Double pendulum

Hiroyuki Inou

September 27, 2018

Abstract

The purpose of this article is to give a readable formula of the differential equation for double spherical pendulum (three-dimensional) in spherical coordinate. Since each spherical coordinate has singularities at poles, we need to use several spherical coordinates to numerically solve the differential equation. Hence we describe two coordinates and how to use those coordinates to compute.

1 Planer double pendulum

It is easy to find differential equations describing planer double pendulum in the web. However, in order to clarify our idea of obtaining the differential equation for double spherical pendulum, we first explain how to obtain the differential equation for planer double pendulum.

Roughly speaking, it can be obtained by solving the Lagrangian equation

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} - \frac{\partial L}{\partial \theta_i} = 0 \tag{1.1}$$

where

$$L = T - V, \tag{1.2}$$

$$T = \frac{1}{2}m_1\|\dot{\boldsymbol{x}}_1\|^2 + \frac{1}{2}m_2\|\dot{\boldsymbol{x}}_2\|^2, \qquad V = m_1 g y_1 + m_2 g y_2, \tag{1.3}$$

$$\boldsymbol{q}_i = \begin{pmatrix} p_i \\ q_i \end{pmatrix} = \begin{pmatrix} l_i \sin\theta_i \\ -l_i\cos\theta_1 \end{pmatrix},$$

$$\boldsymbol{x}_1 = \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \boldsymbol{q}_1, \qquad \boldsymbol{x}_2 = \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \boldsymbol{q}_1 + \boldsymbol{q}_2.$$

The constants $l_i = \|\boldsymbol{q}_i\|$ and m_i $(i = 1, 2)$ are the length and the mass of pendulums respectively, and g is the gravitational constant. figure?

In the following, we consider the Lagrangian $L = L(\boldsymbol{q}_1, \boldsymbol{q}_2, \dot{\boldsymbol{q}}_1, \dot{\boldsymbol{q}}_2)$ as a function of $\boldsymbol{q}_i$ and $\dot{\boldsymbol{q}}_i$. Thus we have

$$\frac{\partial L}{\partial \theta_i} = \frac{\partial L}{\partial \boldsymbol{q}_i}\cdot\frac{\partial \boldsymbol{q}_i}{\partial \theta_i} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\cdot\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \theta_i}, \qquad \frac{\partial L}{\partial \dot{\theta}_i} = \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\cdot\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{\theta}_i}. \tag{1.4}$$

where

$$\frac{\partial L}{\partial \boldsymbol{q}_i} = \begin{pmatrix} \frac{\partial L}{\partial p_i} \\ \frac{\partial L}{\partial q_i} \end{pmatrix}, \qquad \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i} = \begin{pmatrix} \frac{\partial L}{\partial \dot{p}_i} \\ \frac{\partial L}{\partial \dot{q}_i} \end{pmatrix}. \tag{1.5}$$

Therefore,

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\theta}_i} = \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right)\cdot\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{\theta}_i} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\cdot\left(\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{\theta}_i}\right). \tag{1.6}$$

Now we need to express each term by θ_1, θ_2 and the constants above. First we calculate derivatives for $\boldsymbol{q}_i$ and $\dot{\boldsymbol{q}}_i$:

$$\frac{\partial \boldsymbol{q}_i}{\partial \theta_i} = \begin{pmatrix} l_i \cos\theta_i \\ l_i \sin\theta_i \end{pmatrix} = J\boldsymbol{q}_i, \tag{1.7}$$

$$\dot{\boldsymbol{q}}_i = \begin{pmatrix} l_i \dot{\theta}_i \cos\theta_i \\ l_i \dot{\theta}_i \sin\theta_i \end{pmatrix} = \dot{\theta}_i J\boldsymbol{q}_i, \tag{1.8}$$

$$\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \theta_i} = \dot{\theta}_i J \frac{\partial \boldsymbol{q}_i}{\partial \theta_i} = \dot{\theta}_i J^2 \boldsymbol{q}_i = -\dot{\theta}_i \boldsymbol{q}_i, \tag{1.9}$$

$$\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{\theta}_i} = J\boldsymbol{q}_i, \tag{1.10}$$

$$\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{\theta}_i} = J\dot{\boldsymbol{q}}_i = -\dot{\theta}_i \boldsymbol{q}_i, \tag{1.11}$$

$$\ddot{\boldsymbol{q}}_i = \ddot{\theta}_i J\boldsymbol{q}_i + \dot{\theta}_i J\dot{\boldsymbol{q}}_i = \ddot{\theta}_i J\boldsymbol{q}_i - \dot{\theta}_i^2 \boldsymbol{q}_i \tag{1.12}$$

where $J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$.

Since

$$\begin{aligned} L = T - V &= \frac{1}{2} m_1 \|\dot{\boldsymbol{q}}_1\|^2 + \frac{1}{2} m_2 \|\dot{\boldsymbol{q}}_1 + \dot{\boldsymbol{q}}_2\|^2 - m_1 g q_1 - m_2 g (q_1 + q_2) \\ &= \frac{1}{2}(m_1 + m_2)\|\dot{\boldsymbol{q}}_1\|^2 + \frac{1}{2} m_2 \|\dot{\boldsymbol{q}}_2\|^2 + m_2 \dot{\boldsymbol{q}}_1 \cdot \dot{\boldsymbol{q}}_2 - (m_1 + m_2) g q_1 - m_2 g q_2, \end{aligned} \tag{1.13}$$

derivatives of the Lagrangian are:

$$\frac{\partial L}{\partial \boldsymbol{q}_1} = \begin{pmatrix} 0 \\ (m_1 + m_2) g \end{pmatrix} \tag{1.14}$$

$$\frac{\partial L}{\partial \boldsymbol{q}_2} = \begin{pmatrix} 0 \\ m_2 g \end{pmatrix}, \tag{1.15}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} = (m_1 + m_2)\dot{\boldsymbol{q}}_1 + m_2 \dot{\boldsymbol{q}}_2 = (m_1 + m_2)\dot{\theta}_1 J\boldsymbol{q}_1 + m_2 \dot{\theta}_2 J\boldsymbol{q}_2, \tag{1.16}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} = m_2 \dot{\boldsymbol{q}}_1 + m_2 \dot{\boldsymbol{q}}_2 = m_2 \dot{\theta}_1 J\boldsymbol{q}_1 + m_2 \dot{\theta}_2 J\boldsymbol{q}_2, \tag{1.17}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} &= (m_1 + m_2)\ddot{\boldsymbol{q}}_1 + m_2 \ddot{\boldsymbol{q}}_2 \\ &= (m_1 + m_2)\ddot{\theta}_1 J\boldsymbol{q}_1 + m_2 \ddot{\theta}_2 J\boldsymbol{q}_2 - (m_1 + m_2)\dot{\theta}_1^2 \boldsymbol{q}_1 - m_2 \dot{\theta}_2^2 \boldsymbol{q}_2, \end{aligned} \tag{1.18}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} &= m_2 \ddot{\boldsymbol{q}}_1 + m_2 \ddot{\boldsymbol{q}}_2 \\ &= m_2 \ddot{\theta}_1 J\boldsymbol{q}_1 + m_2 \ddot{\theta}_2 J\boldsymbol{q}_2 - m_2 \dot{\theta}_1^2 \boldsymbol{q}_1 - m_2 \dot{\theta}_2^2 \boldsymbol{q}_2. \end{aligned} \tag{1.19}$$

