STRAIGHTENING MAPS FOR POLYNOMIALS WITH ONE BOUNDED CRITICAL ORBIT

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Abstract. Let f_* be a polynomial of degree d (≥ 3) with all critical points escaping to infinity except one periodic critical point with multiplicity $d_* - 1$. We show that f_* can be tuned by an arbitrary monic centered polynomial Q of degree d_* with connected Julia set. This generalizes a celebrate result on the existence of the baby Mandelbrot sets by Douady–Hubbard. In particular, if $d = 3$, we prove that $\mathcal{E}(f_*)$ is homeomorphic to the Mandelbrot set, where $\mathcal{E}(f_*)$ is the set of all the cubic polynomials that are externally conjugate to f_*.

1. Introduction

The terminology of tuning was first introduced by Douady–Hubbard [7] to explain the existence of the baby Mandelbrot sets. Recall that the Mandelbrot set $\mathcal{M}$ consists of all the parameters $c \in \mathbb{C}$ for which the quadratic polynomial $f_c(z) = z^2 + c$ has a connected Julia set. The following result was proved by Douady–Hubbard [7] (see also [8]).

Theorem 1.1 (Tuning). *Let $c_0 \in \mathcal{M}$ be a complex number so that the critical point 0 has period p_0 for f_{c_0}. Then there exists a homeomorphism $\tau\colon \mathcal{M} \to \mathcal{M}$ onto its image with the following properties.*

- $\tau(0) = c_0$;
- *for every $c \in \mathcal{M}$, $f_{\tau(c)}^{p_0}$ has a quadratic-like restriction which is hybrid equivalent to f_c.*

Roughly speaking, tuning is a procedure to replace the closure of each bounded Fatou component of f_{c_0} by the filled Julia set of another polynomial.

The first author and Kiwi [10] defined a natural analogy of the map $\chi = \tau^{-1}$ for higher degree polynomials with connected Julia sets, which is called the *straightening map*. Given a post-critically finite hyperbolic polynomial f_0 of degree $d \geq 2$ and fix an *internal angle system*, they defined a straightening map χ from a certain subset $\mathcal{R}(f_0)$ of $\mathcal{C}_d$ into a space which consists of generalized polynomials with fiberwise connected Julia sets, where $\mathcal{C}_d$ is the set of all the monic centered polynomials of degree d with connected Julia sets. They also proved that straightening maps are always injective. In [19], Shen and the second author proved that the straightening map is a bijection when f_0 is primitive.

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Roughly speaking, straightening map is defined for any family of renormalizable maps (in the sense of polynomial-like mappings) by virtue of the Straightening Theorem by Douady–Hubbard [7]. In this paper, we consider straightening maps for polynomials that all but one of the critical orbit escape. Let f_* be a polynomial with exactly one non-escaping critical point c_* (counted without multiplicity) and at least one unbounded critical orbit. Branner–Hubbard [4] showed that the Julia set $J(f_*)$ of f_* is totally disconnected with zero area if the non-escaping critical point c_* is non-degenerate and the critical tableau is aperiodic. While in this paper, we assume that the non-escaping critical point c_* is periodic and has multiplicity d_*-1. Let $\mathcal{E}(f_*)$ be the set of all the monic centered polynomials that are externally conjugate to f_*, i.e., holomorphically conjugate to f_* on the respective basins of infinity. Fix an internal angle system (see Section 3 for definition), we show that there exists a well defined map χ from $\mathcal{E}(f_*)$ into a space $\mathcal{C}_{d_*}$ which consists of all the monic centered polynomials of degree d_* with connected Julia sets. The aim of this paper is to show the following.

Main Theorem. *Let f_* be a polynomial of degree $d(\geq 3)$ such that all critical points escape to infinity except one periodic critical point c_* with multiplicity $d_* - 1 < d - 1$. Assume that the component of the filled Julia set containing c_* is homeomorphic to a closed disk. Fix an internal angle system for f_* and let $\chi\colon \mathcal{E}(f_*) \to \mathcal{C}_{d_*}$ be the straightening map with respect to the internal angle system. Then χ is a bijection and $\mathcal{E}(f_*)$ is compact and connected.*

Moreover, there exists a positive integer $\ell_* > 0$ such that f^{ℓ_*} has a polynomial-like restriction which is hybrid equivalent to $\chi(f)$ for all $f \in \mathcal{E}(f_*)$.

It was proved by the first author [9] that the straightening map is discontinuous in general. Therefore, the bijectivity is usually optimal for straightening maps.

Nonetheless, straightening maps for quadratic-like families are always continuous [7]. Therefore, for $d = 3$ we have the following.

Theorem 1.2. *Let f_* be a monic centered cubic polynomial with exactly one escaping critical point. Assume that f_* has a periodic critical point c_* and the component of the filled Julia set containing c_* is homeomorphic to a closed disk. Then there exists a homeomorphism χ from $\mathcal{E}(f_*)$ onto the Mandelbrot set $\mathcal{M}$.*

Note that the assumption that the component of the filled Julia set in Main Theorem is equivalent that the component coincides with the closure of the immediate basin of c_*, and also that the period of c_* is equal to the period of the component. This assumption is not essential indeed; we replace the non-singleton connected components of the filled Julia set by tuning, not the closures of the bounded Fatou components, and it does not matter what the dynamics is on them.

The proof of the Main Theorem is based on quasiconformal surgery and pullback argument.

The paper is organized as follows. In Section 2, we recall the definitions of equipotentials and external rays. The notions of internal angle systems, external markings and straightening maps are defined in Section 3. We also introduce the notion of *weak pseudo-conjugacy* and prove that two maps are externally equivalent if and only if they are weakly pseudo-conjugate up to arbitrary depth (Lemma 3.3). This leads to a result that every $f \in \mathcal{E}(f_*)$ is renormalizable, which enables us to define the straightening map. In Section 4, we apply quasiconformal surgery to prove the surjectivity of the straightening map. More precisely, we prove that f_*

can be tuned by any polynomial in $\mathcal{C}_{d_*}$, where f_* is as in the Main Theorem. In Section 5, we use a standard pullback argument to show that the straightening map is injective. Combine with the results in Section 4, we complete the proof the Main Theorem.

2. Equipotentials and external rays

Let $\mathcal{P}_d$ denote the set of all monic centered polynomials of degree $d \geq 2$. For each $f \in \mathcal{P}_d$, let $\mathrm{Crit}(f)$ denote the set of all critical points of f. The *Green function* G_f for f is defined as follows:

$$G_f(z) = \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ = \max(\log, 0)$. It is well-known that G_f is a subharmonic function and $G_f(z) > 0$ exactly on the *basin of infinity* of f:

$$D_f(\infty) := \{z \mid f^n(z) \to \infty\} = \{z \mid G_f(z) > 0\}.$$

We say $z \in D_f(\infty)$ is a *critical point* of G_f if $\nabla G_f(z) = 0$. Let $\mathrm{Crit}(G_f)$ denote the set of all critical points of G_f. Then $\mathrm{Crit}(G_f)$ coincides with the union of the backward orbits of the escaping critical points of f:

$$\mathrm{Crit}(G_f) = \{z \in D_f(\infty) \mid \exists k \geq 0 \text{ such that } f^k(z) \in \mathrm{Crit}(f)\}. \tag{2.1}$$

For $h > 0$, the *equipotential curve of height h of f* is defined as

$$E_f(h) = \{z \mid G_f(z) = h\}.$$

Let us denote $\Omega_f(h)$ the unbounded component of $\mathbb{C} \setminus E_f(h)$. In other words,

$$\Omega_f(h) = \{z \mid G(z) > h\}.$$

The complement $K(f) = \mathbb{C} \setminus D_f(\infty)$ of the basin of infinity is called the *filled Julia set* of f. The common boundary of $K(f)$ and $D_f(\infty)$ is called the *Julia set* of f and denoted by $J(f)$.

