COMBINATORIAL NON-RIGIDITY FOR INFINITELY RENORMALIZABLE UNICRITICAL CUBIC POLYNOMIALS

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Abstract. It is well known that there are no real cubic polynomials f with two distinct critical points such that f is infinitely renormalizable and all the renormalizations of f are cubic-like maps. This due to the real bounds. In this paper, we show that there does exist an infinitely renormalizable complex cubic polynomial F with two distinct critical points such that all the renormalizations are cubic-like. As a consequence, F has no a priori (complex) bounds.

Moreover, F is combinatorially equivalent to a unicritical polynomial. In particular, such a unicritical polynomial is not combinatorially rigid in the space of cubic polynomials. We further prove that the combinatorial class of such a polynomial contains a continuum.

1. Introduction

Infinitely renormalizable real polynomials have been deeply studied for decades. Real bounds and complex bounds have been built for such maps [17]. A remarkable result proved in this century is on the rigidity of the infinitely renormalizable real polynomials [13]. While the rigidity of the infinitely complex polynomials remains open since the combinatorics for complex polynomials are more complicated than those for real polynomials. As we all know, critical points play an important role in holomorphic dynamics. The increase of the critical points will make the combinatorics intricate. Unicritical polynomials can be treated more easily than multi-critical polynomials. Fortunately, the combinatorics for infinitely renormalizable real polynomials are not too complicated due to the real bounds. Indeed, all the deep renormalizations of infinitely renormalizable real polynomials are compositions of unicritical maps. (See [19, 20].) As a corollary, deep renormalizations of an infinitely renormalizable real cubic polynomial with two distinct critical points cannot be cubic. One may naturally ask whether there exists an infinitely renormalizable complex cubic polynomial with two distinct critical points such that all the renormalizations are cubic-like.

The aim of the paper is to answer the question above affirmatively.

Main Theorem. *There exists an infinitely renormalizable complex cubic polynomial F with two distinct critical points $\omega \neq \omega'$ such that all the renormalizations of F are cubic-like maps. More precisely, $\omega, \omega' \in \bigcap_n K_n$ is non-trivial, where K_n is the filled-Julia set of the n-th renormalization of F.*

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The Main Theorem is also related to the combinatorial rigidity problem:

Definition 1.1 (Combinatorial equivalence)**.** We say two polynomials f and g with connected Julia sets and no indifferent cycles are *combinatorially equivalent* if f and g have the same rational lamination.

See § 2.1 for the definition of rational laminations.

Definition 1.2 (Combinatorial rigidity)**.** We say a polynomial f with connected Julia set and no indifferent cycle is *combinatorially rigid* if a polynomial g of the same degree as f with connected Julia set and no indifferent cycle is combinatorially equivalent to f, then the composition of the Böttcher coordinates

$$\phi_g \circ \phi_f^{-1} \colon (\mathbb{C} \setminus K(f)) \to (\mathbb{C} \setminus K(g))$$

extends to a quasiconformal homeomorphism on the Riemann sphere $\hat{\mathbb{C}}$.

McMullen conjectured that every polynomial with connected Julia set and no indifferent cycle is combinatorially rigid [14]. A counterexample is first given by Henriksen [4].

Here we give another counterexample of combinatorial rigidity; in fact, the existence of such a polynomial F in the Main Theorem implies that there exists an infinitely renormalizable cubic unicritical polynomial $P(z) = z^3 + c$ combinatorially equivalent to F. Clearly, F and P are not combinatorially rigid (see Corollary 4.2 for more detail).

Contrary to the fact that our counterexample is unicritical, Henriksen's counterexample is far from unicritical; it is a cubic polynomial with infinitely many capture renormalizations, namely, each renormalization is quadratic and contains only one critical point, and the other critical point outside is eventually captured by the filled Julia set of the renormalization. Its combinatorial class contains (and probably is equal to) a quasiconformally homeomorphic image, contained in a 1-dimensional analytic set, of the combinatorial class in the Julia set of the quadratic polynomial hybrid equivalent to the first quadratic renormalization (see also [7]).

More strongly, we prove there exists a continuum connecting P and F in the combinatorial class (Theorem 4.3). In addition, the important fact is that F is constructed as a perturbation of $z(1+z)^2$. Hence such a continuum has a definite diameter, and furthermore, we can have F infinitely renormalizable not only in the sense of polynomial-like maps, but also in the sense of the near-parabolic renormalization [10], which should allow us to study the dynamics of F further.

The combinatorial rigidity conjecture for the unicritical family $z^d + c$ is still open for $d \geq 2$. Our example does not contradict this; we are working in a bigger family of the whole cubic polynomials, and the intersection of the combinatorial class with the uncritical family can be a singleton.

2. Preliminaries

Let f be a polynomial. The *filled Julia set* $K(f)$ of f is defined as:

$$K(f) = \{z \in \mathbb{C} \mid \sup_{n \in \mathbb{N}} |f^n(z)| < \infty\}.$$

We call the boundary $J(f)$ of $K(f)$ *the Julia set* of f.

In this paper, we only concern about the monic centered polynomials with connected Julia set. We use $\mathcal{C}_3$ to denote the set of all such maps, that is,

$$\mathcal{C}_3 = \{f \mid \ f \text{ is a monic centered cubic polynomial and } J(f) \text{ is connected}\}.$$

2.1. **External rays and equipotential curves.** For any $f \in \mathcal{C}_3$, there exists a unique conformal map $\varphi : \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ with the following properties.

- $\varphi(z)/z \to 1$ as $z \to \infty$.
- $\varphi(f(z)) = (\varphi(z))^3$ for all $z \in \mathbb{C} \setminus K(f)$.

The *Green function* of f is defined as

$$G_f(z) = \begin{cases} \log|\varphi(z)|, & \text{if } z \notin K(f); \\ 0 & \text{otherwise.} \end{cases}$$

For any $s > 1$, we call the set $G_f^{-1}(s)$ the *equipotential curve* of level s and denote it by $E(s)$.

For $t \in \mathbb{R}/\mathbb{Z}$, the *external ray* $R_f(t)$ of angle t is defined as

$$R_f(t) = \varphi^{-1}(\{re^{2\pi i t} \mid r > 1\}).$$

If the limit

$$x = \lim_{r\to 1} \varphi^{-1}(r \exp(2\pi i\theta))$$

exists, then we say $R_f(\theta)$ *lands at* x and θ is the *landing angle for* x.

The *rational lamination* of f, denoted by $\lambda(f)$, is the equivalence relation on $\mathbb{Q}/\mathbb{Z}$ so that $\theta_1 \sim \theta_2$ if and only if $R_f(\theta_1)$ and $R_f(\theta_2)$ land at the same point.

2.2. **Yoccoz puzzle.** Fix an $f \in \mathcal{C}_3$. We say that a finite set Z is f-*admissible* if the following hold:

- Z is the union of some repelling periodic cycles.
- for any $z \in Z$, there exist at least two external rays landing at z.

Fix an f-admissible set Z. Let $\Gamma_0 = \Gamma_0^Z$ denote the union of all the external rays landing on Z, the set Z itself and the equipotential curve $E(1)$. For each $n \geq 1$, define $\Gamma_n = f_0^{-n}(\Gamma_0)$. A bounded component of $\mathbb{C} \setminus \Gamma_n$ is called a *puzzle piece* of depth n.

2.3. **External markings and hybrid conjugacies.** We say $F : U \to V$ is a *cubic-like map* if F is a holomorphic proper map of degree 3 and $U \Subset V$ are Jordan disks in $\mathbb{C}$. Similar to polynomials, we define the *filled Julia set* $K(F)$ of F as

$$K(F) = \bigcap_{n\in\mathbb{N}} F^{-n}(U).$$

By an *admissible path to* $K(F)$ we mean a continuous map $\gamma : [0,1] \to V$ so that $\gamma(0,1] \subset V \setminus K(F)$ and $\gamma(0)$ is a fixed point of F. We say two admissible paths γ_0 and γ_1 to $K(F)$ are *homotopic* if there exists a continuous map $\tilde{\gamma} : [0,1]\times[0,1] \to V$ such that

- $t \mapsto \tilde{\gamma}(s,t)$ is an admissible path to $K(F)$ for all $s \in [0,1]$;
- $\tilde{\gamma}(0,t) = \gamma_0(t)$ and $\tilde{\gamma}(1,t) = \gamma_1(t)$ for all $t \in [0,1]$;
- $\tilde{\gamma}(s,0) = \gamma_0(0)$ for all $s \in [0,1]$.

We say an admissible path γ is *F-invariant* if $F(\gamma \cap U) = \gamma$. A homotopy class of F-invariant admissible paths to $K(F)$ is called *an external marking of F*. An *externally marked cubic-like map* $(F, [\gamma])$ is a cubic-like map F with an external marking $[\gamma]$. By abuse of notation, We often omit $[\gamma]$ and simply call g an externally marked polynomial.

The fixed point $\gamma(0)$ is called the *externally marked fixed point* for $(F, [\gamma])$. Note that every externally marked fixed point has zero combinatorial rotation number.

Note that the external ray $R_f(0)$ of angle 0 naturally induces an external marking of a given cubic polynomial f. This external marking is called *the standard external marking of f*. In the following, every monic cubic polynomial f is externally marked by the standard external marking unless otherwise stated.

Two cubic-like maps $F : U \to V$ and $g : U' \to V'$ are said to be *quasiconformally conjugate* if there exists a quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $\phi \circ F = G \circ \phi$ near $K(F)$. We say that F and G are *hybrid equivalent* if they are quasiconformally conjugate and there exists a quasiconformal conjugacy ϕ so that $\bar{\partial} f = 0$ a.e. on $K(F)$. The following theorem is a special case of [9, Theorem A].