The inner products of above vectors are:

$$\frac{\partial L}{\partial \boldsymbol{q}_1} \cdot \frac{\partial \boldsymbol{q}_1}{\partial \theta_1} = (m_1 + m_2) g l_1 \sin\theta_1, \tag{1.20}$$

$$\begin{aligned} \frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} \cdot \frac{\partial \dot{\boldsymbol{q}}_1}{\partial \theta_1} &= ((m_1 + m_2)\dot{\theta}_1 J\boldsymbol{q}_1 + m_2 \dot{\theta}_2 J\boldsymbol{q}_2) \cdot (-\dot{\theta}_1 \boldsymbol{q}_1) \\ &= -m_2 \dot{\theta}_1 \dot{\theta}_2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2, \end{aligned} \tag{1.21}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} \cdot \frac{\partial \dot{\boldsymbol{q}}_1}{\partial \dot{\theta}_1} = (m_1 + m_2)\ddot{\theta}_1 l_1^2 + m_2 \ddot{\theta}_2 \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + m_2 \dot{\theta}_2^2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2, \tag{1.22}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} \cdot \frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_1}{\partial \dot{\theta}_1} = -m_2 \dot{\theta}_1 \dot{\theta}_2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2, \tag{1.23}$$

and

$$\frac{\partial L}{\partial \boldsymbol{q}_2} \cdot \frac{\partial \boldsymbol{q}_2}{\partial \theta_2} = m_2 g l_2 \sin\theta_2, \tag{1.24}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} \cdot \frac{\partial \dot{\boldsymbol{q}}_2}{\partial \theta_2} = m_2 \dot\theta_1 \dot\theta_2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2, \tag{1.25}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} \cdot \frac{\partial \dot{\boldsymbol{q}}_2}{\partial \dot\theta_2} = m_2 \ddot\theta_1 \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + m_2 \ddot\theta_2 l_2^2 - m_2 \dot\theta_1^2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2, \tag{1.26}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} \cdot \frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_2}{\partial \dot\theta_2} = -m_2 \dot\theta_1 \dot\theta_2 \boldsymbol{q}_2 \cdot J\boldsymbol{q}_1 = m_2 \dot\theta_1 \dot\theta_2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2 \tag{1.27}$$

Here we use the following properties:

$$\boldsymbol{q}_i \cdot \boldsymbol{q}_i = l_i^2, \qquad \boldsymbol{q} \cdot J\boldsymbol{q} = 0, \tag{1.28}$$

$$(J\boldsymbol{q}) \cdot J\boldsymbol{q}' = \boldsymbol{q} \cdot \boldsymbol{q}', \qquad \boldsymbol{q} \cdot J\boldsymbol{q}' = (J\boldsymbol{q}) \cdot J^2\boldsymbol{q} = -\boldsymbol{q}' \cdot J\boldsymbol{q}. \tag{1.29}$$

Let $\Delta\theta = \theta_1 - \theta_2$. Since

$$\boldsymbol{q}_1 \cdot \boldsymbol{q}_2 = l_1 l_2 (\sin\theta_1 \sin\theta_1 + \cos\theta_1 \cos\theta_2) = l_1 l_2 \cos\Delta\theta, \tag{1.30}$$

$$\boldsymbol{q}_1 \cdot J\boldsymbol{q}_2 = l_1 l_2 (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2) = l_1 l_2 \sin\Delta\theta, \tag{1.31}$$

we have

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} &= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1}\right) \cdot \frac{\partial \dot{\boldsymbol{q}}_1}{\partial \dot\theta_1} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} \cdot \left(\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_1}{\partial \dot\theta_1}\right) - \frac{\partial L}{\partial \boldsymbol{q}_1} \cdot \frac{\partial \boldsymbol{q}_1}{\partial \theta_1} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} \cdot \frac{\partial \dot{\boldsymbol{q}}_1}{\partial \theta_1} \\ &= (m_1 + m_2)\ddot\theta_1 l_1^2 + m_2 \ddot\theta_2 \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + m_2 \dot\theta_2^2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2 - (m_1 + m_2) g l_1 \sin\theta_1, \end{aligned} \tag{1.32}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot\theta_1} - \frac{\partial L}{\partial \theta_1} &= \left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2}\right) \cdot \frac{\partial \dot{\boldsymbol{q}}_2}{\partial \dot\theta_2} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} \cdot \left(\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_2}{\partial \dot\theta_2}\right) - \frac{\partial L}{\partial \boldsymbol{q}_2} \cdot \frac{\partial \boldsymbol{q}_2}{\partial \theta_2} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} \cdot \frac{\partial \dot{\boldsymbol{q}}_2}{\partial \theta_2} \\ &= m_2 \ddot\theta_1 \boldsymbol{q}_1 \cdot \boldsymbol{q}_2 + m_2 \ddot\theta_2 l_2^2 - m_2 \dot\theta_1^2 \boldsymbol{q}_1 \cdot J\boldsymbol{q}_2 - m_2 g l_2 \sin\theta_2. \end{aligned} \tag{1.33}$$

Now introduce a new constant $\mu = 1 + \frac{m_1}{m_2}$. Then the Lagrange equation (1.1) is equivalent to the following:

$$\begin{pmatrix} \mu l_1^2 & l_1 l_2 \cos\Delta\theta \\ l_1 l_2 \cos\Delta\theta & l_2^2 \end{pmatrix} \begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = \begin{pmatrix} -\dot\theta_2^2 l_1 l_2 \sin\Delta\theta - \mu g l_1 \sin\theta_1 \\ \dot\theta_1^2 l_1 l_2 \sin\Delta\theta - g l_2 \sin\theta_2 \end{pmatrix}. \tag{1.34}$$

Therefore, we obtain the desired equation

$$\begin{pmatrix} \ddot\theta_1 \\ \ddot\theta_2 \end{pmatrix} = \begin{pmatrix} \mu l_1^2 & l_1 l_2 \cos\Delta\theta \\ l_1 l_2 \cos\Delta\theta & l_2^2 \end{pmatrix}^{-1} \begin{pmatrix} -\dot\theta_2^2 l_1 l_2 \sin\Delta\theta - \mu g l_1 \sin\theta_1 \\ \dot\theta_1^2 l_1 l_2 \sin\Delta\theta - g l_2 \sin\theta_2 \end{pmatrix}. \tag{1.35}$$

2 Double spherical pendulum

Now consider three-dimensional double pendulum. The standard spherical coordinate (for this system) is as follows:

$$\boldsymbol{q}_i = \begin{pmatrix} p_i \\ q_i \\ r_i \end{pmatrix} = \begin{pmatrix} l_i \sin\theta_i \cos\phi_i \\ l_i \sin\theta_i \sin\phi_i \\ -l_i \cos\theta_i \end{pmatrix}, \qquad \boldsymbol{x}_i = \begin{pmatrix} x_i \\ y_i \\ z_i \end{pmatrix}. \tag{2.1}$$

But for numerical computation, it is better to use several coordinate system (see Section 4), so we first give the differential equation for the double spherical pendulum in a general setting:

$$\boldsymbol{q}_i = \begin{pmatrix} p_i(u_i, v_i) \\ q_i(u_i, v_i) \\ r_i(u_i, v_i) \end{pmatrix} \qquad (i = 1, 2). \tag{2.2}$$