Set $R_f := \max\{G_f(v) \mid v \text{ is a critical value of } f\}$. The *Böttcher map for f* is a unique conformal map

$$\phi_f \colon \{z \in \mathbb{C} \mid G_f(z) > R_f\} \to \{z \mid |z| > e^{R_f}\}$$

such that $\phi_f(z)/z \to 1$ as $z \to \infty$ and

$$\phi_f \circ f(z) = (\phi_f(z))^d$$

on $\{z \in \mathbb{C} \mid G_f(z) > R_f\}$.

The gradient flow $\frac{\nabla G_f}{\|G_f\|^2}$ for the Green function G_f on $D_f(\infty)$ has critical points precisely at those of G_f. We say an arc $\mathcal{R}$ is *an external ray* for f if it is a trajectory of $\frac{\nabla G_f}{\|G_f\|^2}$ that does not meet any critical points of G_f. Each external ray $\mathcal{R}$ has a well defined *external angle* $\theta \in \mathbb{R}/\mathbb{Z}$, namely, $2\pi\theta$ is the asymptotic argument of the trajectory $\mathcal{R}$ at the infinity. An external ray of angle θ can be parameterized by a smooth function $\mathbb{R}_+ \to \mathbb{C}$, $s \mapsto \psi(s)$ such that $G_f(\psi(s)) = s$ for all $s \in \mathbb{R}_+$ and $\psi(s) = \phi_f^{-1}(e^{s+i2\pi\theta})$ for s large.

For each integer $d \geq 2$, let $\mathcal{P}_d^+$ denote the set of all monic centered polynomials f of degree d with $R_f > 0$. In other words, $\mathcal{P}_d^+$ is a set of $f \in \mathcal{P}_d$ with at least one escaping critical point.

3. Weak pseudo-conjugacies and Renormalizations

Throughout the paper, we assume that $f_* \in \mathcal{P}_d^+$ is a polynomial of degree $d \geq 3$ with all critical points escaping to infinity except one periodic critical point c_*. Let us denote $d_* - 1$ the multiplicity of c_* respectively.

In this section, we define a weakly pseudo-topological class $\mathcal{E}(f_*)$ of f_* which turns out to be a compact subset of $\mathcal{P}_d$. We show that the maps in $\mathcal{E}(f_*)$ can be characterized in the following: $f \in \mathcal{E}(f_*)$ if and only if $\phi_f^{-1} \circ \phi_{f_*}$ extends to a global conformal conjugacy $\Phi_f \colon \mathbb{C} \setminus K(f_*) \to \mathbb{C} \setminus K(f)$ between $f_*|_{\mathbb{C}\setminus K(f_*)}$ and $f|_{\mathbb{C}\setminus K(f)}$. This characterization implies that all the maps in $\mathcal{E}(f_*)$ are renormalizable. This gives a nice correspondence between the dynamics of maps in $\mathcal{E}(f_*)$ and that of maps in $\mathcal{C}_{d_*}$, which enables us to define a straightening map from $\mathcal{E}(f_*)$ into $\mathcal{C}_{d_*}$.

3.1. **External conjugacy, puzzles and weak pseudo-conjugacy.** We introduce the notion of puzzles and weak pseudo-conjugacy, and prove that weak pseudo-conjugacy is in fact equivalent to external conjugacy.

First recall the definition of external conjugacy.

Definition 3.1. We say two maps $f, \tilde{f} \in \mathcal{P}_d$ are *externally conjugate* if there exists a conformal map $\Phi \colon D_f(\infty) \to D_{\tilde{f}}(\infty)$ such that

$$\Phi \circ f = \tilde{f} \circ \Phi$$

and $\Phi(z) = \phi_{\tilde{f}}^{-1} \circ \phi_f^{-1}$ near infinity.

The conformal map Φ is called the *external conjugacy* between f and $\tilde{f}$.

If f and $\tilde{f}$ have connected Julia sets, then they are externally conjugate. In general, external equivalence means that all the escaping critical points of f and $\tilde{f}$ coincide in terms of (the extension of) the Böttcher coordinate.

Now we construct puzzles and introduce the notion of weak pseudo-conjugacy. Fix $r_0 > 0$. Let $f \in \mathcal{P}_d^+$ and for every $n \in \mathbb{N}$, we denote by $\Gamma_n^f = E_f(r_0/d^n)$ the equipotential curve of height r_0/d^n. An *f-puzzle piece of depth n* is a bounded connected component of $\mathbb{C} \setminus \Gamma_n^f$.

To simplify the argument, we always assume that Γ_n^f consists only of simple closed curves. In fact, if this is not the case, it follows that there exists some $z \in \mathrm{Crit}(G_f)$ such that $G_f(z) = r_0 d^k$ for some k. Such r_0 is discrete in $\mathbb{R}_+ = \{r \in \mathbb{R} \mid r > 0\}$ by (2.1), so we need only replace r_0 with $r_0 + \varepsilon$ for small ε.

Note that every puzzle piece is an open quasidisk. It is fairly easy to see the following Markov property holds: For any two distinct f-puzzle pieces, either they have disjoint closures or one is compactly contained in the other.

Let f and $\tilde{f}$ be two polynomials in $\mathcal{P}_d^+$. For $n \in \mathbb{N}$, we say f and $\tilde{f}$ *are weakly pseudo-conjugate up to depth n* if there exists a homeomorphism $H \colon \mathbb{C} \to \mathbb{C}$ with the following properties:

- $H = \phi_{\tilde{f}}^{-1} \circ \phi_f$ in a neighborhood of infinity, where ϕ_f and $\phi_{\tilde{f}}$ are Böttcher map for f and $\tilde{f}$ respectively;
- $H \circ f = \tilde{f} \circ H$ outside the union of all the f-puzzle pieces of depth n.

Such a homeomorphism H is called a *weak pseudo-conjugacy up to depth n* between f and $\tilde{f}$. Let us mention that the method of weak pseudo-conjugacy was successfully used in [2] to study the combinatorial rigidity for non-renormalizable unicritical polynomials.

We say f and $\tilde{f}$ *are weakly pseudo-conjugate* if f and $\tilde{f}$ are weakly pseudo-conjugate up to an arbitrary depth. Let $\mathcal{E}(f_*)$ denote the set of all the monic centered polynomials which are weakly pseudo-conjugate f_*. We call $\mathcal{E}(f_*)$ the *weakly pseudo-topological class* of f_*.

Lemma 3.2. *Assume that f and $\tilde{f}$ are weakly pseudo-conjugate up to depth n for some $n \in \mathbb{N}$. There exists a unique conformal map*

$$\Phi_n \colon \Omega_f(r_0/d^n) \to \Omega_{\tilde{f}}(r_0/d^n)$$

such that $H_n = \Phi_n$ on $\Omega_f(r_0/d^n)$ for any weak pseudo-conjugacy H_n up to depth n between f and $\tilde{f}$.

Proof. The uniqueness follows once the existence is proved since all the weak pseudo-conjugacies up to depth n between f and $\tilde{f}$ coincide in a neighborhood of infinity.

Let H_n be an arbitrary weak pseudo-conjugacy up to depth n between f and $\tilde{f}$. It suffices to show that H_n is holomorphic in $\Omega_f(r_0/d^n)$. First note that there exists a neighborhood U of infinity such that H_n is holomorphic in U since $H_n = \phi_{\tilde{f}}^{-1} \circ \phi_f$ near infinity. Set $E_n := \bigcup_{j \in \mathbb{N}} \tilde{f}^{-j}(\mathrm{Crit}(\tilde{f})) \cap \Omega_{\tilde{f}}(r_0/d^n)$. Clearly, $\# E_n < \infty$ and hence E_n is a discrete set. For any $w \in \Omega_f(r_0/d^n) \setminus H_n^{-1}(E_n)$, there exists a smallest $k > 0$ such that $f^k(w) \in U$. As $H_n(w) \notin E_n$, there exists a neighborhood $W \ni H_n(w)$ such that $g = \tilde{f}^k|_W$ is conformal on W. Note that $H_n \circ f^k = \tilde{f}^k \circ H_n$ holds on $\Omega_f(r_0/d^n) \setminus H_n^{-1}(E_n)$. Thus $H_n = g^{-1} \circ H_n \circ f^k$ on $H_n^{-1}(W) \ni w$, which implies H_n is holomorphic at w. Since $H_n^{-1}(E_n)$ is discrete, H_n is holomorphic everywhere on $\Omega_f(r_0/d^n)$. $\square$

Lemma 3.3. *Two polynomials f and $\tilde{f}$ are weakly pseudo-conjugate if and only if they are externally conjugate.*

Proof. For convenience, let us denote $W_n = \Omega_f(r_0/d^n)$ and $\widetilde{W}_n = \Omega_{\tilde{f}}(r_0/d^n)$ respectively for all $n \in \mathbb{N}$.