Theorem 2.1 (Straightening)**.** *Let F be a cubic-like map with connected Julia set and let $[\gamma]$ be an external marking of F. There exists a unique cubic polynomial $f \in \mathcal{C}_3$ such that there exists a hybrid conjugacy between F and f sending the external marking $[\gamma]$ of F to the standard external marking of f.*

3. Straightening and Tuning

Recall that $\mathcal{C}_3$ denotes the connectedness locus of monic and centered cubic polynomials. We always consider a polynomial $f \in \mathcal{C}_3$ externally marked by the standard external marking. Then $\mathcal{C}_3$ is isomorphic to the family of affine conjugacy classes of externally marked cubic polynomials with connected filled Julia sets.

For any $q \geq 2$, fix a unicritical cubic polynomial $P_q(z) = z^3 + c_q$ such that

- P_q is monic and centered;
- The unique critical point of P_q is periodic of period q.
- P_q has a repelling fixed point α_{P_q} with rotation number $1/q$.
- α_{P_q} is the landing point of $R_{P_q}(1/(3^q - 1))$.

Note that P_q is the center of the $1/q$-satellite copy of the multibrot set $\mathcal{M}_3$ attached to the main hyperbolic component, namely, at

$$z^3 + \frac{e^{\pi i/q}(3 - e^{2\pi i/q})}{3\sqrt{3}},$$

which has a parabolic fixed point of multiplier $e^{2\pi i/q}$ at $\frac{e^{\pi i/q}}{\sqrt{3}}$. The limit

$$\lim_{q \to \infty} \frac{e^{\pi i/q}(3 - e^{2\pi i/q})}{3\sqrt{3}} = \frac{2}{3\sqrt{3}}$$

is the cusp of the main hyperbolic component.

Let $\mathcal{C}(q)$ denote the set of *combinatorially renormalizable maps* with respect to $\lambda(P_q)$, that is,

$$\mathcal{C}(q) = \{f \in \mathcal{C}_3 \mid \lambda(f) \supset \lambda(P_q)\}.$$

There exists a subset $\mathcal{R}(q) \subset \mathcal{C}(q)$ and a well defined straightening map

$$\chi_q : \mathcal{R}(q) \to \mathcal{C}_3$$

with the following dynamical property. For any $f \in \mathcal{R}(q)$, f is renormalizable in the sense of polynomial-like mapping and its renormalization is hybrid equivalent to $\chi_q(f)$ respecting external markings. See [9, 11] for more details.

In this paper, we only consider *cubic renormalizations*; hence a map $f \in \mathcal{C}_3$ is *renormalizable* if there exists a cubic-like restriction $f^q \colon U \to V$ of some iterate of f with connected filled Julia set such that U contains both of the critical points. We call the $f^q \colon U \to V$ a *renormalization* of period q.

Lemma 3.1. *Let $f \in \mathcal{C}_3$ satisfy the following:*

1. *There exists a fixed point α of rotation number $1/q$ for f.*
2. *There exists a cubic renormalization $f^q \colon U \to V$ of period p. Let K be the filled Julia set of the renormalization.*
3. *An external angle θ of α for f satisfies the following: The connected component of $\mathbb{C} \setminus (R_f(\theta) \cup R_f(3\theta) \cup \{\alpha\})$ containing $R_f(\theta+\varepsilon)$ for sufficiently small ε contains $K \setminus \{\alpha\}$.*

Then by replacing f by $-f(-z)$ if necessary, we have the following:

- *$3\theta = 1/(3^q - 1)$ in $\mathbb{Q}/\mathbb{Z}$, and*
- *$f \in \mathcal{R}(q)$.*

Note that the third assumption is not only for θ, but also for the renormalization because there are "crossed" renormalizations [14] for which such θ does not exist.

Proof. First observe that since the combinatorial rotation number of α is $1/q$, $\theta, 3\theta, 3^2\theta, \dots, 3^{q-1}\theta$ are ordered counterclockwise.

$$R_f(\theta), R_f(3\theta), \dots, R_f(3^{q-1}\theta)$$

as well as the landing point α divide the plane into q sectors. Let S_k be the sector bounded by $R_f(3^{k-1}\theta)$ and $R_f(3^k\theta)$. Then S_1 contains both of the critical points and $f \colon S_k \to S_{k+1}$ is a homeomorphism for $k = 2, \dots, q$ (indices are understood in modulo q here).

Therefore, the intervals $[3^{k-1}\theta, 3^k\theta]$ $(k = 1, \dots, q)$ partition $\mathbb{R}/\mathbb{Z}$ and their lengths ℓ_k satisfy

$$3\ell_1 = 2 + \ell_2, \qquad 3\ell_k = \ell_{k+1} \quad (k = 2, \dots, q).$$

Hence it follows that

$$1 = \sum_{k=1}^{q} \ell_k = \ell_2 \frac{3^q - 1}{2},$$

so

$$2 \cdot 3\theta \equiv \ell_2 = \frac{2}{3^q - 1} \mod 1.$$

Therefore, we have

$$3\theta = \frac{1}{3^q - 1}, \text{ or } \frac{1}{3^q - 1} + \frac{1}{2}.$$

By taking a conjugacy by $z \mapsto -z$ (in other words, by replacing f by $-f(-z)$) for the latter case, we have $3\theta = \frac{1}{3^q-1}$.

Since $K \subset S_1$, it is easy to see that $f \in \mathcal{R}(q)$. $\square$

We proceed to define $\mathcal{C}(\underline{q_n})$ for any finite sequence $\underline{q_n} = (q_1, q_2, \cdots, q_n)$, $q_j \geq 2$ for all j. By the main theorem of [11] (or [9, Theorem B and C]), there exists a unique hyperbolic and post-critically finite cubic polynomial $P_{q_1,q_2} \in \mathcal{C}(q_1)$ such that $\chi_{q_1}(P_{q_1,q_2}) = P_{q_2}$. We then define $P_{\underline{q_n}}$, inductively such that $P_{\underline{q_n}} \in \mathcal{C}(q_1)$ and

$\chi_{q_1}(P_{\underline{q_n}}) = P_{\sigma(\underline{q_n})}$, where $\sigma(q_1, q_2, \cdots, q_n) = (q_2, \cdots, q_n)$ is the shift map. Finally, let

$$\mathcal{C}(\underline{q_n}) = \{f \in \mathcal{C}_3 \mid \lambda(f) \supset \lambda(P_{\underline{q_n}})\}.$$

Lemma 3.2. *For any infinite sequence $\underline{q} = (q_1, q_2, \cdots)$ with $q_j \geq 2$ for all j, let $\underline{q_n} = (q_1, q_2, \cdots, q_n)$ for all $n \geq 1$. Then*

$$\mathcal{C}(\underline{q_n}) \subset \mathcal{C}(\underline{q_{n-1}}).$$

Proof. By construction, $P_{\underline{q_n}} \in \mathcal{C}(\underline{q_{n-1}})$. Hence the lemma follows from the definition of $\mathcal{C}(\underline{q_n})$. □

Lemma 3.3. *For any infinite sequence $\underline{q} = (q_1, q_2, \cdots)$ with $q_j \geq 2$ for all j, let $\underline{q_n} = (q_1, q_2, \cdots, q_n)$ for all $n \geq 1$. If a sequence $\{f_n\}$ of cubic polynomials with $f_n \in \mathcal{C}(\underline{q_n})$ converges to f, then*

$$f \in \mathcal{C}(\underline{q}) = \bigcap_n \mathcal{C}(\underline{q_n})$$

Moreover, f is infinitely renormalizable. In particular, $\mathcal{C}(\underline{q})$ is a nonempty compact set.

Note that $\mathcal{C}(\overline{q_n})$ is not compact. This lemma is essentially proved in [8] in a more general situation, but we provide a simpler proof under our current situation.

Proof. For $n \in \mathbb{N}$, Let $Q_n = P_{\underline{q_n}}$. Note that Q_n has a periodic point α_n with period $s_n := q_1 q_2 \cdots q_{n-1}$ and rotation number $1/q_n$. Let Θ_n be the set of all external angles of α_n for Q_n. For each $n \in \mathbb{Z}_+$, fix an angle $\theta_n \in \Theta_n$ and let β_n be the landing point of the external ray $R_f(\theta_n)$. By [6, Lemma B.1], it suffices to show that each β_n is repelling.

We first show that β_1 is repelling. To obtain a contradiction, we assume that β_1 is parabolic.

Case 1. β_2 is repelling. It follows from [6, Lemma B.1] that $R_f(t)$ lands at β_2 for all $t \in \Theta_2$. We use all the external rays with angles in Θ_2 and the equipotential curve of level 1 to make Yoccoz puzzle. Note that there exist critical puzzle pieces $Y_{Q_2}^{(kq_1q_2)}$ of depth kq_1q_2 ($k \in \mathbb{Z}_+$) such that the first return map

$$Q_2^{q_1q_2} : Y_{Q_2}^{(kq_1q_2)} \to Y_{Q_2}^{((k-1)q_1q_2)}$$

is a 3-to-1 covering map. Since $f_n \in \mathcal{C}(Q_2)$, the corresponding first return map

$$f_n^{q_1q_2} : Y_{f_n}^{(kq_1q_2)} \to Y_{f_n}^{((k-1)q_1q_2)}$$

is also a 3-to-1 covering map. Thus $f^{q_1q_2} : \overline{Y_f^{(kq_1q_2)}} \to \overline{Y_f^{((k-1)q_1q_2)}}$ is a 3-to-1 covering. This implies the orbits of two critical points of f can never escape $\bigcup_{j=0}^{q_1q_2-1} \overline{f^j(Y_f^{(q_1q_2)})}$. However the parabolic fixed point β_1 must attract at least one critical point of f. Hence β_1 has to lie in $\bigcup_{j=0}^{q_1q_2-1} \overline{f^j(Y_f^{(q_1q_2)})}$. By the definition of β_1 and β_2, β_1 cannot belong to the boundary of any puzzle piece. So $\beta_1 \in \bigcup_{j=0}^{q_1q_2-1} f^j(Y_f^{(q_1q_2)})$. This contradicts with the fact that

$$f^{q_1q_2} : Y_f^{(q_1q_2)} \to Y_f^{(0)}$$

is a first return map.