Then the Lagrange equation becomes

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right)\cdot\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{u}_i} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\cdot\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{u}_i} - \frac{\partial L}{\partial \boldsymbol{q}_i}\cdot\frac{\partial \boldsymbol{q}_i}{\partial u_i} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\cdot\frac{\partial \dot{\boldsymbol{q}}_i}{\partial u_i} = 0, \tag{2.3}$$

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right)\cdot\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{v}_i} + \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\cdot\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{v}_i} - \frac{\partial L}{\partial \boldsymbol{q}_i}\cdot\frac{\partial \boldsymbol{q}_i}{\partial v_i} - \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\cdot\frac{\partial \dot{\boldsymbol{q}}_i}{\partial v_i} = 0, \tag{2.4}$$

where

$$\frac{\partial L}{\partial \boldsymbol{q}_i} = \begin{pmatrix} \frac{\partial L}{\partial p_i} \\ \frac{\partial L}{\partial q_i} \\ \frac{\partial L}{\partial r_i} \end{pmatrix}, \qquad \frac{\partial L}{\partial \dot{\boldsymbol{q}}_i} = \begin{pmatrix} \frac{\partial L}{\partial \dot{p}_i} \\ \frac{\partial L}{\partial \dot{q}_i} \\ \frac{\partial L}{\partial \dot{r}_i} \end{pmatrix}. \tag{2.5}$$

Since

$$\dot{\boldsymbol{q}}_i = \dot{u}_i \frac{\partial \boldsymbol{q}_i}{\partial u_i} + \dot{v}_i \frac{\partial \boldsymbol{q}_i}{\partial v_i}, \tag{2.6}$$

we have

$$\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{u}_i} = \frac{\partial q_i}{\partial u_i}, \tag{2.7}$$

$$\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{v}_i} = \frac{\partial q_i}{\partial v_i}. \tag{2.8}$$

Hence it follows that

$$\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{u}_i} = \frac{d}{dt}\frac{\partial \boldsymbol{q}_i}{\partial u_i} = \dot{u}_i \frac{\partial^2 \boldsymbol{q}_i}{\partial u_i^2} + \dot{v}_i \frac{\partial^2 \boldsymbol{q}_i}{\partial u_i \partial v_i} = \frac{\partial \dot{\boldsymbol{q}}_i}{\partial u_i}, \tag{2.9}$$

$$\frac{d}{dt}\frac{\partial \dot{\boldsymbol{q}}_i}{\partial \dot{v}_i} = \frac{d}{dt}\frac{\partial \boldsymbol{q}_i}{\partial v_i} = \dot{u}_i \frac{\partial^2 \boldsymbol{q}_i}{\partial u_i \partial v_i} + \dot{v}_i \frac{\partial^2 \boldsymbol{q}_i}{\partial v_i^2} = \frac{\partial \dot{\boldsymbol{q}}_i}{\partial v_i}. \tag{2.10}$$

Thus the equations (2.3) and (2.4) reduce to

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right)\cdot\frac{\partial \boldsymbol{q}_i}{\partial u_i} - \frac{\partial L}{\partial \boldsymbol{q}_i}\cdot\frac{\partial \boldsymbol{q}_i}{\partial u_i} = 0, \tag{2.11}$$

$$\left(\frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_i}\right)\cdot\frac{\partial \boldsymbol{q}_i}{\partial k v_i} - \frac{\partial L}{\partial \boldsymbol{q}_i}\cdot\frac{\partial \boldsymbol{q}_i}{\partial v_i} = 0. \tag{2.12}$$

Since the Lagrange equation is linear on L, we may divide L by m_2, or equivalently, we may assume $m_2 = 1$, then L is of the following form:

$$L = \frac{1}{2}\mu\|\dot{\boldsymbol{q}}_1\|^2 + \frac{1}{2}\|\dot{\boldsymbol{q}}_2\|^2 + \dot{\boldsymbol{q}}_1 \cdot \dot{\boldsymbol{q}}_2 - \mu g r_1 - g r_2 \tag{2.13}$$

(compare (1.13). Recall that $\mu = 1 + \frac{m_1}{m_2}$). Then we have

$$\frac{\partial L}{\partial \boldsymbol{q}_1} = \begin{pmatrix} 0 \\ 0 \\ -\mu g \end{pmatrix}, \tag{2.14}$$

$$\frac{\partial L}{\partial \boldsymbol{q}_2} = \begin{pmatrix} 0 \\ 0 \\ -g \end{pmatrix}, \tag{2.15}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} = \mu\dot{\boldsymbol{q}}_1 + \dot{\boldsymbol{q}}_2 = \mu\dot{\theta}_1\frac{\partial \boldsymbol{q}_1}{\partial u_1} + \mu\dot{\phi}_1\frac{\partial \boldsymbol{q}_1}{\partial v_1} + \dot{\theta}_2\frac{\partial \boldsymbol{q}_2}{\partial u_2} + \dot{\phi}_2\frac{\partial \boldsymbol{q}_2}{\partial v_2}, \tag{2.16}$$

$$\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} = \dot{\boldsymbol{q}}_1 + \dot{\boldsymbol{q}}_2 = \dot{\theta}_1\frac{\partial \boldsymbol{q}_1}{\partial u_1} + \dot{\phi}_1\frac{\partial \boldsymbol{q}_1}{\partial v_1} + \dot{\theta}_2\frac{\partial \boldsymbol{q}_2}{\partial u_2} + \dot{\phi}_2\frac{\partial \boldsymbol{q}_2}{\partial v_2}. \tag{2.17}$$

Thus we have

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_1} &= \mu\ddot{u}_1\frac{\partial \boldsymbol{q}_1}{\partial u_1} + \mu\ddot{v}_1\frac{\partial \boldsymbol{q}_1}{\partial v_1} + \ddot{u}_2\frac{\partial \boldsymbol{q}_2}{\partial u_2} + \ddot{v}_2\frac{\partial \boldsymbol{q}_2}{\partial v_2} \\ &\quad + \mu\dot{u}_1^2\frac{\partial^2 \boldsymbol{q}_1}{\partial u_1^2} + 2\mu\dot{u}_1\dot{v}_1\frac{\partial^2 \boldsymbol{q}_1}{\partial u_1 \partial v_1} + \mu\dot{v}_1^2\frac{\partial^2 \boldsymbol{q}_1}{\partial v_1^2} \\ &\quad + \dot{u}_2^2\frac{\partial^2 \boldsymbol{q}_2}{\partial u_2^2} + 2\dot{u}_2\dot{v}_2\frac{\partial^2 \boldsymbol{q}_2}{\partial u_2 \partial v_2} + \dot{v}_2^2\frac{\partial^2 \boldsymbol{q}_2}{\partial v_2^2}, \end{aligned} \tag{2.18}$$

$$\begin{aligned} \frac{d}{dt}\frac{\partial L}{\partial \dot{\boldsymbol{q}}_2} &= \ddot{u}_1\frac{\partial \boldsymbol{q}_1}{\partial u_1} + \ddot{v}_1\frac{\partial \boldsymbol{q}_1}{\partial v_1} + \ddot{u}_2\frac{\partial \boldsymbol{q}_2}{\partial u_2} + \ddot{v}_2\frac{\partial \boldsymbol{q}_2}{\partial v_2} \\ &\quad + \dot{u}_1^2\frac{\partial^2 \boldsymbol{q}_1}{\partial u_1^2} + 2\dot{u}_1\dot{v}_1\frac{\partial^2 \boldsymbol{q}_1}{\partial u_1 \partial v_1} + \dot{v}_1^2\frac{\partial^2 \boldsymbol{q}_1}{\partial v_1^2} \\ &\quad + \dot{u}_2^2\frac{\partial^2 \boldsymbol{q}_2}{\partial u_2^2} + 2\dot{u}_2\dot{v}_2\frac{\partial^2 \boldsymbol{q}_2}{\partial u_2 \partial v_2} + \dot{v}_2^2\frac{\partial^2 \boldsymbol{q}_2}{\partial v_2^2} \end{aligned} \tag{2.19}$$