Assume that f and $\tilde{f}$ are weakly pseudo-conjugate. For any $n \in \mathbb{N}$, by Lemma 3.2, there exists a unique conformal map $\Phi_n \colon W_n \to \widetilde{W}_n$ such that $\tilde{f} \circ \Phi_n = \Phi_n \circ f$ on W_n and $\Phi_n = \phi_{\tilde{f}}^{-1} \circ \phi_f$ in a neighborhood of infinity. Since $W_{n+1} \supset W_n$ and Φ_{n+1} coincides with Φ_n near the infinity, we conclude that

$$\Phi_{n+1}|_{W_n} = \Phi_n \tag{3.1}$$

for all $n \in \mathbb{N}$. Now we define $\Phi \colon D_f(\infty) \to D_{\tilde{f}}(\infty)$ as $\Phi = \Phi_n$ on W_n for all $n \in \mathbb{N}$. It follows that Φ is well defined from equation (3.1) and the fact that $\mathbb{C} \setminus K(f) = \bigcup\limits_{n \in \mathbb{N}} W_n$. One can easily check Φ is the desired conformal map.

Now assume that f and $\tilde{f}$ are externally conjugate. Let Φ be the conformal map from $D_f(\infty)$ onto $D_{\tilde{f}}(\infty)$ such that $\tilde{f} \circ \Phi = \Phi \circ f$ on $D_f(\infty)$ and $\Phi = \phi_{\tilde{f}}^{-1} \circ \phi_f$ in a neighborhood of infinity.

Claim 1. *$G_{\tilde{f}} \circ \Phi = G_f$ on $D_f(\infty)$.*

Since $\Phi = \phi_{\tilde{f}}^{-1} \circ \phi_f$ near infinity, there exists $R > 0$ sufficiently large such that Φ maps $\Omega_f(R)$ onto $\Omega_{\tilde{f}}(R)$ and

$$G_{\tilde{f}}(\Phi(z)) = G_f(z) \tag{3.2}$$

for all $z \in \Omega_f(R)$. For every $w \in D_f(\infty)$, there exists a smallest positive integer k such that $f^k(w) \in \Omega_f(R)$ since w lies in the basin of infinity $D_f(\infty)$. It follows from equation (3.2) that

$$G_{\tilde{f}}(\tilde{f}^k \circ \Phi(w)) = G_{\tilde{f}}(\Phi \circ f^k(w)) = G_f(f^k(w)) = d^k G_f(w).$$

This implies $G_{\tilde{f}}(\Phi(w)) = d^{-k} G_{\tilde{f}}(\tilde{f}^k \circ \Phi(w)) = G_f(w)$.

We conclude from Claim 1 that Φ maps $\Omega_f(s)$ onto $\Omega_{\tilde{f}}(s)$ for all $s > 0$, hence that $\Phi(W_n) = \widetilde{W}_n$ for all $n \in \mathbb{N}$.

It remains to show that for any $n \in \mathbb{N}$, there exists a homeomorphism $H_n \colon \mathbb{C} \to \mathbb{C}$ such that $H_n = \Phi$ on W_n. Let D be an arbitrary component of $\mathbb{C} \setminus \overline{W_n}$. Note that both ∂W_n and $\partial \widetilde{W}_n$ consist of finitely many quasicircles. Thus $\Phi \colon W_n \to \widetilde{W}_n$ extends to a quasisymmetric map from ∂D onto $\Phi(\partial D)$. Let D' denote the bounded component $\mathbb{C} \setminus \Phi(\partial D)$. By Beurling-Ahlfors extension, we can extend $\Phi \colon \partial D \to \partial D'$ to a quasiconformal map (and hence a homeomorphism) $h_D \colon D \to D'$. Finally, we define $H_n = \Phi$ on $\overline{W}_n$ and $H_n = h_D$ on D, where D runs over all the component of $\mathbb{C} \setminus \overline{W_n}$. □

Let $f \in \mathcal{E}(f_*)$ and let $\Phi_f \colon D_{f_*}(\infty) \to D_f(\infty)$ be the external conjugacy. We call Φ_f *the relative Böttcher map between* f_* *and* f.

Lemma 3.4. *If f and $\tilde{f}$ are weakly pseudo-conjugate up to depth n for some $n \in \mathbb{N}$, then there exists a quasiconformal weak pseudo-conjugacy H_n between f and $\tilde{f}$ up to depth n.*

Proof. By Lemma 3.2, there exists a conformal map $\Phi_n \colon \Omega_f(r_0/d^n) \to \Omega_{\tilde{f}}(r_0/d^n)$ such that $\tilde{f} \circ \Phi_n = \Phi_n \circ f$ on $\Omega_f(r_0/d^n)$. We finish the proof by declaring that Φ_n can extend to a quasiconformal homeomorphism $H_n \colon \mathbb{C} \to \mathbb{C}$. This follows by the same method as in the proof of Lemma 3.3. □

Theorem 3.5. *$\mathcal{E}(f_*)$ is a compact set.*

Proof. First of all, $\mathcal{E}(f_*)$ is contained in a compact set. Indeed, there exists a constant $M > 0$ such that $G_f(c) \le M$ for each $f \in \mathcal{E}(f_*)$ and $c \in \mathrm{Crit}(f)$. By [3, Proposition 3.6], $\mathcal{E}(f_*)$ lies in a compact set. Therefore, it suffices to show $\mathcal{E}(f_*)$ is a closed set.

Let $\{f_k\}$ be an arbitrary sequences in $\mathcal{E}(f_*)$ such that $f_k \to f \in \mathcal{P}_d$, we show that $f \in \mathcal{E}(f_*)$. For each $k \in \mathbb{N}$ and $n \in \mathbb{N}_+$, set $X_{k,n} := \{z \mid G_{f_k}(z) > 1/n\}$ and $X_n := \{z \mid G_{f_*}(z) > 1/n\}$. Note that for each $n \in \mathbb{N}$, $\overline{X_{k,n}} \subset D_f(\infty)$ for k sufficiently large. Thus $\Phi_{f_k}|_{X_n}(X_n) \subset D_f(\infty)$ for k large, where Φ_{f_k} is the relative Böttcher map between f_* and f_k. It follows from Montel's Theorem that $\{\Phi_{f_k}|_{X_n}\}$ is pre-compact. Without loss of generality, we assume that for each $n \in \mathbb{N}$, $\Phi_{f_k}|_{X_n}$ converges to a holomorphic map $\psi_n \colon X_n \to D_f(\infty)$. Since $\Phi'_{f_k}(\infty) = 1$ for all $k \in \mathbb{N}$, we conclude $\psi'_n(\infty) = 1 \ne 0$ and hence ψ_n is a univalent map (holomorphic injection). Clearly $f \circ \psi_n = \psi_n \circ f_*$ on X_n. We must have $\psi_n = \phi_f^{-1} \circ \phi_{f_*}$ near infinity as $\psi'_n(\infty) = 1$. Thus $\psi_{n'} = \psi_n$ on X_n for all $n' > n$. Let us define $\psi \colon D_{f_*}(\infty) \to D_f(\infty)$ so that $\psi|_{X_n} = \psi_n$ for all $n \in \mathbb{N}$. Obviously, ψ is a univalent map and $f \circ \psi = \psi \circ f_*$. Since $\psi = \phi_f^{-1} \circ \phi_{f_*} \colon \{z \mid G_{f_*} > L\} \to \{z \mid G_f > L\}$ is a conformal map for L large and $f \circ \psi = \psi \circ f_*$ on $D_{f_*}(\infty)$, we conclude that $\psi \colon D_{f_*}(\infty) \to D_f(\infty)$ is also a conformal map. Therefore ψ is a relative Böttcher map between f_* and f. By Lemma 3.3, $f \in \mathcal{E}(f_*)$. □

3.2. **Principal nests and Renormalizations.** Recall that a *polynomial-like map* is a holomorphic proper map between two topological disks such that the domain of the definition is compactly contained in its image. See [7, 13] for more details.