Case 2. β_2 is parabolic. Then β_3 must be repelling since a cubic polynomial can have at most two parabolic cycles. By using a same argument in case 1, one can show that this is impossible.

Similarly, one can show that β_n is repelling for all $n \in \mathbb{Z}_+$.

□

Definition 3.4. Let f and g are cubic polynomials with connected Julia sets. Let α be a fixed point of f, and β be a β-fixed point of g, i.e., a fixed point with combinatorial rotation number 0. Let p/q be an irreducible fraction with $q \geq 2$. We say (f, α) is a *p/q-rotatory* of (g, β) if there exist topological disks $U \Subset V$ with the following properties.

- The rotation number of α is p/q and $\alpha \in U$.
- $f^q : U \to V$ is a cubic-like map with connected Julia set K.
- Both of the critical points of f lie in U.
- There exists a hybrid conjugacy h between $f^q|U$ and g such that $h(\alpha) = \beta$.

If g is externally marked, we further require the following: There exists $\theta \in \mathbb{Q}/\mathbb{Z}$ such that

- the external ray $R_f(\theta)$ lands at x;
- $K \setminus \{x\}$ is contained in a component of
$$\mathbb{C} \setminus \left(\bigcup_{n=0}^{p-1} f^n(R_f(\theta)) \cup \{x\} \right);$$
- the external marking for $f^q \colon U \to V$ defined by $R_f(\theta)$ (note that this does not depend on the choice of θ) corresponds to the standard external marking by the hybrid conjugacy h.

Note that no rotatory exists if β is parabolic. Otherwise, rotatories always exist:

Theorem 3.5. *Let p/q be an irreducible fraction with $q \geq 2$ and let g be an externally marked cubic polynomial with connected Julia set. If the externally marked fixed point is repelling, then there exists a p/q-rotatory of (g, β).*

Indeed, a quasiconformal surgery construction is given in [11, Theorem 4.1] under more general settings.

In the following, we only consider the case $p = 1$; i.e., $1/q$-rotatories. Then by Lemma 3.1, the following further holds:

Corollary 3.6. *Under the assumption of the theorem, consider further the case $p = 1$. Then there exists a $1/q$-rotatory (f, α) of (g, β) such that*

1. *the combinatorial rotation number of α for f is $1/q$,*
2. *the external ray $R_f(\frac{1}{3^q-1})$ lands at α, and*
3. *$f \in \mathcal{R}(q)$.*

We recursively apply this corollary to construct infinitely renormalizable polynomials. However, since we want to start with parabolic maps, The first step is done explicitly as follows:

For $q \geq 2$, let

$$\widehat{g}_q(z) = e^{2\pi i/q} z\,(1+z)^2\,. \tag{3.1}$$

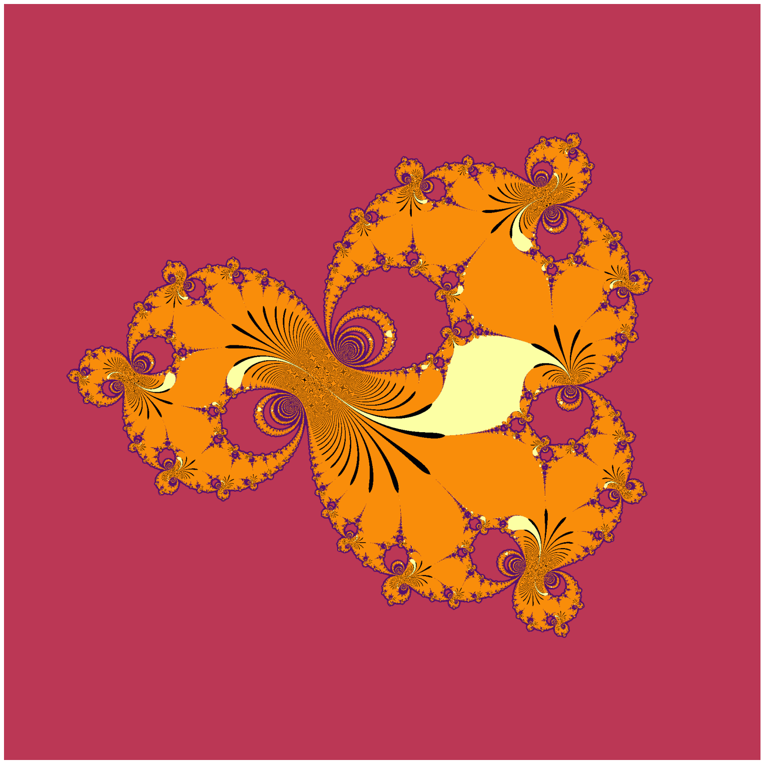

FIGURE 1. The Julia set of $\hat{g}_{128}$. The biggest yellow region is the filled Julia set of the combinatorial (or parabolic-like) renormalization. There is a critical point located at the thinnest part of the region, which the preimage of the parabolic fixed point of multiplier $e^{2\pi i/128}$ at the origin.

Note that $\widehat{g}_q$ has a parabolic fixed point 0 with rotation number $1/q$ and there exists a critical point -1 satisfies

$$\widehat{g}_q(-1) = 0.$$

Let $\widehat{\beta}_q = e^{-\pi i/q} - 1$ be the repelling fixed point of $\widehat{g}_q$ that is close to 0. Note that there is a unique external ray landing at $\widehat{\beta}_q$. Such an external ray induces an external marking κ_q of $\widehat{g}_q$. By Theorem 2.1, there exists a unique cubic polynomial $g_q \in \mathcal{C}_3$ so that it is hybrid equivalent to $\widehat{g}_q$ and there exists a hybrid conjugacy between g_q and $\widehat{g}_q$ that sends the standard external marking of g_q to the external marking κ_q of $\widehat{g}_q$. Consequently, g_q and $\widehat{g}_q$ are affinely conjugate. Let β_q and α_q be the fixed point of g_q which corresponds to $\widehat{\beta}_q$ and 0 respectively by the affine conjugacy between $\widehat{g}_q$ and g_q. Let ω_q and ω'_q be the critical points of g_q such that ω_q lies in the parabolic basin of α_q and $g_q(\omega'_q) = \alpha_q$.

Let $g_\infty = \lim_{q\to\infty} g_q$. It is a monic centered cubic polynomial affinely conjugate to $\widehat{g}_\infty = \lim \widehat{g}_q = z(1+z)^2$.

Lemma 3.7. *Let f be a monic and centered cubic polynomial. Assume that f has a parabolic fixed point x_0 with multiplier $e^{2\pi i/q}$ for some $q \in \mathbb{N}$ and there exists a* *critical point c which satisfies*

$$f(c) = x_0.$$

Then f is affinely conjugate to g_q.

Proof. Let $\varphi : \mathbb{C} \to \mathbb{C}$ be an affine map such that $\varphi(x_0) = 0$ and $\varphi(c) = -1$. Then it is easy to see

$$\widehat{g}_q = \varphi \circ f \circ \varphi^{-1}.$$

□

Lemma 3.8 (QC-rigidity). *Let g_q be the polynomial as above. Assume that $f \in \mathcal{C}_3$ is quasiconformally conjugate to g_q. If there exists a quasiconformal conjugacy ϕ sending the standard external markings of f to that of g_q, then $f = g_q$.*

Proof. Since g_q and f are quasiconformally conjugate, f has a parabolic fixed point x_0 with multiplier $e^{2\pi i/q}$ and there exists a critical point c_0 of f such that $f(c_0) = x_0$. By Lemma 3.7, f is affinely conjugate to g_q. The conclusion follows since the conjugacy respects the standard external marking of f and g_q. □

Lemma 3.9. *For any $p, q \geq 2$, there exists a monic and centered cubic polynomial $g_{p,q} \in \mathcal{R}(p)$ such that*

- *$g_{p,q}$ has a repelling fixed point $\alpha_{p,q}$ with rotation number $1/p$;*
- *$(g_{p,q}, \alpha_{p,q})$ is a $1/p$-rotatory of (g_q, β_q).*

Moreover, as q tends to infinity,

- *$g_{p,q}$ converges to g_p.*
- *Let $\alpha'_{p,q}$ be the parabolic periodic point of $g_{p,q}$ of period p whose immediate basin contains a critical point. Then both $\alpha_{p,q}$ and $\alpha'_{p,q}$ converge to α_p.*

Proof. First of all, for any $p, q \geq 2$, it follows from Corollary 3.6 that a $1/p$-rotatory $(g_{p,q}, \alpha_{p,q})$ exists and $g_{p,q} \in \mathcal{R}(p)$. Let $\omega_{p,q}$ and $\omega'_{p,q}$ be the critical points of $g_{p,q}$ such that $\omega_{p,q}$ lies in the immediate basin of the parabolic periodic point $\alpha'_{p,q}$, and $g_{p,q}(\omega'_{p,q}) = g_{p,q}(\alpha'_{p,q})$.