Applying these, we obtain that the reduced Lagrange equations (2.11) and (2.12) is equivalent to the following:

$$A\begin{pmatrix} \ddot{u}_1 \\ \ddot{v}_1 \\ \ddot{u}_2 \\ \ddot{v}_2 \end{pmatrix} + B\begin{pmatrix} \dot{u}_1^2 \\ 2\dot{u}_1\dot{v}_1 \\ \dot{v}_1^2 \\ \dot{u}_2^2 \\ 2\dot{u}_2\dot{v}_2 \\ \dot{v}_2^2 \end{pmatrix} + \begin{pmatrix} \mu g\frac{\partial r_1}{\partial u_1} \\ \mu g\frac{\partial r_1}{\partial v_1} \\ g\frac{\partial r_2}{\partial u_2} \\ g\frac{\partial r_2}{\partial v_2} \end{pmatrix} = 0, \tag{2.20}$$

where

$$A = \begin{pmatrix} \mu A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{2.21}$$

$$B = \begin{pmatrix} \mu B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \tag{2.22}$$

$$A_{ij} = \begin{pmatrix} \frac{\partial \boldsymbol{q}_j}{\partial u_j} \cdot \frac{\partial \boldsymbol{q}_i}{\partial u_i} & \frac{\partial \boldsymbol{q}_j}{\partial v_j} \cdot \frac{\partial \boldsymbol{q}_i}{\partial u_i} \\ \frac{\partial \boldsymbol{q}_j}{\partial u_j} \cdot \frac{\partial \boldsymbol{q}_i}{\partial v_i} & \frac{\partial \boldsymbol{q}_j}{\partial v_j} \cdot \frac{\partial \boldsymbol{q}_i}{\partial v_i} \end{pmatrix}, \tag{2.23}$$

$$B_{ij} = \begin{pmatrix} \frac{\partial^2 \boldsymbol{q}_j}{\partial u_j^2} \cdot \frac{\partial \boldsymbol{q}_i}{\partial u_i} & \frac{\partial^2 \boldsymbol{q}_j}{\partial u_j \partial v_j} \cdot \frac{\partial \boldsymbol{q}_i}{\partial u_i} & \frac{\partial^2 \boldsymbol{q}_j}{\partial v_j^2} \cdot \frac{\partial \boldsymbol{q}_i}{\partial u_i} \\ \frac{\partial^2 \boldsymbol{q}_j}{\partial u_j^2} \cdot \frac{\partial \boldsymbol{q}_i}{\partial v_i} & \frac{\partial^2 \boldsymbol{q}_j}{\partial u_j \partial v_j} \cdot \frac{\partial \boldsymbol{q}_i}{\partial v_i} & \frac{\partial^2 \boldsymbol{q}_j}{\partial v_j^2} \cdot \frac{\partial \boldsymbol{q}_i}{\partial v_i} \end{pmatrix} \tag{2.24}$$

3 Standard spherical coordinate

Now we consider the standard coordinate system given by (2.1). Since we have two angular coordinates, the derivatives of $\boldsymbol{q}_i$ and $\dot{\boldsymbol{q}}_i$ become complicated. The angular derivatives of $\boldsymbol{q}_i$ (up to second order) are as follows:

$$\frac{\partial \boldsymbol{q}_i}{\partial \theta_i} = \begin{pmatrix} l_i \cos\theta_i \cos\phi_i \\ l_i \cos\theta_i \sin\phi_i \\ l_i \sin\theta_i \end{pmatrix}, \qquad \frac{\partial \boldsymbol{q}_i}{\partial \phi_i} = \begin{pmatrix} -l_i \sin\theta_i \sin\phi_i \\ l_i \sin\theta_i \cos\phi_i \\ 0 \end{pmatrix}, \tag{3.1}$$

$$\frac{\partial^2 \boldsymbol{q}_i}{\partial \theta_i^2} = \begin{pmatrix} -l_i \sin\theta_i \cos\phi_i \\ -l_i \sin\theta_i \sin\phi_i \\ l_i \cos\theta_i \end{pmatrix} = -\boldsymbol{q}_i, \qquad \frac{\partial^2 \boldsymbol{q}_i}{\partial \phi_i^2} = -\begin{pmatrix} l_i \sin\theta_i \cos\phi_i \\ l_i \sin\theta_i \sin\phi_i \\ 0 \end{pmatrix} = -\begin{pmatrix} p_i \\ q_i \\ 0 \end{pmatrix}, \tag{3.2}$$

$$\frac{\partial^2 \boldsymbol{q}_i}{\partial \theta_i \partial \phi_i} = \begin{pmatrix} -l_i \cos\theta_i \sin\phi_i \\ l_i \cos\theta_i \cos\phi_i \\ 0 \end{pmatrix}. \tag{3.3}$$

We need inner products of those vectors. First, there are many orthogonal pairs:

$$\frac{\partial \boldsymbol{q}_i}{\partial \theta_i} \perp \frac{\partial \boldsymbol{q}_i}{\partial \phi_i}, \frac{\partial \boldsymbol{q}_i}{\partial \theta_i} \perp \frac{\partial^2 \boldsymbol{q}_i}{\partial \theta_i^2}, \quad \frac{\partial \boldsymbol{q}_i}{\partial \theta_i} \perp \frac{\partial^2 \boldsymbol{q}_i}{\partial \theta_i \partial \phi_i}, \quad \frac{\partial \boldsymbol{q}_i}{\partial \phi_i} \perp \frac{\partial^2 \boldsymbol{q}_i}{\partial \theta_i^2}, \quad \frac{\partial \boldsymbol{q}_i}{\partial \phi_i} \perp \frac{\partial^2 \boldsymbol{q}_i}{\partial \phi_i^2}.$$

Non vanishing products are as follows.

$$\frac{\partial \boldsymbol{q}_i}{\partial \theta_i} \cdot \frac{\partial \boldsymbol{q}_i}{\partial \theta_i} = l_i^2, \tag{3.4}$$

$$\frac{\partial \boldsymbol{q}_i}{\partial \phi_i} \cdot \frac{\partial \boldsymbol{q}_i}{\partial \phi_i} = l_i^2 \sin^2\theta_i, \tag{3.5}$$

$$\frac{\partial \boldsymbol{q}_i}{\partial \phi_i} \cdot \frac{\partial^2 \boldsymbol{q}_i}{\partial \theta_i \partial \phi_i} = l_i^2 \sin\theta_i \cos\theta_i = \frac{1}{2} l_i^2 \sin(2\theta_i), \tag{3.6}$$

$$\frac{\partial \boldsymbol{q}_i}{\partial \theta_i} \cdot \frac{\partial^2 \dot{\boldsymbol{q}}_i}{\partial \phi_i^2} = -l_i^2 \sin\theta_i \cos\theta_i = -\frac{1}{2} l_i^2 \sin(2\theta_i). \tag{3.7}$$