Two polynomial-like maps $g_i \colon U_i \to V_i$ $(i = 1, 2)$ are said to be *quasiconformally equivalent* if there exists a quasiconformal map φ from a neighborhood of $K(g_1)$ onto a neighborhood of $K(g_2)$ such that $\varphi \circ g_1 = g_2 \circ \varphi$ near $K(g_1)$. We say g_1 and g_2 are *hybrid equivalent* if they are quasiconformally equivalent and the quasiconformal map φ above can be taken such that $\bar{\partial}\varphi = 0$ almost everywhere on $K(g_1)$.

Definition 3.6. Let $f \in \mathcal{P}_d$ and fix a critical point $c \in \mathrm{Crit}(f)$. We say f is *renormalizable at c* if there exist topological disks $U \Subset V$ containing c and a positive integer p which satisfy the following:

(1) $f^p|_U \colon U \to V$ is a polynomial-like map with connected Julia set;
(2) for each $c' \in \mathrm{Crit}(f)$, there exists at most one $0 \leq j < p$ such that $c' \in f^j(U)$;
(3) $p > 1$ or $\mathrm{Crit}(f) \not\subset U$.

Such a polynomial-like map in (1) is called a *pre-renormalization* of f. While there are several choices for U and V, such a pre-renormalization is unique up to hybrid equivalence.

Let Y_*^0 be the f_*-puzzle piece of depth 0 which contains the critical point c_*. We define the *principal nest for f_** as following:

$$Y_*^0 = I_0 \Supset I_1 \Supset \cdots \Supset I_k \Supset \cdots,$$

where I_{k+1} is the component of the first return domain to I_k which contains the critical point c_*. Let $m(k)$ denote the depth of the puzzle piece I_k for all $k \in \mathbb{N}$ and let ℓ_k denote the first return time of I_{k+1} to I_k. Then $\{\ell_k\}$ is a non-decreasing sequence of positive integers. Let p_* be the period of c_* for f_*. Since ℓ_k divides p_*, the sequence is eventually constant and we set $\ell_* := \sup_{k \in \mathbb{N}} \ell_k$, which also divides p_*. Moreover, if we assume the component of $K(f_*)$ containing c_* is homeomorphic to a closed disk as in the Main Theorem, then we have $\ell_* = p_*$.

There exists $\kappa_* \geq 0$ such that

$$f_*^{\ell_*}|_{I_{k+1}} \colon I_{k+1} \to I_k$$

is a polynomial-like map with connected Julia set for all $k \geq \kappa_*$. Such an ℓ_* is called the renormalization period of f_*.

From now on, we assume that the renormalization period of f_* is ℓ_* throughout the paper.

Lemma 3.7. *Assume that $f \in \mathcal{E}(f_*)$. Then there exists a critical point $c \in \mathrm{Crit}(f)$ such that f is renormalizable at c. More precisely,*

$$f^{\ell_*}|_{H_{k+1}(I_{k+1})} \colon H_{k+1}(I_{k+1}) \to H_{k+1}(I_k) = H_k(I_k)$$

is a polynomial-like map with connected Julia set for all $k \geq \kappa_*$, where H_n is a weak pseudo-conjugacy between f_* and f up to depth $m(n)$ for all $n \in \mathbb{N}$.

Proof. For convenience, let us denote $I'_k = H_k(I_k)$ for all $k \in \mathbb{N}$. By Lemma 3.3, there exists a conformal map $\Phi \colon D_f(\infty) \to D_{\tilde{f}}(\infty)$ such that $H_k = \Phi$ on $\partial I'_k$ for all $k \in \mathbb{N}$. We thus conclude that $H_k(I_{k'}) = H_{k'}(I_{k'}) = I'_{k'}$, and hence $I'_k = H_k(I_k) \Supset H_k(I_{k'}) = I'_{k'}$ for all $k' \geq k$.

Claim. *For every $k \in \mathbb{N}$, $f^{\ell_k} \colon I'_{k+1} \to I'_k$ is a holomorphic proper map.*

It suffices to show $f^{\ell_k}\colon \partial I'_{k+1} \to \partial I'_k$ is a covering (by virtue of the Argument Principle). Note that $f_*^{\ell_k}\colon \partial I_{k+1} \to \partial I_k$ is a covering. Indeed, $f_*^{\ell_k}\colon I_{k+1} \to I_k$ is a holomorphic proper map since this is a first return map. Thus the corresponding boundary map is a covering. Since Φ conjugates $f_*^{\ell_k}|_{\partial I_{k+1}}$ to $f^{\ell_k}|_{\partial I'_{k+1}}$, we conclude

$$f^{\ell_k}\colon \partial I'_{k+1} \to f^{\ell_k}(\partial I'_{k+1}) = \Phi(\partial I_k) = \partial I'_k$$

is also a covering.

Note that $\ell_k \equiv \ell_*$ when $k \geq \kappa_*$ by the definition of κ_*. It follows from the Claim above that $f^{\ell_*}|_{I'_{k+1}}\colon I'_{k+1} \to I'_k$ is a polynomial-like map for all $k \geq \kappa_*$. Since each $\overline{I'_n}$ is connected compact set and $I'_n \Supset I'_{n+1}$, the small filled Julia set $K(f^{\ell_*}|_{I'_{k+1}}) = \bigcap_{n\geq k+1} I'_n$ is a connected compact set. In particular, $f^{\ell_*}|_{I'_{\kappa_*+1}}\colon I'_{\kappa_*+1} \to I'_{\kappa_*}$ is a pre-renormalization for f. To see this, it remains to check there exists a critical point $c \in \mathrm{Crit}(f)$ such that $c \in I'_{\kappa_*+1}$. Indeed, $f_*\colon \partial I_{\kappa_*+1} \to f_*(\partial I_{\kappa_*+1})$ is a covering of degree $d_* \geq 2$ since I_{κ_*+1} contains only one critical point c_* of f_*. Since Φ conjugates $f_*|_{\partial I_{\kappa_*+1}}$ to $f|_{\partial I'_{\kappa_*+1}}$, we conclude

$$f\colon \partial I'_{\kappa_*+1} \to f(\partial I'_{\kappa_*+1}) = \Phi(f_*(\partial I_{\kappa_*+1}))$$

is also a covering of degree $d_* \geq 2$. Let B be the bounded component of $\mathbb{C} \setminus \Phi(f_*(\partial I_{\kappa_*+1}))$. By the argument principle, $f\colon I'_{\kappa_*+1} \to B$ is a branched covering of degree $d_* \geq 2$. It follows that there exists a critical point of f lying I'_{κ_*+1} from the Riemann-Hurwitz formula and the fact that I'_{κ_*+1} and B are simply connected domains. ☐

3.3. **External markings and Straightening.** For every polynomial $f \in \mathcal{E}(f_*)$, by Lemma 3.7,

$$f^{\ell_*}|_{H_{\kappa_*+1}(I_{\kappa_*+1})}\colon H_{\kappa_*+1}(I_{\kappa_*+1}) \to H_{\kappa_*+1}(I_{\kappa_*})$$

gives a pre-renormalization for f, where H_{κ_*+1} is a weak pseudo-conjugacy between f_* and f up to depth $m(\kappa_*+1)$. We call such a pre-renormalization for f *the canonical pre-renormalization for* f, and denote it by T_f. By Douady–Hubbard straightening theorem [7], this canonical pre-renormalization T_f is hybrid equivalent to a polynomial P with a connected Julia set. The polynomial P is uniquely determined up to affine conjugacy. For monic centered quadratic polynomials, each affine conjugacy class is a singleton. However in general, there can be several monic centered polynomials which are affine conjugate, hence we introduce the notion of *external marking* so that the polynomial P hybrid equivalent to T_f respecting their external markings becomes unique.