The task now is to show that $g_{p,q}$ converges to g_p. Since the connectedness locus $\mathcal{C}_3$ of cubic polynomials is compact, any subsequence of $\{g_{p,q}\}_{q\geq 2}$ has a convergent subsequence. It suffices to show that the limit function of any convergent subsequence of $\{g_{p,q}\}_{q\geq 2}$ is just g_p.

To this end, we assume that $\{g_{p,q}\}_{g\geq 2}$ itself converges to some cubic polynomial Q and show that $Q = g_p$. By passing to a subsequence, we may further assume that the following holds.

(1) As $q \to \infty$, $\alpha_{p,q}$ converges to a fixed point α of Q and $\alpha'_{p,q}$ converges to a fixed point α' of Q^p. Moreover, $(Q^p)'(\alpha') = 1$.
(2) $\omega'_{p,q}$ converges to a critical point c of Q such that $Q^p(c) = Q(\alpha')$.

We claim that $\alpha = \alpha'$. Assume by contradiction that $\alpha \neq \alpha'$. Since one of the critical point of Q is in the basin of α' and the other is eventually mapped to α', α must be a repelling fixed point of Q. As there are p external rays landing at α_p with rotation number $1/p$, there are also p external rays landing at α with the same rotation number $1/p$. Clearly, $g_{p,q}$ and Q are immediately renormalizable with respect to the fixed point α_p and α respectively. More precisely, there exists quasidisks $U_q \Subset V$ and $U \Subset V$ such that

- $g^p_{p,q} : U_q \to V$ is hybrid equivalent to g_q;

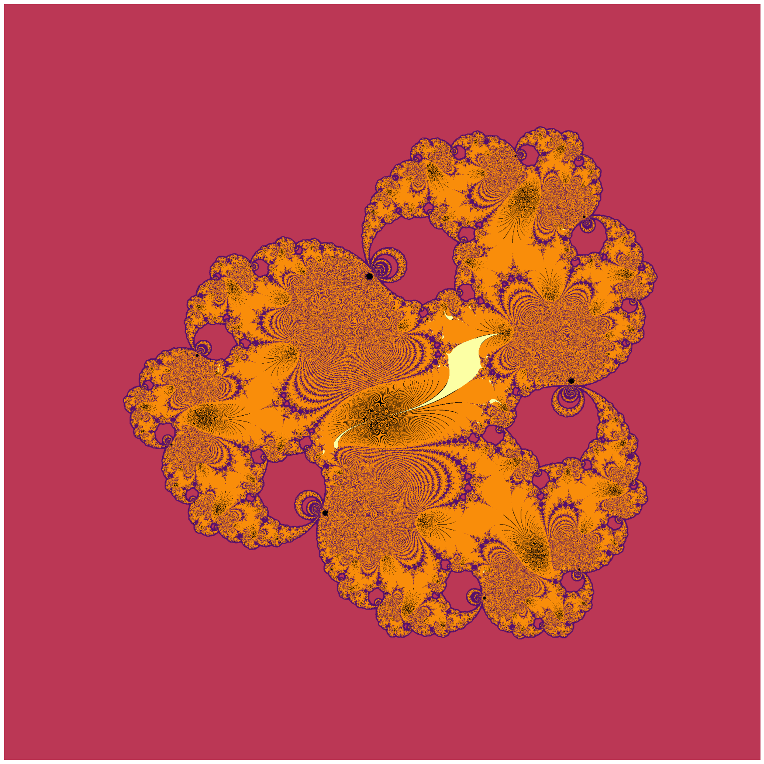

FIGURE 2. The Julia set of $\lambda_0 z(1 + b_0 z + z^2)$ with $\lambda_0 = 0.9987956669... + 0.04906469347...i$ and $b_0 = 1.776923343... + 0.09663115176...i$, which is affinely conjugate to $g_{128,128}$. This is a $1/128$-rotatory of g_{128}, and the yellow regions correspond to the yellow region in Figure 1.

- $Q^p : U \to V$ is hybrid equivalent to g_∞.

Let h_q be a hybrid conjugacy between $g_{p,q}^p|U$ and g_q. Since $g_{p,q}^p$ converges to Q^p, we may assume that the dilatation of h_q is uniformly bounded from above. Thus h_q lies in a compact family. Without loss of generality, we may assume that h_q converges to a quasiconformal map h. Note that $h_q(\alpha_q) = \beta_q$ and $h_q(\alpha'_q) = 0$. Let $q \to \infty$, we have $h(\alpha) = 0 = h(\alpha')$. This lead to a contradiction since $\alpha \neq \alpha'$.

Since the rotation number of $\alpha_{p,q}$ is $1/p$ and $\alpha = \alpha'$ is parabolic, $Q'(\alpha) = e^{2\pi i/p}$ by the Yoccoz inequality [5]. Therefore, by the condition (2) above, Q is affinely conjugate to g_p by Lemma 3.7. Since Q and g_p are monic centered cubic polynomials, Q is either equal to g_p or $-g_p(-z)$. Hence $\alpha = \alpha'$ is the unique parabolic fixed point for Q.

Since $g_{p,q}$ is a $1/p$-rotatory, $\alpha_{p,q}$ is the landing point of $R_{g_{p,q}}\left(\frac{1}{3^p-1}\right)$. Moreover, since

$$\alpha = \lim_{q\to\infty} \alpha_{p,q} \in \limsup_{q\to\infty} R_{g_{p,q}}\left(\frac{1}{3^p-1}\right)$$

and there is no parabolic fixed point other than α, it follows that α is the landing point of $R_Q(\frac{1}{3^p-1})$. Hence Q is in fact equal to g_p and we have proved

$$\lim_{q\to\infty} g_{p,q} = Q = g_p. \qquad \square$$

Now we repeatedly apply Corollary 3.6 to construct a sequence of finitely renormalizable polynomials. For any given integer $n \geq 1$ and $\underline{q_n} = (q_1, q_2, \cdots, q_n)$, we define $g_{\underline{q_n}} \in \mathcal{C}(\underline{q_n})$ and a sequence of periodic points $(\beta_{\underline{q_n}}, \alpha^{(1)}_{\underline{q_n}}, \ldots, \alpha^{(n)}_{\underline{q_n}})$ by induction on n:

For $n = 1$, we identify $\underline{q_1} = (q)$ with q, and let g_q and β_q be defined as above. Let $\alpha^{(1)}_q = \alpha_q$.

For $n = 2$, let us denote $\underline{q_2} = (q_1, q_2) = (p, q)$. Let $g_{\underline{q_2}} = g_{p,q}$ be as in Lemma 3.9. Let $\beta_{\underline{q_2}}$ be the landing point of 0-external ray and let

$$\alpha^{(1)}_{q_1,q_2} = \alpha_{p,q}, \qquad\qquad \alpha^{(2)}_{q_1,q_2} = \alpha'_{p,q},$$

where $\alpha_{p,q}$ and $\alpha'_{p,q}$ are as in the proof of Lemma 3.9.

For $n \geq 3$, let $(g_{\underline{q_n}}, \alpha^{(1)}_{\underline{q_n}})$ be a $1/q_1$-rotatory of $(g_{\sigma(\underline{q_n})}, \beta_{\sigma(\underline{q_n})})$. (Recall that σ is the shift map.) By taking an affine conjugacy if necessary, we may assume $g_{\underline{q_n}} \in \mathcal{R}(q_1)$ and $\chi(g_{\underline{q_n}}) = g_{\sigma(\underline{q_n})}$. Let $\beta_{\underline{q_n}}$ be the landing point of 0-external ray for $g_{\underline{q_n}}$. Take a hybrid conjugacy h from a renormalization $g^{q_1}_{\underline{q_n}} : U \to V$ to $g_{\sigma(\underline{q_n})}$ and for $k = 2, \ldots, n$, let

$$\alpha^{(k)}_{\underline{q_n}} = h^{-1}(\alpha^{(k-1)}_{\sigma(\underline{q_n})}).$$

Observe that $\alpha^{(k)}_{\underline{q_n}}$ is of period $\prod_{j=1}^{k-1} q_j$ and $\alpha^{(n)}_{\underline{q_n}}$ is a parabolic periodic point whose immediate basin contains a critical point. Let $\omega_{\underline{q_n}}$ and $\omega'_{\underline{q_n}}$ be the critical points of $g_{\underline{q_n}}$ such that

- $\omega_{\underline{q_n}}$ lies in the immediate basin of $\alpha^{(n)}_{\underline{q_n}}$, and
- $g_{\underline{q_n}}(\omega'_{\underline{q_n}}) = g_{\underline{q_n}}(\alpha^{(n)}_{\underline{q_n}})$.

Definition 3.10. We say a polynomial $f \in \mathcal{C}_3$ is *quasiconformally rigid* if there is no polynomial $g \neq f \in \mathcal{C}_3$ such that there exists a quasiconformal conjugacy between f and g which respects the standard external markings of f and g.

The following lemma will be one of the main ingredients in the proof of the main theorem.

Lemma 3.11. *Fix $n \geq 1$. For $\underline{q_n} = (q_1, q_2, \cdots, q_n)$, let $\underline{q_n}k = (q_1, q_2, \cdots, q_n, k)$ be the concatenation of $\underline{q_n}$ and k. Then*

- *$g_{\underline{q_n}}$ is quasiconformally rigid.*
- *$g_{\underline{q_n}k}$ converges to $g_{\underline{q_n}}$ as $k \to \infty$.*

Proof. The proof is by induction on n. For $n = 1$, the conclusion follows from Lemma 3.8 and Lemma 3.9.