The following are dot products of derivatives of $\boldsymbol{q}_1$ with those of $\boldsymbol{q}_2$:

$$\begin{aligned}
\frac{\partial \boldsymbol{q}_1}{\partial \theta_1} \cdot \frac{\partial \boldsymbol{q}_2}{\partial \theta_2} &= l_1 l_2 (\cos\theta_1 \cos\theta_2 (\cos\phi_1 \cos\phi_2 + \sin\phi_1 \sin\phi_1) + \sin\theta_1 \sin\theta_2) \\
&= l_1 l_2 (\cos\theta_1 \cos\theta_2 \cos\Delta\phi + \sin\theta_1 \sin\theta_2),
\end{aligned} \tag{3.8}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \theta_1} \cdot \frac{\partial \boldsymbol{q}_2}{\partial \phi_2} = l_1 l_2 \cos\theta_1 \sin\theta_2 \sin\Delta\phi, \tag{3.9}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \phi_1} \cdot \frac{\partial \boldsymbol{q}_2}{\partial \phi_2} = l_1 l_2 \sin\theta_1 \sin\theta_2 \cos\Delta\phi, \tag{3.10}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \theta_1} \cdot \frac{\partial^2 \boldsymbol{q}_2}{\partial \theta_2^2} = l_1 l_2 (\sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \cos\Delta\phi), \tag{3.11}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \theta_1} \cdot \frac{\partial^2 \boldsymbol{q}_2}{\partial \theta_2 \partial \phi_2} = l_1 l_2 \cos\theta_1 \cos\theta_2 \sin\Delta\phi, \tag{3.12}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \theta_1} \cdot \frac{\partial^2 \boldsymbol{q}_2}{\partial \phi_2^2} = -l_1 l_2 \cos\theta_1 \sin\theta_2 \cos\Delta\phi, \tag{3.13}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \phi_1} \cdot \frac{\partial^2 \boldsymbol{q}_2}{\partial \theta_2^2} = l_1 l_2 \sin\theta_1 \sin\theta_2 \sin\Delta\phi, \tag{3.14}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \phi_1} \cdot \frac{\partial^2 \boldsymbol{q}_2}{\partial \theta_2 \partial \phi_2} = l_1 l_2 \sin\theta_1 \cos\theta_2 \cos\Delta\phi, \tag{3.15}$$

$$\frac{\partial \boldsymbol{q}_1}{\partial \phi_1} \cdot \frac{\partial^2 \boldsymbol{q}_2}{\partial \phi_2^2} = l_1 l_2 \sin\theta_1 \sin\theta_2 \sin\Delta\phi, \tag{3.16}$$

where $\Delta\phi = \phi_1 - \phi_2$.

3.1 The Lagrange equation for the standard spherical coordinate

Hence it follows that the equation (2.20) is of the following form:

$$A \begin{pmatrix} \ddot\theta_1 \\ \ddot\phi_1 \\ \ddot\theta_2 \\ \ddot\phi_2 \end{pmatrix} + B \begin{pmatrix} \dot\theta_1^2 \\ 2\dot\theta_1\dot\phi_1 \\ \dot\phi_1^2 \\ \dot\theta_2^2 \\ 2\dot\theta_2\dot\phi_2 \\ \dot\phi_2^2 \end{pmatrix} + \begin{pmatrix} \mu g l_1 \sin\theta_1 \\ 0 \\ g l_2 \sin\theta_2 \\ 0 \end{pmatrix} = \mathbf{0}, \tag{3.17}$$

where

$$A = \begin{pmatrix} \mu A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{3.18}$$

$$A_{ii} = l_i^2 \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\theta_i \end{pmatrix}, \tag{3.19}$$

$$A_{12} = l_1 l_2 \begin{pmatrix} \cos\theta_1 \cos\theta_2 \cos\Delta\phi + \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\theta_2 \sin\Delta\phi \\ -\sin\theta_1 \cos\theta_2 \sin\Delta\phi & \sin\theta_1 \sin\theta_2 \cos\Delta\phi \end{pmatrix}, \tag{3.20}$$

$$A_{21} = {}^{\mathrm{t}}A_{12}, \tag{3.21}$$

$$\tag{3.22}$$

and

$$B = \begin{pmatrix} \mu B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{3.23}$$

$$B_{ii} = l_i^2 \begin{pmatrix} 0 & 0 & -\sin\theta_i \cos\theta_i \\ 0 & \sin\theta_i \cos\theta_i & 0 \end{pmatrix}, \tag{3.24}$$

$$B_{12} = l_1 l_2 \begin{pmatrix} \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \cos\Delta\phi & \cos\theta_1 \cos\theta_2 \sin\Delta\phi & -\cos\theta_1 \sin\theta_2 \cos\Delta\phi \\ \sin\theta_1 \sin\theta_2 \sin\Delta\phi & \sin\theta_1 \cos\theta_2 \cos\Delta\phi & \sin\theta_1 \sin\theta_2 \sin\Delta\phi \end{pmatrix}, \tag{3.25}$$

$$B_{21} = l_1 l_2 \begin{pmatrix} \cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \cos\Delta\phi & -\cos\theta_1 \cos\theta_2 \sin\Delta\phi & -\sin\theta_1 \cos\theta_2 \cos\Delta\phi \\ -\sin\theta_1 \sin\theta_2 \sin\Delta\phi & \cos\theta_1 \sin\theta_2 \cos\Delta\phi & -\sin\theta_1 \sin\theta_2 \sin\Delta\phi \end{pmatrix}. \tag{3.26}$$

3.2 Reduction

Observe that the second rows of A and B are multiples of $\sin\theta_1$ and the fourth rows are multiples of $\sin\theta_2$. Moreover, the second column of A is a multiple of $\sin\theta_1$ and the fourth column is a multiple of $\sin\theta_2$. Therefore, we can rewrite the equation as follows:

$$\tilde{A} \begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\phi}_1 \sin\theta_1 \\ \ddot{\theta}_2 \\ \ddot{\phi}_2 \sin\theta_2 \end{pmatrix} + \tilde{B} \begin{pmatrix} \dot{\theta}_1^2 \\ 2\dot{\theta}_1\dot{\phi}_1 \\ \dot{\phi}_1^2 \\ \dot{\theta}_2^2 \\ 2\dot{\theta}_2\dot{\phi}_2 \\ \dot{\phi}_2^2 \end{pmatrix} + \begin{pmatrix} \mu g l_1 \sin\theta_1 \\ 0 \\ g l_2 \sin\theta_2 \\ 0 \end{pmatrix} = \mathbf{0}, \tag{3.27}$$

where

$$\tilde{A} = \begin{pmatrix} \mu l_1^2 I & Q \\ {}^{\mathrm{t}}Q & l_2^2 I \end{pmatrix}, \tag{3.28}$$