Definition 3.8 (Access and external marking)**.** Let $g\colon U \to V$ be a polynomial-like map with a connected filled Julia set. A *path to* $K(g)$ is a continuous map $\gamma\colon [0,1] \to V$ such that $\gamma((0,1]) \subset V \setminus K(g)$ and $\gamma(0) \in J(g)$. We say two paths γ_0 and γ_1 to $K(g)$ are *homotopic* if there exists a continuous map $\tilde{\gamma}\colon [0,1]\times[0,1] \to V$ such that

(1) $t \mapsto \tilde{\gamma}(s,t)$ is a path to $K(g)$ for all $s \in [0,1]$;
(2) $\tilde{\gamma}(0,t) = \gamma_0(t)$ and $\tilde{\gamma}(1,t) = \gamma_1(t)$ for all $t \in [0,1]$;
(3) $\tilde{\gamma}(s,0) = \gamma_0(0)$ for all $s \in [0,1]$.

An *access to* $K(g)$ is a homotopy class of paths to $K(g)$.

An *external marking* of g is an access Γ to $K(g)$ which is forward invariant in the following sense. For every representative γ of Γ, the connected component of $g(\gamma) \cap V$ which intersects $J(g)$ is also a representative of Γ.

The *standard external marking* for a polynomial P with connected Julia set is defined as the homotopy class of the external ray $\mathcal{R}_P(0)$ with angle 0 in the sense of paths to $K(P)$.

Theorem 3.9 (Straightening)**.** *Let g be a polynomial-like map with a connected filled Julia set and let Γ be an external marking of g. There is a unique monic centered polynomial P such that there is a hybrid conjugacy between g and P which sends the external marking Γ to the standard marking of P.*

See [10, Theorem A] for a proof.

In [10, 19], the homotopy class of the external ray with a specific angle is used successfully to determine an external marking for a pre-renormalization. The construction of such external rays depends on a fascinating phenomenon that there exists at least one periodic external ray landing at a given repelling or parabolic periodic point of a polynomial with connected Julia set. Although this is not a phenomenon in general for polynomials with disconnected Julia sets, Petersen-Zakeri [18] showed the following result.

Theorem 3.10 (Pertersen-Zareki)**.** *Let f be a non-linear polynomial and let z_0 be a repelling or parabolic periodic point of f. If the connected component $K(z_0)$ of the filled Julia set $K(f)$ containing z_0 is not a singleton, then there exists at least one periodic external ray $\mathcal{R}$ landing at z_0. Moreover, the period of z_0 divides the period of $\mathcal{R}$.*

Lemma 3.11. *Let T_{f_*} denote the canonical pre-renormalization for f_*. Let γ and γ' be two paths to $K(T_{f_*})$. If $\gamma(0) = \gamma'(0)$, then γ and γ' are homotopic.*

Proof. The conclusion follows immediately from the fact that $K(T_{f*})$ is homeomorphic to a closed disk. □

Let D_* be the Fatou component of f_* which contains the critical point c_*. Clearly, ∂D_* is a Jordan curve since the pre-renormalization T_{f_*} is hybrid equivalent to $z \mapsto z^{d_*}$. By *an internal angle system α for f_** we mean a homeomorphism $\alpha\colon \partial D_* \to \mathbb{R}/\mathbb{Z}$ such that $\alpha \circ f^{\ell_*} = m_{d_*} \circ \alpha$, where $m_{d_*}\colon \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z},\ x \mapsto d_* x \mod 1$. Note that $\alpha^{-1}(0)$ is a repelling fixed point for T_{f_*}. Let $K(\alpha^{-1}(0))$ denote the connected component of $K(f_*)$ that contains $\alpha^{-1}(0)$. As $K(T_{f_*}) \subset K(\alpha^{-1}(0))$, we know $K(\alpha^{-1}(0))$ is not a singleton. By Theorem 3.10, there exists an $f_*^{\ell_*}$-fixed (f_*-)external ray $\mathcal{R}$ landing at $\alpha^{-1}(0)$. Obviously, the external ray $\mathcal{R}$ determines an external marking for T_{f_*}. It follows from Lemma 3.11 that two distinct external rays landing at $\alpha^{-1}(0)$ determines a common external marking for T_{f_*}. We denote such an external marking by Γ_α and call it *the external marking for T_{f_*} induced by α*.

Fix an internal angle system α for f_* and choose an f_*-external ray $\mathcal{R}$ landing at $\alpha^{-1}(0)$. For every $f \in \mathcal{E}(f_*)$, by definition, there exists a conformal map $\Phi_f\colon D_{f_*}(\infty) \to D_f(\infty)$ such that $f \circ \Phi_f = \Phi_f \circ f_*$ on $D_{f_*}(\infty)$. So $\Phi_f(\mathcal{R})$ is an f-external ray and hence a path to $K(T_f)$. Therefore $\Phi_f(\mathcal{R})$ determines an external marking for the canonical pre-renormalization T_f. Such an external marking is called *the external marking for T_f induced by α* and denoted by $\Gamma_\alpha(f)$.

The following proposition shows that it is well-defined; i.e., the external marking for T_f does not depend on the choice of $\mathcal{R}$ landing at $\alpha^{-1}(0)$.

Proposition 3.12. *Let α be an internal angle system for f_* and let $f \in \mathcal{E}(f_*)$. If $\mathcal{R}$ and $\mathcal{R}'$ are two distinct f_*-external rays landing at $\alpha^{-1}(0)$, then $\Phi_f(\mathcal{R})$ and $\Phi(\mathcal{R}')$ are f-external rays landing at a common point.*

Proof. Since f and f_* are weakly pseudo-conjugate, by Lemma 3.4 and Lemma 3.7, there exists a quasiconformal map $\widehat{h}\colon \mathbb{C} \to \mathbb{C}$ with the following properties

- $\widehat{h} = \Phi_f$ on $\{z \mid G_{f_*}(z) \geq r_0/d^{m(\kappa_*+1)}\}$, where $m(\kappa_*+1)$ is the depth of the puzzle piece I_{κ_*+1};
- $f^{\ell_*} \circ \widehat{h} = \widehat{h} \circ f_*^{\ell_*}$ on ∂I_{κ_*+1}.

It is not difficult to see there exists a quasiconformal map $h\colon \mathbb{C}\setminus K(T_{f_*}) \to \mathbb{C}\setminus K(T_f)$ such that $h = \widehat{h}$ on $\mathbb{C} \setminus I_{\kappa_*+1}$ and $f^{\ell_*} \circ h = h \circ f_*^{\ell_*}$ on $I_{\kappa_*+1} \setminus K(T_{f_*})$. Indeed, such a quasiconformal map h can be constructed by pulling back $\hat{h}$ by the pre-renormalizations T_{f_*} and T_f. Note that $h(\mathcal{R}) = \Phi_f(\mathcal{R})$ and $h(\mathcal{R}') = \Phi_f(\mathcal{R}')$. Let $\Psi\colon \mathbb{C} \setminus \overline{\mathbb{D}} \to \mathbb{C} \setminus K(T_f)$ be a Riemann mapping. As $\partial K(T_{f_*})$ is a Jordan curve, $\Psi^{-1} \circ h$ can extend to a homeomorphism from $\mathbb{C} \setminus \mathrm{int}(K(T_{f_*}))$ onto $\mathbb{C} \setminus \mathbb{D}$. So $\Psi^{-1} \circ h(\mathcal{R})$ and $\Psi^{-1} \circ h(\mathcal{R}')$ are homotopic paths to $\overline{\mathbb{D}}$. By Lindelöf Theorem ([1, Theorem 3-5]), $h(\mathcal{R})$ and $h(\mathcal{R}')$ landing at a common point. $\square$

Recall that $\mathcal{C}_{d_*}$ is the set of all the monic centered polynomials of degree d_* with connected Julia sets. Fix an internal angle system α for f_*. Then by Theorem 3.9, there exists a well defined map

$$\chi = \chi_\alpha \colon \mathcal{E}(f_*) \to \mathcal{C}_{d_*} \tag{3.3}$$

with the following property: For any $f \in \mathcal{E}(f_*)$, there exists a hybrid conjugacy between the canonical pre-renormalization T_f for f and $\chi(f)$ which sends the external marking $\Gamma_\alpha(f)$ to the standard external marking for $\chi(f)$. Such a map χ is called a *straightening map*. The main goal of this paper is to show the following.