Now suppose $n \geq 2$ and the conclusion holds for $n - 1$, we proceed to prove it also holds for n. The proof will be divided into two parts.

We first show that $g_{\underline{q_n}k}$ converges to $g_{\underline{q_n}}$ as $k \to \infty$. By passing to a subsequence, we may assume that $g_{\underline{q_n}k}$ converges to some $Q \in \mathcal{C}_3$. It suffices to prove that $Q = g_{\underline{q_n}}$.

Claim. *Q lies in $\mathcal{R}(q_1)$ and $\chi(Q) = \chi(g_{\underline{q_n}})$, where $\chi : \mathcal{R}(q_1) \to \mathcal{C}_3$ is the corresponding straightening map.*

It follows from the Claim and the injectivity of χ ([9, Theorem B]) that $g_{\underline{q_n}} = Q$.

Proof of the Claim. Let us consider two sequences

$$\underline{q'_{n-1}} = \sigma(\underline{q_n}) = (q_2, q_3, \cdots, q_n),$$
$$\underline{q'_{n-1}}k = \sigma(\underline{q_n}k) = (q_2, q_3, \cdots, q_n, k).$$

Set $h_{n,k} = g_{\underline{q'_{n-1}}k}$ and $h_* = g_{\underline{q'_{n-1}}}$. By hypothesis, h_* is quasiconformally rigid and $h_{n,k} \to h_*$ as $k \to \infty$. Let $\theta = \frac{1}{3^{q_1}-1}$ be a landing angle of $\alpha^{(1)}_{\underline{q_n}k}$ for $f_{\underline{q_n}k}$, which is a repelling fixed point of $g_{\underline{q_n}k}$ with rotation number $1/q_1$. By passing to a subsequence, we may assume that $\alpha^{(1)}_{\underline{q_n}k}$ converges to a fixed point α of Q as $k \to \infty$. By a similar argument in the proof of Lemma 3.3, one can show that α is a repelling fixed point of Q. It follows from [6, Lemma B.1] that θ is an external angle of α for Q and $R_{g_{\underline{q_n}k}}(\theta)$ converges to $R_Q(\theta)$ in the Hausdorff topology.

Let $\theta' = 3^{q_1-1}\theta$ and let $S_{\underline{q_n}k}$ be the sector bounded by $R_{g_{\underline{q_n}k}}(\theta) \cup R_{g_{\underline{q_n}k}}(\theta') \cup \{\alpha^{(1)}_{\underline{q_n}k}\}$. Since $g_{\underline{q_n}k} \in \mathcal{C}(\underline{q_1})$, any critical point ω of $g_{\underline{q_n}k}$ satisfies $g^{jq_1}_{\underline{q_n}k}(\omega) \in \overline{S_{\underline{q_n}k}}$. Therefore the rational lamination of Q contains that of P_{q_1}, so it follows that

$$Q \in \mathcal{C}(q_1) = \{f \in \mathcal{C}_3 \mid \lambda(f) \supset \lambda(P_{q_1})\}.$$

By [9, Lemma 5.13], $Q \in \mathcal{R}(q_1)$ and there exists an analytic family of cubic-like renormalizations near Q. Then as in [3, Chapter II, Section 7], we may choose hybrid conjugacies ϕ_n between $\chi(g_{\underline{q_n}k}) = h_{n,k}$ and the renormalization of $g_{\underline{q_n}k}$ so that the maximal dilatation of ϕ_n is uniformly bounded. Passing to a further subsequence, we see that $\chi(Q)$ is quasiconformally conjugate to $\lim_{k\to\infty} \chi(g_{\underline{q_n}k})$ respecting the standard external markings. Note that

$$\lim_{k\to\infty} \chi(g_{\underline{q_n}k}) = \lim_{k\to\infty} h_{n-1,k} = h_* = \chi(g_{\underline{q_n}}).$$

Thus $\chi(Q)$ is quasiconformally conjugate to $\chi(g_{\underline{q_n}})$ respecting the standard external markings. By hypothesis, $\chi(g_{\underline{q_n}}) = h_*$ is quasiconformally rigid. Hence $\chi(g_{\underline{q_n}}) = \chi(Q)$.

Now we proceed to prove that $g_{\underline{q_n}}$ is quasiconformally rigid. Assume that there exists a $\widetilde{f} \in \mathcal{C}_3$ and a quasiconformal conjugacy ψ between $\widetilde{f}$ and $g_{\underline{q_n}}$ which respects the standard external markings. We must have $\lambda(g_{\underline{q_n}}) = \lambda(\widetilde{f})$. By the definition of $g_{\underline{q_n}}$, $g_{\underline{q_n}}$ lies in $\mathcal{R}(q_1)$ and $\chi(g_{\underline{q_n}}) = h_*$. Thus $\widetilde{f}$ also lies in $\mathcal{R}(q_1)$. Note that conjugacy ψ naturally induced a quasiconformal conjugacy between $\chi(\widetilde{f})$ and $\chi(g_{\underline{q_n}}) = h_*$ respecting the standard external markings. Since h_* is quasiconformally rigid, we have $\chi(\widetilde{f}) = h_* = \chi(g_{\underline{q_n}})$. It follows from the injectivity of χ ([9, Theorem B]) that $\widetilde{f} = g_{\underline{q_n}}$. $\square$

Theorem 3.12. *Let $\underline{q} = (q_1, q_2, \dots)$ satisfy $q_j \geq 2$ for all j. If q_j grows sufficiently fast, then there exists some $g_{\underline{q}} \in \mathcal{C}(\underline{q})$ with two distinct critical points.*

Proof. Fix $q_1 \geq 2$. Recall that g_{q_1} has a repelling fixed point α with rotation number $1/q_1$. Moreover, one critical point ω_1 lies in the parabolic basin and the other ω'_1 is mapped to α by g_{q_1}. In particular, $\omega_1 \neq \omega'_1$. Let $\eta = |\omega_1 - \omega'_1|$.

By Lemma 3.11, if q_n grows sufficiently fast, then the sequence of cubic polynomials $g_{\underline{q_n}}$ constructed as above has the following properties:

- for any $n \geq 2$, $g_{\underline{q_n}} \in \mathcal{C}(\underline{q_n})$ where $\underline{q_n} = (q_1, q_2, \cdots, q_n)$.
- for any $n \geq 2$, $g_{\underline{q_n}}$ has two critical points ω_n and ω'_n.
- for any $n \geq 2$, $|\omega_n - \omega_{n-1}| < \frac{\eta}{10^n}$ and $|\omega'_n - \omega'_{n-1}| < \frac{\eta}{10^n}$.

Passing to a subsequence, we may assume that $g_{\underline{q_n}}$ converges to a polynomial $g_{\underline{q}} \in \mathcal{C}_3$ and ω_n (resp. ω'_n) converges to a critical point ω (resp. ω') of $g_{\underline{q}}$. Note that

$$|\omega - \omega_1| \leq \sum |\omega_n - \omega_{n-1}| < \frac{\eta}{10} \text{ and } |\omega' - \omega'_1| \leq \sum |\omega'_n - \omega'_{n-1}| < \frac{\eta}{10}.$$

Hence $\omega \neq \omega'$.

It follows from Lemma 3.3 that $g_{\underline{q}} \in \mathcal{C}(\underline{q})$. Clearly, $g_{\underline{q}}$ is infinitely cubic-like renormalizable. $\square$

4. Combinatorial non-rigidity

For any infinite sequence $\underline{q} = (q_1, q_2, \cdots)$ with $q_j \geq 2$ for all j, there exists a unicritical polynomial $P_{\underline{q}}(z) = z^3 + c_{\underline{q}} \in \mathcal{C}(\underline{q})$. Hence under the assumption of Theorem v3.12, we have obtained two distinct polynomials $P_{\underline{q}}$ and $g_{\underline{q}}$ in $\mathcal{C}(\underline{q})$.

Lemma 4.1. *The combinatorial class of $P_{\underline{q}}$ is equal to $\mathcal{C}(\underline{q})$, i.e.,*

$$\mathcal{C}(\underline{q}) = \{f \in \mathcal{C}_3 \mid \lambda(f) = \lambda(P_{\underline{q}})\}.$$

The following holds from the lemma and Theorem 3.12:

Corollary 4.2. *If $\underline{q} = (q_1, q_2, \dots)$ satisfy Theorem 3.12, then $P_{\underline{q}}$ is not combinatorially rigid.*

Proof. Let $g_{\underline{q}} \in \mathcal{C}(\underline{q})$ be as in Theorem 3.12.

By the lemma, we have $\lambda(g_{\underline{q}}) = \lambda(P_{\underline{q}})$. Now assume the composition of the Böttcher coordinates

$$\phi_{g_{\underline{q}}} \circ \phi_{P_{\underline{q}}} \colon (\mathbb{C} \setminus K(P_{\underline{q}})) \to (\mathbb{C} \setminus K(g_{\underline{q}}))$$

extends to a quasiconformal homeomorphism h on $\hat{\mathbb{C}}$.

Since both $P_{\underline{q}}$ and $g_{\underline{q}}$ do not have any bounded Fatou component, we have

$$\overline{(\mathbb{C} \setminus K(P_{\underline{q}}))} = \overline{(\mathbb{C} \setminus K(g_{\underline{q}}))} = \hat{\mathbb{C}},$$

hence $h \colon \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a quasiconformal conjugacy from $P_{\underline{q}}$ to $g_{\underline{q}}$.