$$Q => l_1 l_2 \begin{pmatrix} \cos\theta_1 \cos\theta_2 \cos\Delta\phi + \sin\theta_1 \sin\theta_2 & \cos\theta_1 \sin\Delta\phi \\ -\cos\theta_2 \sin\Delta\phi & \cos\Delta\phi \end{pmatrix}, \tag{3.29}$$

$$\tilde{B} = \begin{pmatrix} \mu \tilde{B}_{11} & \tilde{B}_{12} \\ \tilde{B}_{21} & \tilde{B}_{22} \end{pmatrix}, \tag{3.30}$$

$$\tilde{B}_{ii} = l_i^2 \begin{pmatrix} 0 & 0 & -\sin\theta_i \cos\theta_i \\ 0 & \cos\theta_i & 0 \end{pmatrix} \tag{3.31}$$

$$\tilde{B}_{12} = l_1 l_2 \begin{pmatrix} \sin\theta_1 \cos\theta_2 - \cos\theta_1 \sin\theta_2 \cos\Delta\phi & \cos\theta_1 \cos\theta_2 \sin\Delta\phi & -\cos\theta_1 \sin\theta_2 \cos\Delta\phi \\ \sin\theta_2 \sin\Delta\phi & \cos\theta_2 \cos\Delta\phi & \sin\theta_2 \sin\Delta\phi \end{pmatrix}, \tag{3.32}$$

$$\tilde{B}_{21} = l_1 l_2 \begin{pmatrix} \cos\theta_1 \sin\theta_2 - \sin\theta_1 \cos\theta_2 \cos\Delta\phi & -\cos\theta_1 \cos\theta_2 \sin\Delta\phi & -\sin\theta_1 \cos\theta_2 \cos\Delta\phi \\ -\sin\theta_1 \sin\Delta\phi & \cos\theta_1 \cos\Delta\phi & -\sin\theta_1 \sin\Delta\phi \end{pmatrix}, \tag{3.33}$$

(3.34)

and I is the identity matrix.

3.3 Computing $\tilde{A}^{-1}$

We need $\tilde{A}^{-1}$ to numerically solve the equation (3.27). One may simply use some library to compute inverse matrices, but we also give a simple formula for $\tilde{A}^{-1}$. First observe that for square matrices Q, R of the same size,

$$\begin{pmatrix} \beta I & -Q \\ -R & \alpha I \end{pmatrix} \begin{pmatrix} \alpha I & Q \\ R & \beta I \end{pmatrix} = \begin{pmatrix} \alpha\beta I - QR & O \\ O & \alpha\beta I - RQ \end{pmatrix}. \tag{3.35}$$

Hence let

$$\hat{A} = \begin{pmatrix} l_2^2 I & -Q \\ -{}^{\mathrm{t}}Q & \mu l_1^2 I \end{pmatrix}, \tag{3.36}$$

then we have

$$\tilde{A} = \begin{pmatrix} \mu l_1^2 I & Q \\ {}^{\mathrm{t}}Q & l_2^2 I \end{pmatrix}, \qquad \hat{A}\tilde{A} = \begin{pmatrix} \mu l_1^2 l_2^2 I - Q\,{}^{\mathrm{t}}Q & O \\ O & \mu l_1^2 l_2^2 I - {}^{\mathrm{t}}QQ \end{pmatrix}. \tag{3.37}$$

Let

$$D_1 = \mu l_1^2 l_2^2 I - Q\,{}^{\mathrm{t}}Q, \qquad D_2 = \mu l_1^2 l_2^2 I - {}^{\mathrm{t}}QQ, \qquad \hat{D} = \begin{pmatrix} D_1^{-1} & O \\ O & D_2^{-1} \end{pmatrix}. \tag{3.38}$$

Then it follows that

$$\tilde{A}^{-1} = \hat{D}\hat{A}. \tag{3.39}$$

Namely, the Lagrange equation is equivalent to

$$\begin{pmatrix} \ddot{\theta}_1 \\ \ddot{\phi}_1 \sin\theta_1 \\ \ddot{\theta}_2 \\ \ddot{\phi}_2 \sin\theta_2 \end{pmatrix} = -\hat{D}\hat{A}\left(\tilde{B} \begin{pmatrix} \dot{\theta}_1^2 \\ 2\dot{\theta}_1\dot{\phi}_1 \\ \dot{\phi}_1^2 \\ \dot{\theta}_2^2 \\ 2\dot{\theta}_2\dot{\phi}_2 \\ \dot{\phi}_2^2 \end{pmatrix} + \begin{pmatrix} \mu g l_1 \sin\theta_1 \\ 0 \\ g l_2 \sin\theta_2 \\ 0 \end{pmatrix} \right). \tag{3.40}$$

3.4 Energy

The energy is given by:

$$\begin{aligned}
\frac{E}{m_2} &= \frac{1}{m_2}(T+V) = \frac{1}{2}\frac{m_1}{m_2}\|\dot{\boldsymbol{q}}_1\|^2 + \frac{1}{2}\|\dot{\boldsymbol{q}}_1 + \dot{\boldsymbol{q}}_2\|^2 + \mu g r_1 + g r_2 && (3.41) \\
&= \frac{1}{2}(\mu\|\dot{\boldsymbol{q}}_1\|^2 + \|\dot{\boldsymbol{q}}_2\|^2) + \dot{\boldsymbol{q}}_1 \cdot \dot{\boldsymbol{q}}_2 + \mu g r_1 + g r_2 \\
&= \frac{1}{2}(\mu l_1^2(\dot{\theta}_1^2 + \dot{\phi}_1^2 \sin^2\theta_1) + l_2^2(\dot{\theta}_2^2 + \dot{\phi}_2^2 \sin^2\theta_2)) + g(\mu r_1 + r_2) && (3.42) \\
&\quad + \dot{\theta}_1\dot{\theta}_2 \frac{\partial \boldsymbol{q}_1}{\partial\theta_1}\cdot\frac{\partial \boldsymbol{q}_2}{\partial\theta_2} + \dot{\theta}_1\dot{\phi}_2 \frac{\partial \boldsymbol{q}_1}{\partial\theta_1}\cdot\frac{\partial \boldsymbol{q}_2}{\partial\phi_2} + \dot{\phi}_1\dot{\theta}_2 \frac{\partial \boldsymbol{q}_1}{\partial\phi_1}\cdot\frac{\partial \boldsymbol{q}_2}{\partial\theta_2} + \dot{\phi}_1\dot{\phi}_2 \frac{\partial \boldsymbol{q}_1}{\partial\phi_1}\cdot\frac{\partial \boldsymbol{q}_2}{\partial\phi_2} \\
&= \frac{1}{2}(\mu l_1^2(\dot{\theta}_1^2 + \dot{\phi}_1^2 \sin^2\theta_1) + l_2^2(\dot{\theta}_2^2 + \dot{\phi}_2^2 \sin^2\theta_2)) - g(\mu l_1 \cos\theta_1 + l_2\cos\theta_2) \\
&\quad + l_1 l_2(\dot{\theta}_1\dot{\theta}_2(\cos\theta_1\cos\theta_2\cos\Delta\phi + \sin\theta_1\sin\theta_2) + \dot{\theta}_1\dot{\phi}_2\cos\theta_1\sin\theta_2\sin\Delta\phi \\
&\qquad - \dot{\phi}_1\dot{\theta}_2\sin\theta_1\cos\theta_2\sin\Delta\phi + \dot{\phi}_1\dot{\phi}_2\sin\theta_1\sin\theta_2\cos\Delta\phi).
\end{aligned}$$

4 Yet another coordinates

Since the standard spherical coordinate has singularities at the poles, numerical computation by the above equation (3.40) does not work well near the poles; i.e., when $\sin\theta_1$ or $\sin\theta_2$ is close to zero, the derivatives of ϕ_i can be big. Because of this, the numerical computation can diverge.