Theorem 3.13. *Fix an internal angle system α for f_*. Then the straightening map $\chi_\alpha\colon \mathcal{E}(f_*) \to \mathcal{C}_{d_*}$ is a bijection.*

We divide the proof into two parts: surjectivity and injectivity. In Section 4, we use quasiconformal surgery to verify that χ is surjective. In Section 5, we use a standard pullback argument to prove that χ is injective.

4. Tuning and Quasiconformal surgery

The aim of this section is to show that the straightening map defined by (3.3) is surjective. To this end, we consider the inverse problem for straightening.

Definition 4.1 (Tuning)**.** Fix an internal angle system α for f_* and let $Q \in \mathcal{C}_{d_*}$. We say f_* can be *tuned by* Q if there exists a polynomial $f \in \mathcal{E}(f_*)$ such that there exists a hybrid conjugacy between the canonical pre-renormalization T_f for f and Q which sends the external marking $\Gamma_\alpha(f)$ to the standard external marking for P.

The following theorem implies the surjectivity of χ.

Theorem 4.2. *Fix an internal angle system α for f_*. The polynomial f_* can be tuned by any map in $\mathcal{C}_{d_*}$.*

We use quasiconformal surgery to show the following.

Theorem 4.3. *Fix an internal angle system α for f_* and fix an f_*-external ray $\mathcal{R}$ landing at $\alpha^{-1}(0)$. For any $Q \in \mathcal{C}_{d_*}$ and $k > \kappa_*$, there exists a polynomial $f \in \mathcal{P}_d$ and a quasiconformal map $h\colon \mathbb{C} \to \mathbb{C}$ with the following properties:*

- *f and f_* are weakly pseudo-conjugate up to depth $m(k)$, where $m(k)$ is the depth of the puzzle piece I_k;*
- *h is a weak pseudo-conjugacy up to depth $m(k)$ between f_* and f;*
- *$F := f^{\ell_*}\colon h(I_{k+1}) \to h(I_k)$ is a polynomial-like map hybrid equivalent to Q;*
- *the external ray $\mathcal{R}_f$ for f of the same angle as $\mathcal{R}_{f_*}$ is smooth and defines an external marking for F;*
- *there exists a hybrid conjugacy between F and Q sending the external marking $[\mathcal{R}_f]$ to the standard external marking $[\mathcal{R}_Q(0)]$.*

We postpone the proof of Theorem 4.3 to the end of this section. Here, we prove how Theorem 4.2 follows from Theorem 4.3.

Proof of Theorem 4.2. Fix an integer $k > \kappa_*$. Let $Q \in \mathcal{C}_{d_*}$ and let $f \in \mathcal{P}_d$, $\mathcal{R}_f$,h and F be given by Theorem 4.3. It suffices to show that f is weakly pseudo-conjugate to f_*. We show by induction that for each $n \geq m(k)$, there exists a homeomorphism $H_n\colon \mathbb{C} \to \mathbb{C}$ with the following properties:

- $H_n = \phi_f^{-1} \circ \phi_{f_*}$ in a neighborhood of infinity, where ϕ_{f_*} and ϕ_f are Böttcher map for f_* and f respectively;
- $H_n \circ f_* = f \circ H_n$ on $\Omega_{f_*}(r_0/d^n)$.

For $n = m(k)$, set $H_n = h$.

Now assume that H_n has been defined for some $n \geq m(k)$ and let us construct H_{n+1}. For each $Y \subset \mathbb{C}$, let us denote $H_n(Y)$ by Y'. It suffices to construct, for each component Y of $\{z \mid G_{f_*}(z) < r_0/d^n\}$, a homeomorphism $H_{n+1}\colon \overline{Y} \to \overline{Y'}$ so that $f \circ H_{n+1}(z) = H_n \circ f_*(z)$ for $z \in \overline{Y} \cap \{z \mid G_{f_*}(z) \geq r_0/d^{n+1}\}$. Indeed, if Y does not contain the critical point c_* of f_*, then $f_*\colon Y \to f_*(Y)$ is a conformal map, and so is $f\colon Y' \to f(Y')$. In this case, we define

$$H_{n+1}|\overline{Y} = (f|_{\overline{Y'}})^{-1} \circ H_n|_{f_*(\overline{Y})} \circ f_*|_{\overline{Y}}.$$

Assume that Y does contain the critical point c_* of f_*. Then $Y' \Supset K(F)$. Note that Y is the f_*-puzzle piece of depth n containing c_*. For each $m \in \mathbb{N}$, let Y_m denote the f_*-puzzle piece of depth m containing c_*. Since

$$f_*^{\ell_*}\colon \overline{Y} \setminus Y_{n+1} \to \overline{Y_{n-\ell_*}} \setminus Y_{n+1-\ell_*}$$

and

$$f^{\ell_*}\colon \overline{Y'} \setminus Y'_{n+1} \to \overline{Y'_{n-\ell_*}} \setminus Y'_{n+1-\ell_*}$$

are both d_*-to-1 coverings between annuli, there is a homeomorphism $H_{n+1}\colon \overline{Y} \setminus Y_{n+1} \to \overline{Y'} \setminus Y'_{n+1}$ such that $H_{n+1} \circ f_*^{\ell_*} = f^{\ell_*} \circ H_n$ on $\overline{Y} \setminus Y_{n+1}$ and $H_{n+1} = H_n$ on ∂Y. Extending the map H_{n+1} in an arbitrary way to a homeomorphism from $\overline{Y}$ to $\overline{Y'}$, we obtain the desired map $H_{n+1}\colon \overline{Y} \to \overline{Y'}$. This completes the proof. $\square$

4.1. **Quasiconformal surgery.** Now we prove Theorem 4.3 to complete this section.

Fix $k > \kappa_*$ and consider a pre-renormalization

$$f_*^{\ell_*}|_{I_{k+1}} \colon I_{k+1} \to I_k$$

for f_*. By construction, $\partial I_k \subset D_{f_*}(\infty)$. Take an annulus $A_* \subset I_k \cap D_{f_*}(\infty)$ such that ∂I_k is an outer boundary component of ∂A_* and take an annular neighborhood $A'_* \subset D_{f_*}(\infty)$ of ∂I_{k+1}. Let W_* be the bounded component of $\mathbb{C} \setminus \overline{A'_*}$, which is a topological disk. We assume that $\overline{A_*}$ and $\overline{A'_*}$ does not intersect, hence the domain between $\overline{A_*}$ and $\overline{A'_*}$ (in other words, the bounded component of $\mathbb{C} \setminus \overline{A_* \cup A'_* \cup W_*}$) is an annulus. Let us denote this annular domain by $\hat{A}_*$ and denote $m_* = \operatorname{mod} \hat{A}_*$. Note that we take A_* and A'_* contained in the basin of infinity, but $\hat{A}_*$ intersects $K(f_*)$.

Take $\rho_- < \rho_0 < \rho_+$ be such that $d_*\rho_0 > \rho_+$ and $\frac{1}{2\pi}(d_*\rho_- - \rho_+) = m_*$. For any $Q \in \mathcal{C}_{d_*}$, let U, V and W be the topological disks bounded by the equipotential curves $E_Q(\rho_0)$, $E_Q(d_*\rho_0)$ and $E_Q(\rho_-)$ respectively. Then $Q \colon U \to V$ is polynomial-like.

Let $\hat{A}$ be an annulus in $V \setminus U$ defined as follows:

$$\hat{A} = \Omega_Q(\rho_+) - \overline{\Omega_Q(d_*\rho_-)} = \{z \in D_Q(\infty) |\ \rho_+ < G_Q(z) < d_*\rho_-\}$$

Then

$$\operatorname{mod} \hat{A} = \operatorname{mod}\{e^{\rho_+} < |z| < e^{d_*\rho_-}\} = m_*.$$

Recall that the external ray $\mathcal{R}_Q$ of angle 0 defines the standard external marking for Q.