However, there is no topological conjugacy from $P_{\underline{q}}$ to $g_{\underline{q}}$ because $P_{\underline{q}}$ is unicritical and $g_{\underline{q}}$ is not. Therefore, there is no such a quasiconformal extension and the combinatorial non-rigidity for $P_{\underline{q}}$ holds. $\square$

Proof of Lemma 4.1. By definition,

$$\mathcal{C}(\underline{q}) = \{f \in \mathcal{C}_3 \mid \lambda(f) \supset \lambda(P_{\underline{q}_n}) \text{ for any } n\}.$$

Hence it suffices to prove that any $f \in \mathcal{C}(\underline{q})$ and any $\lambda(f)$-equivalent $\theta_1, \theta_2 \in \mathbb{Q}/\mathbb{Z}$, there exists some n such that θ_1 and θ_2 are $\lambda(P_{\underline{q}_n})$-equivalent.

Let us denote $\lambda_n = \lambda(P_{\underline{q}_n})$. Now assume θ_1 and θ_2 are not λ_n-equivalent for a given n. Then they are contained in an infinite λ_n-unlinked class (gap) L (see [9] and [12] for details), hence for the common landing point x of $R_f(\theta_1)$ and $R_f(\theta_2)$ is contained in

$$\bigcup_{j\geq 0} f^{-j}(K_n),$$

where K_n is the filled Julia set of a λ_n-renormalization (in other words, n-th renormalization) of f. In particular, the eventual period of θ_1 under $\theta \mapsto 3\theta$ is greater than the $\prod_{k=1}^{n-1} q_k$ (note that this is the period of $\alpha^{(n)}_{\underline{q}_n}$ for $g_{\underline{q}_n}$).

Hence θ_1 and θ_2 are λ_n-equivalent for sufficiently large n. □

Now we prove the following:

Theorem 4.3. *If $\underline{q} = (q_1, q_2, \dots)$ satisfy Theorem 3.12, then there exists a continuum connecting $P_{\underline{q}}$ and $g_{\underline{q}}$ contained in $\mathcal{C}(\underline{q})$*

Proof. We just modify the construction in Section 3.

First, recall that P_q is the center of the $1/q$-satellite copy of $\mathcal{M}_3$ and $P_{\underline{q}_n} \in \mathcal{C}(\underline{q}_n)$ is the hyperbolic post-critically finite unicritical polynomial such that $\chi_{q_1}(P_{\underline{q}_n}) = P_{\underline{q}_{n-1}}$ for $n \geq 2$.

Now we construct a sequence $(\gamma_n)_n$ of paths with $\gamma_n \subset \overline{H}_n$. First take an arbitrary continuous path $\gamma_1 \colon [0,1] \to \overline{\mathcal{H}_1}$ such that

- $\gamma_1(0) = P_{q_1}$ and $\gamma_1(1) = g_{q_1}$,
- $\gamma_1(t) \in \mathcal{H}_1$ for $t \in [0,1)$.

Then apply the same construction as $g_{\underline{q}_n}$, starting from every map $g \in \gamma_1$, we obtain a continuous path γ_n connecting $P_{\underline{q}_n}$ and $g_{\underline{q}_n}$ in $\overline{\mathcal{H}_n}$.

Then the set

$$\mathcal{K} = \bigcap_{N\geq 0} \overline{\bigcup_{n\geq N} \gamma_n([0,1])}$$

is a continuum containing $P_{\underline{q}}$ and $g_{\underline{q}}$. It follows by Lemma 3.3 that $\mathcal{K} \subset \mathcal{C}(\underline{q})$. □

5. Near-parabolic renormalizations

In this section, we introduce another kind of renormalization, which is called the *near-parabolic renormalization* [10].

We use the notations in Section 3. Let $\underline{q} = (q_1, q_2, \dots)$ satisfy Theorem 3.12 and let $g_{\underline{q}}$ be as in the theorem. As in the proof, $g_{\underline{q}}$ is a subsequential limit of $g_{\underline{q}_n}$, so $g_{\underline{q}}$ has a sequence of periodic points

$$(\beta_{\underline{q}}, \alpha^{(1)}_{\underline{q}}, \alpha^{(2)}_{\underline{q}}, \dots)$$

which is a subsequential limit of $(\beta_{\underline{q}_n}, \alpha^{(1)}_{\underline{q}_n}, \alpha^{(2)}_{\underline{q}_n}, \dots, \alpha^{(n)}_{\underline{q}_n})$. Then $\alpha^{(n-1)}_{\underline{q}}$ and $\alpha^{(n)}_{\underline{q}}$ are fixed points of n-th (polynomial-like) renormalization of $g_{\underline{q}}$. Similarly, let $\omega_{\underline{q}}$ and $\omega'_{\underline{q}}$ be the critical points which are subsequential limits of ω_n and ω'_n respectively. Let

$$p_n = \prod_{j=1}^{n-1} q_j$$

be the period of $\alpha^{(n)}_{\underline{q}}$.

The purpose of this section is to prove the following:

Theorem 5.1. *Let $\underline{q} = (q_1, q_2, \dots)$. If we further assume every q_j is sufficiently large and q_j grows sufficiently fast (that is, possibly even faster than Theorem 3.12), then there exist a universal constant $k \in \mathbb{N}$ and a decreasing sequence $(\Omega_n)_{n\in\mathbb{N}}$ of open sets such that*

1. $\overline{\Omega_{n+1}} \subset \Omega_n$.
2. Ω_n *contains the periodic orbit of* $\alpha_{\underline{q}}^{(n)}$ *for* $g_{\underline{q}}$,
3. $\omega_{\underline{q}} \in \Omega_n$, *but* $\omega'_{\underline{q}} \notin \Omega_n$.
4. $g_{\underline{q}}^j(\omega_{\underline{q}}), g_{\underline{q}}^j(\omega'_{\underline{q}}) \in \Omega_n$ *for* $1 \le j \le p_n - kp_{n-1}$.

In particular, the postcritical set is contained in

$$\bigcap_{n\ge 0} \overline{\Omega_n},$$

hence $\omega'_{\underline{q}}$ *is not recurrent.*

Recall that the *postcritical set* of a given polynomial f of degree at least two is defined by

$$P(f) = \overline{\bigcup_{n\ge 1} f^n(\mathrm{Crit}(f))}$$

where $\mathrm{Crit}(f)$ is the set of critical points of f.

5.1. **Near-parabolic renormalizations.** Let $P(z) = z(1+z)^2$ and $V = \psi_1(\hat{\mathbb{C}}\setminus E)$, where

$$E = \left\{ z = x+iy \in \mathbb{C} \,\middle|\, \left(\frac{x+0.18}{1.24}\right)^2 + \left(\frac{y}{1.04}\right)^2 \le 1 \right\}, \tag{5.1}$$

$$\psi_1(z) = -\frac{4z}{(1+z)^2}.$$

Consider the following family of holomorphic maps:

$$\mathcal{F}_1 = \left\{ h = P\circ\varphi^{-1} \colon \varphi(V) \to \mathbb{C} \,\middle|\, \begin{array}{l} \varphi\colon V \to \varphi(V)\text{: univalent, } \varphi(0)=0,\ \varphi'(0)=1, \\ \varphi \text{ has a quasiconformal extension to } \mathbb{C} \end{array} \right\}.$$

For $h = P\circ\varphi^{-1} \in \mathcal{F}_1$, let $\mathrm{Dom}(f) = \varphi(V)$ be the domain of definition of h.

For a set $X \in \mathbb{C}$, let

$$X \ltimes \mathcal{F}_1 = \{ f(z) = h(e^{2\pi i\alpha} z) \mid \alpha \in X,\ h \in \mathcal{F}_1 \}.$$

For $f(z) = h(e^{2\pi i\alpha}z) \in X \ltimes \mathcal{F}_1$, let

$$\mathrm{Dom}(f) = e^{-2\pi i\alpha}\,\mathrm{Dom}(h) = e^{-2\pi i\alpha}\varphi(V)$$

denote the domain of definition of f.

Now let $\alpha_* > 0$ be small and let

$$\begin{aligned} \blacktriangleleft &= \{\alpha \in \mathbb{C} \mid 0 < |\alpha| < \alpha_*,\ |\arg\alpha| < \tfrac{\pi}{4}\}, \\ \blacktriangleright &= \{\alpha \in \mathbb{C} \mid 0 < |\alpha| < \alpha_*,\ |\arg(-\alpha)| < \tfrac{\pi}{4}\}, \\ \blacktriangleright\!\!\blacktriangleleft &= \blacktriangleright \cup \blacktriangleleft. \end{aligned}$$

In the following, we consider maps of the form $f(z) = h(e^{2\pi i\alpha}h) \in \blacktriangleleft \ltimes \mathcal{F}_1$. The argument for the case $\alpha \in \blacktriangleright$ is parallel. In fact, if $f(z) = h(e^{2\pi i\alpha}z) \in \blacktriangleright \ltimes \mathcal{F}_1$, it is conjugate by complex conjugate to the map of the form

$$\hat{f}(z) = \overline{f(\overline{z})} = \overline{h(e^{2\pi i(-\alpha)}\overline{z})},$$

which lies in $\blacktriangleleft \ltimes \mathcal{F}_1$ since we have $-\alpha \in \blacktriangleleft$ and $\overline{h(\overline{z})} \in \mathcal{F}_1$. Note that V is symmetric with respect to the real axis.