So we give another coordinate system, which can be applied near the poles. Consider

$$\boldsymbol{q}_i = \begin{pmatrix} p_i \\ q_i \\ r_i \end{pmatrix} = l_i \begin{pmatrix} \cos\vartheta_i \\ \sin\vartheta_i\cos\varphi_i \\ \sin\vartheta_i\sin\varphi_i \end{pmatrix}. \tag{4.1}$$

Then

$$\frac{\partial \boldsymbol{q}_i}{\partial \vartheta_i} = l_i \begin{pmatrix} -\sin\vartheta_i \\ \cos\vartheta_i \cos\varphi_i \\ \cos\vartheta_i \sin\varphi_i \end{pmatrix}, \qquad \frac{\partial \boldsymbol{q}_i}{\partial \varphi_i} = l_i \begin{pmatrix} 0 \\ -\sin\vartheta_i \sin\varphi_i \\ \sin\vartheta_i \cos\varphi_i \end{pmatrix}, \tag{4.2}$$

$$\frac{\partial^2 \boldsymbol{q}_i}{\partial \vartheta_i^2} = l_i \begin{pmatrix} -\cos\vartheta_i \\ -\sin\vartheta_i \cos\varphi_i \\ -\sin\vartheta_i \sin\varphi_i \end{pmatrix} = -\boldsymbol{q}_i, \qquad \frac{\partial^2 \boldsymbol{q}_i}{\partial \varphi_i^2} = l_i \begin{pmatrix} 0 \\ -\sin\vartheta_i \cos\varphi_i \\ -\sin\vartheta_i \sin\varphi_i \end{pmatrix} = -\begin{pmatrix} 0 \\ q_i \\ r_i \end{pmatrix}, \tag{4.3}$$

$$\frac{\partial^2 \boldsymbol{q}_i}{\partial \vartheta_i \partial \varphi_i} = l_i \begin{pmatrix} 0 \\ -\cos\vartheta_i \sin\varphi_i \\ \cos\vartheta_i \cos\varphi_i \end{pmatrix}. \tag{4.4}$$

Observe that

$$\begin{pmatrix} \cos\theta \\ \sin\theta\cos\phi \\ \sin\theta\sin\phi \end{pmatrix} = O \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ -\cos\theta \end{pmatrix}, \qquad O = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \tag{4.5}$$

and O is an orthogonal matrix. Since orthogonal matrices preserve the inner products, the matrices A and B in (2.20) are exactly the same as for (θ, ϕ) (just replace (θ, ϕ) by (ϑ, φ)).

Remark. The orthogonal matrix O is orientation-reversing; In order to respect orientations, one may use, for example,

$$\begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ \cos\theta \end{pmatrix}, \quad \begin{pmatrix} \sin\theta\sin\phi \\ \sin\theta\cos\phi \\ -\cos\theta \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \sin\theta\cos\phi \\ -\sin\theta\sin\phi \\ -\cos\theta \end{pmatrix} \tag{4.6}$$

as the standard coordinate.

Therefore the equation (2.20) becomes as follows:

$$A \begin{pmatrix} \ddot\vartheta_1 \\ \ddot\varphi_1 \\ \ddot\vartheta_2 \\ \ddot\varphi_2 \end{pmatrix} + B \begin{pmatrix} \dot\vartheta_1^2 \\ 2\dot\vartheta_1\dot\varphi_1 \\ \dot\varphi_1^2 \\ \dot\vartheta_2^2 \\ 2\dot\vartheta_2\dot\varphi_2 \\ \dot\varphi_2^2 \end{pmatrix} + \begin{pmatrix} \mu g l_1 \cos\vartheta_1 \sin\varphi_1 \\ \mu g l_1 \sin\vartheta_1 \cos\varphi_1 \\ g l_2 \cos\vartheta_2 \sin\varphi_2 \\ g l_2 \sin\vartheta_2 \cos\varphi_2 \end{pmatrix} = 0, \tag{4.7}$$

The matrices A and B are obtained from those in the previous section by replacing (θ, ϕ) by (ϑ, φ).

4.1 Reduction

We can similarly reduce the equation (4.7) by taking out the factor of $\sin\vartheta_i$ from $(2i)$-th column of A and $(2i)$-th row.

Similarly we use the same notation for the matrices in the previous section, replaced (θ, ϕ) by (ϑ, φ). Then (4.7) is equivalent to:

$$\tilde{A} \begin{pmatrix} \ddot\vartheta_1 \\ \ddot\varphi_1 \sin\vartheta_1 \\ \ddot\vartheta_2 \\ \ddot\varphi_2 \sin\vartheta_2 \end{pmatrix} + \tilde{B} \begin{pmatrix} \dot\vartheta_1^2 \\ 2\dot\vartheta_1\dot\varphi_1 \\ \dot\varphi_1^2 \\ \dot\vartheta_2^2 \\ 2\dot\vartheta_2\dot\varphi_2 \\ \dot\varphi_2^2 \end{pmatrix} + \begin{pmatrix} \mu g l_1 \cos\vartheta_1 \sin\varphi_1 \\ \mu g l_1 \cos\varphi_1 \\ g l_2 \cos\vartheta_2 \sin\varphi_2 \\ g l_2 \cos\varphi_2 \end{pmatrix} = 0. \tag{4.8}$$

4.2 Computing $\tilde{A}^{-1}$

The inverse matrix $\tilde{A}^{-1}$ is also equal to the one in Section 3.3 by replacing (θ, ϕ) by (ϑ, φ).

5 Using several spherical coordinates

As already stated, in order to avoid singularities of spherical coordinates, we need to prepare several coordinates and choose a coordinate for which the singularities are not close to the current state.

In this section, we discuss how to choose a nice coordinate depending on the current state. We also give a precise formula for coordinate changes.

5.1 Choosing local coordinates

First of all, the standard spherical coordinate has singularities at the north and south poles, i.e., when $\sin\theta_i = 0$ and $\boldsymbol{q}_i = {}^{\mathrm{t}}(0, 0, \pm l_i)$. Therefore, if $\sin\theta_i$ is small, we need to use another coordinate system.

If we use (ϑ_i, φ_i)-coordinate in the previous section, then it has singularities at $\sin\vartheta_i = 0$ and $\boldsymbol{q}_i = {}^{\mathrm{t}}(\pm 1, 0, 0)$. However, since we have been using same coordinate system for $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$, we have a problem if one of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ is close to north or south pole, and the other is close to ${}^{\mathrm{t}}(\pm 1, 0, 0)$; in other words, if $\sin\theta_1$ and $\sin\vartheta_2$ (or $\sin\vartheta_1$ and $\sin\theta_2$) are small.

To avoid such singularities, we just use rotational symmetry with respect to z-axis. For example, let $\boldsymbol{q}'_i = (q_i, p_i, r_i)$ (equivalently, let $(\theta', \phi' = \frac{\pi}{2} - \theta, \phi)$). Then by symmetry, the differential equation for $\boldsymbol{q}'_i$ is the same as that for $\boldsymbol{q}_i$.