Lemma 4.4. *There exists a quasiconformal homeomorphism $\varphi \colon \overline{V} \to \overline{I_k}$ such that:*

(1) $\varphi(U) = I_{k+1}$ *and* $\varphi(\hat{A}) = \hat{A}_*$
(2) φ *is conformal on* $W \cup \hat{A}$.
(3) $\varphi(Q(z)) = f_*^{\ell_*}(\varphi(z))$ *for* $z \in \partial U$.
(4) $\varphi(\mathcal{R}_Q \cap \partial U) = \mathcal{R} \cap \partial I_{k+1}$, *hence we also have* $\varphi(\mathcal{R}_Q \cap \partial V) = \mathcal{R} \cap \partial I_k$.
(5) $\varphi(\mathcal{R}_Q \cap (\overline{V} \setminus U))$ *is homotopic to* $\mathcal{R} \cap (\overline{I_k} \setminus I_{k+1})$ *in* $\overline{I_k} \setminus I_{k+1}$ *relative to the boundary.*

Proof. First let $\varphi|_U \colon U \to I_{k+1}$ be a Riemann map. Since $\operatorname{mod} \hat{A} = \operatorname{mod} \hat{A}_*$, the annuli $\hat{A}$ and $\hat{A}_*$ are conformally isomorphic. Let $\varphi|_{\hat{A}} \colon \hat{A} \to \hat{A}_*$ be a conformal isomorphism.

Also, let $\varphi|_{\partial V} \colon \partial V \to \partial I_k$ be a diffeomorphism satisfying $\varphi(\mathcal{R}_Q \cap \partial V) = \mathcal{R} \cap \partial I_k$. Then, by pulling back by the dynamics, we can define φ on ∂U satisfying (3) and (4). Finally, we extend φ quasiconformally on $\overline{V}$ such that (5) holds. □

Now let $V_* = f_*(I_{k+1})$. Then $f_*^{\ell_*-1} \colon V_* \to I_k$ is a conformal isomorphism. Let us denote its inverse by $\psi = (f_*^{\ell_*-1}|_{V_*})^{-1}$. Define a quasiregular mapping g as follows:

$$g(z) = \begin{cases} f_*(z), & z \notin I_{k+1}, \\ \psi \circ \varphi \circ Q \circ \varphi^{-1}(z), & z \in I_{k+1}. \end{cases}$$

Then g is holomorphic outside $A'_* \cup (\varphi \circ Q \circ \varphi^{-1})^{-1}(A_*)$.

Define an almost complex structure σ as follows. Let σ_0 be the standard complex structure on $\mathbb{C}$. Let

$$\sigma = \begin{cases} \sigma_0, & \text{on } \Omega_{f_*}(r_0/d^{m(k)}), \\ (g^n)^*\sigma_0, & \text{on } g^{-n}(\Omega_{f_*}(r_0/d^{m(k)})), \\ \sigma_0, & \text{otherwise.} \end{cases}$$

where $m(k)$ is the depth of the puzzle piece I_k. Since A_* and A'_* are contained in $D_{f_*}(\infty)$, the forward orbit of every $z \in \mathbb{C}$ passes $A'_* \cup (\varphi \circ Q \circ \varphi^{-1})^{-1}(A_*)$ at most once. Hence it follows that the dilatation of σ is bounded. By the measurable Riemann mapping theorem, there exists a quasiconformal homeomorphism $h\colon \mathbb{C} \to \mathbb{C}$ such that $h^*\sigma_0 = \sigma$. Then $f = h \circ g \circ h^{-1}\colon \mathbb{C} \to \mathbb{C}$ is a holomorphic map of degree d, hence a polynomial.

Let $\Omega_0 = \Omega_{f_*}(r_0/d^{m(k)})$. Observe that h is conformal and $g = f_*$ on Ω_0. Hence $h|_{\Omega_0}$ is a conformal conjugacy from f_* to f. By taking an affine conjugate, we may assume that $f \in \mathcal{P}_d$ and h is tangent to the identity as $z \to \infty$. This implies h is a weak pseudo-conjugacy up to depth $m(k)$ between f_* and f.

By construction, $F = f^{\ell_*}\colon h(I_{k+1}) \to h(I_k)$ is a polynomial-like map hybrid equivalent to $Q\colon U \to V$. Moreover, there exists a hybrid conjugacy between F and Q sending the external marking $[h \circ \varphi(\mathcal{R}_Q \cap V)]$ to the standard external marking $[\mathcal{R}_Q]$.

Let $\mathcal{R}_f$ be the external ray for f of the same angle as $\mathcal{R}_{f_*}$. Then $h(\mathcal{R}_{f_*} \cap \Omega_0) \subset \mathcal{R}_f$. Since $h(\mathcal{R}_{f_*} \cap \Omega_0)$ does not intersect the post-critical set of f, $\mathcal{R}_f$ is smooth.

We claim that $\mathcal{R}_f$ and $h \circ \varphi(\mathcal{R}_Q \cap V)$ defines the same external marking for F. Since both are invariant curves by f^{ℓ_*} in $h(I_k)$ and they are homotopic relative to the boundary in the fundamental annulus $h(I_k \setminus I_{k+1})$ for the pre-renormalization $f^\ell\colon h(I_{k+1}) \to h(I_k)$, they are homotopic path to the small filled Julia set.

5. Injectivity and existence of baby Mandelbrot sets for cubics

In this section, we proceed to prove the Main Theorem.

Theorem 5.1. *Fix an internal angle system α for f_*. Then the straightening map $\chi\colon \mathcal{E}(f_*) \to C_{d_*}$ is injective.*

Proof. Assume that $f, g \in \mathcal{E}(f_*)$ and $\chi(f) = \chi(g)$. We show that $f = g$. Recall that

$$I_0 \Supset I_1 \Supset \cdots \Supset I_{\kappa_*} \Supset I_{\kappa_*+1} \Supset \cdots$$

is the principle nest for f_*. For each $k \in \mathbb{N}$, set $I_k(f) = \psi_k(I_k)$ (resp. $I_k(g) = \varphi_k(I_k)$) where ψ_k (resp. φ_k) is a weak pseudo-conjugacy up to depth k between f_* and f (resp. g). Let $\Phi := \Phi_g \circ \Phi_f^{-1}$, where Φ_f (resp. Φ_g) is the relative Böttcher map between f_* and f (resp. g).

Let

$$\Omega_j = \Omega_j(f) = \Omega_f(r_0/d^{m(k_*)+j}).$$

Define a quasiconformal homeomorphism H_0 as follows: Let

$$H_0 = \Phi \quad \text{on } \Omega_0.$$

Since $\chi(f) = \chi(g)$, the pre-renormalizations $T_f = f^{\ell_*}\colon I_{k_*+1}(f) \to I_{k_*}(f)$ and $T_g = g^{\ell_*}\colon I_{k_*+1}(g) \to I_{k_*}(g)$ are hybrid equivalent. Take a hybrid conjugacy $h\colon U \to h(U)$ respecting the external markings. We may assume that $U \Subset I_{k_*}$. Let $V =$

$T_f^{-1}(U) \Subset I_{k_*+1}$. Observe that $f^{\ell_*-1}\colon f(V) \to U$ and $g^{\ell_*-1}\colon g(h(V)) \to h(U)$ are univalent. Define H_0 on $\bigcup_{n=1}^{\ell_*} f^n(V)$ by

$$H_0 = (g^{\ell_*-n})^{-1} \circ h \circ f^{\ell_*-n} \quad \text{on } f^n(U).$$

Let $\mathcal{R}$ be an external ray for f as well as its landing point, which is a representative of the external markings. and let $\mathcal{R}'$ be the external ray of the same angle for g with the landing point. Then extend H_0 quasiconformally on $\mathbb{C}$ so that $H_0(\mathcal{R})$ is homotopic to $\mathcal{R}'$ relative to $\Omega_0 \cup K(T_f)$. Let

$$\mathcal{K}(T_f) = \bigcup_{n=0}^{\ell_*-1} f^n(K(T_f)), \qquad \mathcal{K}(T_g) = \bigcup_{n=0}^{\ell_*-1} g^n(K(T_g)).$$

Then H_0 is a quasiconformal conjugacy between f and g on a neighborhood of $\mathcal{K}(T_f)$ and $H_0(\mathcal{K}(T_f)) = \mathcal{K}(T_g)$.