Proposition 5.2. *If α_* is sufficiently small, there exists a constant $\boldsymbol{k} > 0$ and a neighborhood W of the origin such that the following holds:*

For any $f = h(e^{2\pi i\alpha}z) \in \blacktriangleleft \ltimes \mathcal{F}_1$, f has a unique non-zero fixed point $\sigma_f \in W$, and there exist a domain $\mathcal{P}_f \Subset \mathrm{Dom}(f)$, and a univalent map $\Phi_f \colon \mathcal{P}_f \to \mathbb{C}$ such that

1. *The boundary of $\mathcal{P}_f$ is a piecewise smooth curve containing 0, σ_f and $cp_f = -\frac{1}{3}$.*
2. *$\operatorname{Im}\Phi_f(z) \to +\infty$ as $z \to 0$ and $\operatorname{Im}\Phi_f(z) \to -\infty$ as $z \to \sigma_f$ in $\mathcal{P}_f$.*
3. *$\Phi_f(cp_f) = 0$ and*
$$\{w \in \mathbb{C} \mid \operatorname{Re} w \in (0,2)\} \subset \Phi_f(\mathcal{P}_f).$$
4. *$\Phi_f(f(z)) = \Phi_f(z) + 1$, where both sides are defined.*
5. *Φ_f is uniquely determined by the above conditions.*
6. *Φ_f depends holomorphically on (α, h).*
7. *$\Phi_f(\mathcal{P}_f) = \{w \in \mathbb{C} \mid 0 < \operatorname{Re} w < \operatorname{Re}(\frac{1}{\alpha}) - \boldsymbol{k}\}$.*

See [2, Propositions 2.1–2.3]. When $\alpha \in \blacktriangleright$, the same statement holds with (2) replaced by

(2') $\operatorname{Im}\Phi_f(z) \to -\infty$ as $z \to 0$ and $\operatorname{Im}\Phi_f(z) \to +\infty$ as $z \to \sigma_f$ in $\mathcal{P}_f$.

The Φ_f above is called the (normalized) *Fatou coordinate* of f.

For $f \in \blacktriangleright\!\!\blacktriangleleft \ltimes \mathcal{F}_1$, define $\alpha = \alpha(f)$ and $\beta = \beta(f)$ by

$$f'(0) = \exp(2\pi i\alpha(f)), \qquad f'(\sigma_f) = \exp(2\pi i\beta(f)). \tag{5.2}$$

Now let

$$\begin{aligned} A_{f,\pm} &:= \Phi_f^{-1}(\{w \in \mathbb{C} \mid \tfrac{1}{2} \le \operatorname{Re} w \le \tfrac{3}{2},\ \pm\operatorname{Im} w \ge 2\}), \\ C_f &:= \Phi_f^{-1}(\{w \in \mathbb{C} \mid \tfrac{1}{2} \le \operatorname{Re} w \le \tfrac{3}{2},\ -2 \le \operatorname{Im} w \le 2\}). \end{aligned} \tag{5.3}$$

Proposition 5.3. *Under the assumption of Proposition 5.2, there exist a positive integer $k_f^\pm$, domains $A_{f,\pm}^{-k_f^\pm}$ and $C_{f,\pm}^{-k_f^\pm}$ such that*

1. *The fixed point 0 lies in the boundary of $A_{f,+}^{-k_f^+}$, and σ_f lies in the boundary of $A_{f,-}^{-k_f^-}$.*
2. *For $0 \le k < k_f^\pm$, let*
$$A_{f,\pm}^{-k} = f^{k_f^\pm - k}(A_{f,\pm}^{-k_f^\pm}), \qquad C_{f,\pm}^{-k} = f^{k_f^\pm - k}(C_{f,\pm}^{-k_f^\pm}).$$
Then $A_{f,\pm}^0 = A_{f,\pm}$ and $C_{f,\pm}^0 = C_f$.
3. *We have*
$$A_{f,\pm}^{-k_f^\pm}, C_{f,\pm}^{-k_f^\pm} \subset \left\{ z \in \mathcal{P}_f \mid \frac{1}{2} < \operatorname{Re}\Phi_f(z) < \operatorname{Re}\frac{1}{\alpha} - \boldsymbol{k} \right\}.$$
4. *The maps*
$$\begin{aligned} f \colon A_{f,\pm}^k &\to A_{f,\pm}^{k+1} \quad (-k_f^\pm \le k \le -1), \\ f \colon C_{f,\pm}^k &\to C_{f,\pm}^{k+1} \quad (-k_f^\pm \le k \le -2) \end{aligned}$$

are univalent, and

$$f\colon C^{-1}_{f,\pm} \to C^{0}_{f,\pm}$$

is a proper branched covering of degree two.

See [2, Propositions 2.6, 2.7].

The petal $\mathcal{P}_f$ approaches to the fixed points 0 and σ_f as $\operatorname{Im}\Phi_f(z)$ tends to $\pm\infty$. Asymptotically, it approaches the fixed points in logarithmic spirals when $\operatorname{Im}\alpha, \operatorname{Im}\beta \neq 0$:

Proposition 5.4. *There exists a constant k' such that the following holds:*

Under the situation in Proposition 5.2, there exist continuous branches of $\arg w$ *on $\mathcal{P}_f$ and on $\mathcal{P}_f - \sigma_f$ such that*

(1) *For all $\zeta \in (0, \operatorname{Re}\frac{1}{\alpha} - \boldsymbol{k})$,*

$$\lim_{\xi\to+\infty}(\arg\Phi_f^{-1}(\zeta+i\xi) + 2\pi\xi\operatorname{Im}\alpha) = \arg\sigma_f + 2\pi\zeta\operatorname{Re}\alpha + c_f.$$

(2) *For all $\zeta \in (0, \operatorname{Re}\frac{1}{\alpha} - \boldsymbol{k})$,*

$$\lim_{\xi\to-\infty}(\arg(\Phi_f^{-1}(\zeta+i\xi) - \sigma_f) + 2\pi\xi\operatorname{Im}\beta) = \arg\sigma_f + 2\pi\zeta\operatorname{Re}\beta + c'_f.$$

where $\alpha = \alpha(f)$ and $\beta = \beta(f)$, and c_f, c'_f are constants depending on f with $|c_f|, |c'_f| \leq k'(1 - \log|\alpha|))$.

See [2, Proposition 2.4].

Let

$$S_f^{\pm} := A_{f,\pm}^{-k_f^{\pm}} \cup C_{f,\pm}^{-k_f^{\pm}}.$$

Then we can define the *lifted horn maps* $\tilde{\mathcal{E}}_f^{\pm}$ by

$$\tilde{\mathcal{E}}_f^{\pm} := \Phi_f \circ f^{k_f^{\pm}} \circ \Phi_f^{-1}\colon \Phi_f(S_f^{\pm}) \to \mathbb{C}.$$

Then $\tilde{\mathcal{E}}_f^{\pm}$ satisfies the following functional equation

$$\tilde{\mathcal{E}}_f^{\pm}(w+1) = \tilde{\mathcal{E}}_f^{\pm}(w) + 1$$

where both sides are defined (that is, on the "left" boundary arc).

Therefore, there exist well-defined holomorphic maps $\mathcal{E}_f^{\pm}$ defined near the origin such thatp

$$\mathcal{E}_f^{\pm}(z) \circ \operatorname{Exp}^{\pm} = \operatorname{Exp}^{\pm} \circ \tilde{\mathcal{E}}_f^{\pm},$$

where

$$\operatorname{Exp}^{\pm}(w) := -\frac{4}{27}\exp(\pm 2\pi i w).$$

As in the case of parabolic renormalizations, the origin is a removable singularity and they have the forms

$$\mathcal{E}_f^{+}(z) = e^{-2\pi i\frac{1}{\alpha}}z + O(z^2), \qquad \mathcal{E}_f^{-}(z) = e^{-2\pi i\frac{1}{\beta}}z + O(z^2).$$

The maps $\mathcal{E}^{\pm}$ are called the *horn maps*.

Theorem 5.5 ([10])**.** *If α_* is sufficiently small, then for any map $f(z) = h(e^{2\pi i\alpha}z) \in \blacktriangleleft \ltimes \mathcal{F}_1$, the horn maps $\mathcal{E}_f^{\pm}$ can be extended to a map of the form*

$$\mathcal{E}_f^{+}(z) = h_+(e^{-2\pi i\frac{1}{\alpha}}z), \qquad \mathcal{E}_f^{-}(z) = h_-(e^{-2\pi i\frac{1}{\beta}}z)$$

with $h_\pm \in \mathcal{F}_1$.

Moreover, $h_\pm$ depends holomorphically on f.

See [10] for the complex structure on $\mathcal{F}_1$. We only need the continuity of $(\alpha, h) \mapsto h_\pm$. As above, the statement hold also for the case $\alpha \in \blacktriangleright$, by exchanging α and β.

Now we have the *upper* renormalization h_+ and the *lower* renormalization h_-. For near parabolic case, it is rather important which fixed point the renormalization under considerration goes around, hence it is often more convinient to use the notion of *top* and *bottom* renormalizations, following Cheraghi and Shishikura [2]:

Definition 5.6. We denote the extensions of $\mathcal{E}_f^+$ (resp. $\mathcal{E}_f^-$) in the above theorem by $\mathcal{R}_t(f)$ (resp. $\mathcal{R}_b(f)$) and call it the *top* (resp. *bottom*) *near-parabolic renormalizations* of f.

For the case $f \in \blacktriangleright \ltimes \mathcal{F}_1$, We denote the extension $\mathcal{E}^-$ by $\mathcal{R}_t(f)$ and that of $\mathcal{E}^+$ by $\mathcal{R}_b(f)$.

In other words, we define the top and bottom renormalizations in a consistent way by the following property:

- the *top* near-parabolic renormalization is (the extension of) the horn map at the origin, and
- the *bottom* near-parabolic renormalization is (the extension of) the horn map at the bifurcated fixed point σ_f.

