now choose an internal angle system H_k of F_k so that $H_k^{-1}(0) = z_k$.

It remains to check the second property. For each $0 \leq k \leq n$, let α_k be the dividing fixed point of $f_k = \chi_{k-1} \circ \cdots \circ \chi_0(f_0)$ and let $\mathcal{O}_k$ be a cycle of external rays landing at α_k. Let $\widehat{\mathcal{O}}_k = (\phi_{k-1} \circ \cdots \circ \phi_0)^{-1}(\mathcal{O}_k)$. Note that all the rays in $\widehat{\mathcal{O}}_k$ land at a common periodic point α_k' of f_0. Let $\mathcal{S}$ be the collection of external rays landing at $\bigcup \alpha_k'$ and let

$$\widehat{\Theta} = \{(s,t) \mid \mathcal{R}(f_0; s) \text{ and } \mathcal{R}(f_0; t) \in \mathcal{S} \text{ land at a common point}\}.$$

For any $f \in \mathcal{R}(\lambda_{f_0})$, let

$$\hat{K}_f' = \bigcap_{(s,t) \in \widehat{\Theta}} S_f(s,t)$$

where $S_f(s,t)$ is the component of $\mathbb{C} \setminus \overline{\mathcal{R}(f;s) \cup \mathcal{R}(f;t)}$ with the following property. For any $u \in \mathbb{R}/\mathbb{Z}$, the external ray $\mathcal{R}(f;u)$ lies in $S_f(s,t)$ if and only if the external ray $\mathcal{R}(f_0;u)$ lies in the component of $\mathbb{C} \setminus \overline{\mathcal{R}(f_0;s) \cup \mathcal{R}(f_0;t)}$ that contains U_0. By the constructions of χ_k, $f^{q'}|_{\hat{K}_f'}$ extends to be a polynomial-like map that is hybrid equivalent to $\chi_n \circ \cdots \circ \chi_0(f)$. Since $\hat{K}_f \subset \hat{K}_f'$ and both $f^{q'}|_{\hat{K}_f}$ and $f^{q'}|_{\hat{K}_f'}$ extends to a polynomial-like map of a same degree d, it follows from [13, Theorem 5.11] that $\hat{K}_f = \hat{K}_f'$. Thus

$$\chi_{f_0}(f) = \chi_n \circ \cdots \circ \chi_0(f).$$

□

Now we prove the Main Theorem to complete this paper.

Proof of the Main Theorem. Let $\chi_k = \chi_{F_k}$ $(k = 0, \cdots, n)$ be given by Lemma 6.2. By Theorem 6.1 and [16, Main Theorem], χ_k is almost surjective for all $k = 0, \cdots, n$, and hence $\chi_{f_0} = \chi_n \circ \cdots \circ \chi_0$ is almost surjective. □

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