Let K be the maximal dilatation of H_0. By a standard pullback argument, we prove the following:

Claim. *We can construct a sequence of K-quasiconformal homeomorphisms $\{H_j\}_{j\geq 0}$ such that*

1. $H_{j+1} = \Phi$ *on* Ω_j*;*
2. $H_j \circ f = g \circ H_{j+1}$ *on* $\Omega_{j+1} \cup f^{-j}(\mathcal{K}(T_f))$*.*
3. $H_j = H_0$ *on a small neighborhood of* $\mathcal{K}(T_f)$ *(which shrinks to* $\mathcal{K}(T_f)$ *as* $j \to \infty$*).*
4. $H_j(\mathcal{R})$ *is homotopic to* $\mathcal{R}'$ *relative to* $\Omega_j \cup K(T_f)$*.*

In fact, assume we can construct H_j. First let $H_{j+1} = \Phi$ on Ω_{j+1}. Then it is clear that (2) holds on Ω_{j+1}. Recall that each complementary component I of Ω_{j+1} is a puzzle piece of depth $m(k_*)+j+1$ for f by definition. Hence $f(I)$ is a puzzle piece of depth $m(k_*)+j$. By assumption on H_j, we have $g(H_j(I))$ is a puzzle piece for g of the same depth. Let I' be a puzzle piece of depth $m(k_*)+1$ contained in the puzzle piece of depth $m(k_*)+j$ containing $H_j(I)$ (note that $H_j(I)$ is not a puzzle piece for g in general). Observe that

$$g(I') = H_j(f(I)), \qquad \Phi(\partial I) = \partial I'. \tag{5.1}$$

If I does not contain $K(T_f)$, then $f\colon I \to f(I)$ and $g\colon I' \to g(H_j(I))$ are univalent. In this case, we define $H_{j+1} = g^{-1} \circ H_j \circ f$. If I contain $K(T_f)$, then both

$$f\colon I \setminus K(T_f) \to f(I) \setminus f(K(T_f)), \qquad g\colon I' \setminus K(T_g) \to H_j(f(I)) \setminus g(K(T_g))$$

are d_*-to-one covering between annuli. Thus, by (5.1), we can lift

$$H_k\colon f(I) \setminus f(K(T_f)) \to H_j(f(I)) \setminus g(K(T_g))$$

to

$$H_{k+1}\colon I \setminus K(T_f) \to I' \setminus K(T_g)$$

so that $H_{k+1} = \Phi$ on ∂I. Then, by condition (4), $H_j(f(\mathcal{R}))$ and $g(\mathcal{R})$ are homotopic relative to $\Omega_j \cup g(K(T_f))$. Hence it follows that $H_{j+1}(\mathcal{R})$ and $\mathcal{R}'$ are homotopic relative to $\Omega_{j+1} \cup K(T_g)$ by condition (3). Therefore, by letting $H_{k+1} = H_j = H_0$ on $\mathcal{K}(T_f)$, we have a homeomorphism on $\mathbb{C}$. It is easy to check that it is K-quasiconformal map satisfying the conditions. Thus we have proved the claim.

By passing to a subsequence, H_j converges to a quasiconformal homeomorphism H. Then $H = \Phi$ on $D_f(\infty)$ and $\bar{\partial} H \equiv 0$ almost everywhere on $\bigcup_{n\geq 0} f^{-n}\mathcal{K}(T_f)$. Furthermore, since the post-critical set $P(f)$ is contained in $\mathcal{K}(T_f) \cup D_f(\infty)$, the set

$$K(f) \setminus \bigcup_{n\geq 0} f^{-n}(\mathcal{K}(T_f))$$

is of measure zero [12], [15, Theorem 3.9]. Indeed, for any $z \in K(f)\setminus\bigcup_{n\geq 0} f^{-n}(\mathcal{K}(T_f))$,added"Indeed,..." if the orbit of z passes through $I_{\kappa_*+1}(f)$ then it has to pass through $I_{\kappa_*}(f) \setminus I_{\kappa_*+1}(f)$. This implies that $\sup_n \mathrm{d}(f^n(z), P(f)) > 0$ for almost all the $z \in K(f) \setminus \bigcup_{n\geq 0} f^{-n}(\mathcal{K}(T_f))$. By [15, Theorem 3.9], $K(f) \setminus \bigcup_{n\geq 0} f^{-n}(\mathcal{K}(T_f))$ must have Lebesgue measure zero. Therefore, H is 1-quasiconformal, hence affine. Since H is asymptotic to the identity and conjugates f and g, it follows that H is the identity and $f = g$. □

Proof of the Main Theorem. Combine Theorem 4.2 and Theorem 5.1, we conclude that the straightening map χ is a bijection. It remains to show $\mathcal{E}(f_*)$ is connected. The proof is parallel to that of the Main Theorem in [19]. Let X be a nonempty open and closed subset of $\mathcal{E}(f_*)$, we prove that $X = \mathcal{E}(f_*)$. Note that X is compact since $\mathcal{E}(f_*)$ is compact.

We first show that $\chi(X)$ is a closed subset of $\mathcal{C}_{d_*}$. Assume that P_n is a sequence in $\chi(X)$ such that $P_n \to P$ in $\mathcal{C}_{d_*}$, we are going to show that $P \in \chi(X)$. Set $f_n := \chi^{-1}(P_n)$ for all $n \in \mathbb{N}$. By the compactness of X, passing to a subsequence we may assume that f_n converges to some $f \in X$. For each $n \in \mathbb{N}$, let $T_{f_n} : U_n \to V_n$ be the canonical pre-renormalization for f_n. Since $f_n \to f$, we have

$$\inf_{n\in\mathbb{N}} \mathrm{mod}(V_n \setminus \overline{U_n}) > 0.$$

By [13, Proposition 2.5], we may choose hybrid conjugacies h_n between T_{f_n} and P_n respecting the external marking such that the maximal dilatations of h_n are uniformly bounded. Therefore, passing to a further subsequence, we conclude that there exists a quasiconformal conjugacy H between T_f and P respecting the external marking. Thus f is conjugate to $\chi^{-1}(P)$ via a quasiconformal map $h\colon \mathbb{C} \to \mathbb{C}$ which is conformal outside $K(f)$ and $h'(\infty) = 1$. Let $\{g_t\}_{t\in[0,1]}$ be a path in $\mathcal{E}(f_*)$ connecting f and $\chi^{-1}(P)$ such that $g_t = H_t \circ f \circ H_t^{-1}$ where H_t^{-1} is a quasiconformal map with Beltrami differential $t\frac{\bar{\partial}H}{\partial H}$ and $H'(\infty) = 1$. Since X is an open and closed subset of $\mathcal{E}(f_*)$, the whole path $\{g_t\}_{t\in[0,1]}$ lies in X. In particular, $\chi^{-1}(P) \in X$ and hence $P \in \chi(X)$.

If $X \neq \mathcal{E}(f_*)$, then $\mathcal{C}_{d_*} = \chi(\mathcal{E}(f_*))$ is a disjoint union of two closed subset $\chi(X)$ and $\chi(\mathcal{E}(f_*) \setminus X)$. This contradicts the fact that $\mathcal{C}_{d_*}$ is connected [14]. □

As a consequence of the Main Theorem, we have the following.

Theorem 5.2. *Let $f_* \in \mathcal{P}_3^+$. Assume that f_* has a periodic critical point c_*. Then there exists a homeomorphism χ from $\mathcal{E}(f_*)$ onto the Mandelbrot set $\mathcal{M}$.*

Proof. In this case, the internal angle system is unique and we can forget it. Let χ be the straightening map from $\mathcal{E}(f_*)$ onto $\mathcal{M}$. It follows from [16, Proposition 4.7] and the compactness of $\mathcal{E}(f_*)$ that χ is a homeomorphism. □

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Declarations

Conflict of interest. The authors have no relevant financial or nonfinancial interests to disclose.

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