Therefore, if $\boldsymbol{q}_1$ (resp. $\boldsymbol{q}_2$) is close to the north or south pole, and if $\boldsymbol{q}_2$ (resp. $\boldsymbol{q}_1$) is close to ${}^{\mathrm{t}}(1, 0, 0)$, then we can use (ϑ, φ)-coordinate for $\boldsymbol{q}'_i$ (say, (ϑ', φ')-coordinate) to avoid singularities.

As a conclusion, we can avoid singularities of local coordinates by the following algorithm:

1. If $|\sin\theta_1|$ and $|\sin\theta_2|$ are not so small (e.g., $|\sin\theta_i| \geq 0.5$), then use (θ, ϕ)-coordinate;
2. if $|\sin\theta_1|$ or $|\sin\theta_2|$ is small, then;
   (a) if $|\sin\theta_i|$ is not small and $|\sin\phi_i|$ is small, then use (ϑ', φ')-coordinate (the differential equation is exactly the same as for (ϑ, φ)-coordinate);
   (b) otherwise, use (ϑ, φ)-coordinate.

Remark.

- Even if we consider triple (or more but not too much) pendulum, we can avoid singularities by using rotational symmetry, i.e., consider another coordinate $(\theta', \phi') = (\theta + \alpha, \phi)$ by choosing an appropriate α, and then convert it to (ϑ, φ)-coordinate.
- We may also use different coordinates for $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ and write down the corresponding differential equation again. But note that in this case we need to calculate the inner products of derivatives of $\boldsymbol{q}_1$ and $\boldsymbol{q}_2$ for such a coordinate system to get a precise formula for the matrices A, B and so on.

5.2 Coordinate changes

Here we compute the coordinate changes between (θ, ϕ)-coordinate and (ϑ, φ)-coordinate.

In the following, we omit the suffix $i = 1, 2$ here.

First we solve the following equation:

$$\begin{pmatrix} p \\ q \\ r \end{pmatrix} = l \begin{pmatrix} \sin\theta\cos\phi \\ \sin\theta\sin\phi \\ -\cos\theta \end{pmatrix} = l \begin{pmatrix} \cos\vartheta \\ \sin\vartheta\cos\varphi \\ \sin\vartheta\sin\varphi \end{pmatrix}. \tag{5.1}$$

Normally one can use the function `atan2(y,x)` to get the argument of (x, y). Hence

$$\begin{aligned}
\varphi &= \operatorname{atan2}(r, q) = \operatorname{atan2}(-\cos\theta, \sin\theta\sin\phi), && (5.2)\\
\vartheta &= \operatorname{atan2}(r, p\sin\varphi) = \operatorname{atan2}(-\cos\theta, \sin\theta\cos\phi\sin\varphi) && (5.3)\\
&= \operatorname{atan2}(q, p\cos\varphi) = \operatorname{atan2}(\sin\theta\sin\phi, \sin\theta\cos\phi\cos\varphi)\\
&= \operatorname{atan2}\left(\sqrt{q^2+r^2}, p\right) = \operatorname{atan2}\left(\sqrt{\sin^2\theta\sin^2\phi+\cos^2\theta}, \sin\theta\cos\phi\right).
\end{aligned}$$

Note that some formulas do not work when $\sin\varphi$ or $\cos\varphi$ is zero.

Similarly,

$$\begin{aligned}
\phi &= \operatorname{atan2}(q, p) = \operatorname{atan2}(\sin\vartheta\cos\varphi, \cos\vartheta), && (5.4)\\
\theta &= \operatorname{atan2}(p, -r\cos\phi) = \operatorname{atan2}(\cos\vartheta, -\sin\vartheta\sin\varphi\cos\phi) && (5.5)\\
&= \operatorname{atan2}(q, -r\sin\phi) = \operatorname{atan2}(\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi\sin\phi)\\
&= \operatorname{atan2}\left(\sqrt{p^2+q^2}, -r\right) = \operatorname{atan2}\left(\sqrt{\cos^2\vartheta+\sin^2\vartheta\cos^2\varphi}, -\sin\vartheta\sin\varphi\right).
\end{aligned}$$

We also need to change the derivatives. Recall that

$$\dot{\boldsymbol{q}} = \begin{pmatrix}\dot{p}\\ \dot{q}\\ \dot{r}\end{pmatrix} = \dot{\theta}\frac{\partial\boldsymbol{q}}{\partial\theta} + \dot{\phi}\frac{\partial\boldsymbol{q}}{\partial\phi} = \dot{\vartheta}\frac{\partial\boldsymbol{q}}{\partial\vartheta} + \dot{\varphi}\frac{\partial\boldsymbol{q}}{\partial\varphi}. \tag{5.6}$$

Thus

$$\begin{aligned}
\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\theta} &= l^2\dot{\theta}, && (5.7)\\
\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\phi} &= l^2\dot{\phi}\sin^2\theta, && (5.8)\\
\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\vartheta} &= l^2\dot{\vartheta}, && (5.9)\\
\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\varphi} &= l^2\dot{\varphi}\sin^2\vartheta. && (5.10)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\dot{\theta} &= \frac{1}{l^2}\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\theta} = \frac{1}{l^2}\left(\dot{\vartheta}\frac{\partial\boldsymbol{q}}{\partial\vartheta} + \dot{\varphi}\frac{\partial\boldsymbol{q}}{\partial\varphi}\right)\cdot\frac{\partial\boldsymbol{q}}{\partial\theta}, && (5.11)\\
\dot{\phi} &= \frac{1}{l^2}\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\phi} = \frac{1}{l^2\sin^2\theta}\left(\dot{\vartheta}\frac{\partial\boldsymbol{q}}{\partial\vartheta} + \dot{\varphi}\frac{\partial\boldsymbol{q}}{\partial\varphi}\right)\cdot\frac{\partial\boldsymbol{q}}{\partial\phi}. && (5.12)
\end{aligned}$$

It follows that $(\dot{\theta}, \dot{\phi})$ can be computed from $(\dot{\vartheta}, \dot{\varphi})$ and (θ, ϕ). Note that we may assume $l = 1$ for conversion between (θ, ϕ) and (ϑ, φ).

Similarly, $(\dot{\vartheta}, \dot{\varphi})$ can be computed by the following:

$$\begin{aligned}
\dot{\vartheta} &= \frac{1}{l^2}\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\vartheta} = \frac{1}{l^2}\left(\dot{\theta}\frac{\partial\boldsymbol{q}}{\partial\theta} + \dot{\phi}\frac{\partial\boldsymbol{q}}{\partial\phi}\right)\cdot\frac{\partial\boldsymbol{q}}{\partial\vartheta}, && (5.13)\\
\dot{\varphi} &= \frac{1}{l^2}\dot{\boldsymbol{q}}\cdot\frac{\partial\boldsymbol{q}}{\partial\varphi} = \frac{1}{l^2\sin^2\vartheta}\left(\dot{\theta}\frac{\partial\boldsymbol{q}}{\partial\theta} + \dot{\phi}\frac{\partial\boldsymbol{q}}{\partial\phi}\right)\cdot\frac{\partial\boldsymbol{q}}{\partial\varphi}. && (5.14)
\end{aligned}$$