Recall the neighborhood W of 0 in Proposition 5.2. We may assume $f \in \{\alpha\} \times \mathcal{F}_1$ contains a unique non-zero fixed point σ_f in W not only if $\alpha \in \blacktriangleright\!\blacktriangleleft$ but also if $\alpha \in \mathbb{D}_{\alpha_*} = \{\alpha \in \mathbb{C} \mid |\alpha| < \alpha_*\}$. Thus $\beta(f) = \frac{1}{2\pi i} \log f'(\sigma_f)$ is defined even in this case.

Lemma 5.7. *There exists some $\alpha_{**} < \alpha_*$ such that for $f \in \{\alpha\} \ltimes \mathcal{F}_1$ with $|\alpha| < \alpha_*$, if $\beta \in \mathbb{R}$ and $|\beta| < \alpha_{**}$, then $\alpha = \alpha(f) \in \blacktriangleright\!\blacktriangleleft$.*

In particular, the top and the bottom near-parabolic renormalizations are defined for f.

See also [2, Lemma 6.8].

Proof. For $f \in \blacktriangleright\!\blacktriangleleft \ltimes \mathcal{F}_1$, let

$$I(f) := \frac{1}{2\pi i} \int_{\partial W} \frac{dz}{z - f(z)}.$$

By the precompactness of $\mathcal{F}_1$, $I(f)$ depends continuously (even holomorphically) on f and is bounded (see [2, Lemma 3.24] for more detail).

Thus the lemma follows easily by the equality

$$I(f) = \frac{1}{1 - f'(0)} + \frac{1}{1 - f'(\sigma_f)}.$$

In fact, if $\beta > 0$ is sufficiently small, then $I(f, \sigma_f)) = \frac{1}{1 - f'(\sigma_f)}$ is large and

$$\frac{\operatorname{Re} I(f, \sigma_f)}{\operatorname{Im} I(f, \sigma_f)}$$

is close to 0. Hence

$$I(f, 0) = \frac{1}{1 - f'(0)} = I(f) - I(f, \sigma_f)$$

has large negative imaginary part and uniformly bounded real part. This implies $\alpha \in \blacktriangleright\!\blacktriangleleft$.

The case $\beta < 0$ is similar. $\square$

Note that the assumption that $\beta \in \mathbb{R}$ is too strong, and we only need to assume that $|\operatorname{Re}\beta| > k|\operatorname{Im}\beta|$ for some $k > 1$. Also, if $\beta > 0$, then we have $\operatorname{Re}\alpha < 0$, and vice versa.

6. Deep renormalizations

Here we consider the case that a given map $f \in \blacktriangleright\!\!\blacktriangleleft \ltimes \mathcal{F}_1$ is several times near-parabolic renormalizable, especially the case of bottom renormalizations.

In this section, we fix $f = f_0 \in \blacktriangleright\!\!\blacktriangleleft \ltimes \mathcal{F}_1$, which is (at least) twice near-parabolic renormalizable. More precisely, $f_1 := \mathcal{R}_*(f)$ $(* = t, b)$ again lies in $\blacktriangleright\!\!\blacktriangleleft \ltimes \mathcal{F}_1$. Let $s_0 \in \{+,-\}$ be such that $\mathcal{R}_*(f) = \mathcal{E}_f^{s_0}$ near the origin.

As in the previous section, there exists a domain $S_0 := S_f^{s_0} \subset \mathcal{P}_f$ such that

$$\operatorname{Exp}^{s_0} \circ \mathcal{R}_*(f) = \Phi_f \circ f^{k_f^{s_0}} \circ \Phi_f^{-1} \circ \operatorname{Exp}^{s_0} \quad (* = t, b).$$

on $\operatorname{Exp}^{s_0}(S_0)$.

Then since $f^{k_f^{s_0}}(S_0) = A_{f,s_0} \cup C_f$, it follows by (5.3), Proposition 5.3 and Proposition 5.2 (7) that

$$\mathcal{U}_0 := \bigcup_{n=0}^{l_0} f^n(S_0)$$

is a neighborhood of 0 (the case $* = t$) or σ_f (the case $* = b$), where

$$l_0 := k_f^{s_0} + \left\lfloor \operatorname{Re}\frac{1}{\alpha} - \boldsymbol{k} \right\rfloor.$$

Now we further assume f is N-times near-parabolic renormalizable, and the j-th renormalization is bottom for $j = 2, \dots, N$. Namely, for each $j = 2, 3, \dots, N$, we assume $f_{j-1} \in \blacktriangleright\!\!\blacktriangleleft \ltimes \mathcal{F}_1$, hence $f_j := \mathcal{R}_b(f_{j-1})$ is well-defined.

Let $s_j \in \{+,-\}$ satisfy $f_j = \mathcal{E}_{f_j}^{s_j}$ and let

$$S_j := S_{f_j}^{s_j}, \qquad l_j := k_{f_j^{s_j}} + \left\lfloor \operatorname{Re}\frac{1}{\alpha(f_j)} - \boldsymbol{k} \right\rfloor,$$
$$\mathcal{U}_j := \bigcup_{n=0}^{l_j} f_j^n(S_j).$$

Since f_j is the bottom renormalization of f_{j-1} for $j \geq 2$, $\mathcal{U}_j$ is a neighborhood of $\sigma_{j-1} := \sigma_{f_{j-1}}$.

We want to define a set $\mathcal{U}_j^m$ $(0 \leq m < j \leq N)$ in the phase space for f_m which is a neighborhood of a periodic orbit of f_m and "corresponds to" $\mathcal{U}_j =: \mathcal{U}_j^{j-1}$.

To this end, we want to define an appropriate inverse branch η_j of $\operatorname{Exp}^{s_j} \circ \Phi_j$ on $\mathcal{U}_{j+1}$, where $\Phi_j = \Phi_{f_j}$ is the Fatou coordinate for f_j. Observe that $\mathcal{P}_{j+1}$ approaches the fixed point at 0 in a logarithmic spiral, the image of every inverse branch of $\operatorname{Exp}^{\pm}$ on $\mathcal{P}_{j+1}$ does not contained in the vertical strip $\Phi_j(\mathcal{P}_j)$. However, this spiral part at 0 is not contained in $\mathcal{U}_j$, and we have the following:

Lemma 6.1. *There exists a universal constant $C > 0$ such that we can choose a contiuous branch of the argument on $\mathcal{U}_j$ such that $|\arg z| < C$ for all $z \in \mathcal{U}_j$.*

In particular, if α_{j-1} is sufficiently large, then there exists a continuous inverse branch η_j of $\operatorname{Exp}^{s_{j-1}}$ on $\mathcal{U}_j$ such that

$$\eta_j(\mathcal{U}_j) \subset \mathcal{P}_{j-1}.$$

Proof. We may assume $\alpha_j = \alpha(f_j) > 0$. First we recall some results in [18] and [10]. Consider the *pre-Fatou coordinate* $z = \tau_j(w)$ defined by

$$\tau_j(w) := \frac{\sigma_j}{1 - e^{-2\pi i \alpha_j w}}.$$

Let $F_j(w)$ be the lift of f_j by τ_j such that

$$F_j(w) = w + 1 + o(1) \quad \text{as } \operatorname{Im} w \to +\infty.$$

Then there exists a domain $\tilde{\mathcal{P}}_j$ such that τ_j maps $\tilde{\mathcal{P}}_j)$ conformally to $\mathcal{P}_j$.

Recall that $\tau_j(w)$ converges locally uniformly to $-\frac{1}{w}$ as $\alpha_j \to 0$ [18, § 3.3.2], and $F_j(w)$ also converges to

$$\mathcal{F}_1^Q = \left\{ H = Q \circ \varphi^{-1} \colon \varphi(V) \to \mathbb{C} \;\middle|\; \begin{array}{l} \varphi\colon V^Q \to \varphi(V^Q)\text{: univalent,} \\ \varphi(\infty) = \infty,\ \lim_{z\to\infty} \frac{\varphi(z)}{z} = 1, \\ \varphi \text{ has a quasiconformal extension to } \hat{\mathbb{C}} \end{array} \right\},$$

up to affine conjugacy, where $V^Q = \hat{C} \setminus E$ and

$$Q(z) = z\frac{\left(1 + \frac{1}{z}\right)^6}{\left(1 - \frac{1}{z}\right)^4}.$$

Moreover, if $F_j \to H \in \mathcal{F}_1^Q$ as $\alpha_j \to 0$ (by taking an affine conjugate, and passing to a subsequence if necessary), the Fatou coordinate $\Phi_j \circ \tau_j$ for F_j converges to an attracting Fatou coordinates $\Phi_{H,\mathrm{attr}}$ of H locally uniformly on some domain $\mathrm{Dom}(\Phi_{H,\mathrm{attr}})$ containing a right half plane.

Moreover, $\Phi_{H,\mathrm{attr}}(cv_H) = 1$ where $cv_H = 27$ is the unique critical value, and the image $\Phi_{H,\mathrm{attr}}(\mathrm{Dom}(\Phi_{H,\mathrm{attr}}))$ contains the right half plane $\mathbb{H}_+ = \{z \in \mathbb{C} \mid \operatorname{Re} z > 0\}$ [10, Propositions 5.6, 5.7]. Also,

%%%

This essentially follows from [10, §5].

In fact, let $h \in \mathcal{F}_1$ (i.e., consider the parabolic case $\alpha = 0$) and let $H = \psi_0 \circ f \circ \psi_0^{-1}$ where $\psi_0(z) = -\frac{4}{z}$. The map H can be written as $H = Q \circ \phi^{-1}$ where $Q(z) = z\frac{\left(1+\frac{1}{z}\right)^6}{\left(1-\frac{1}{z}\right)^4}$ and ϕ is a univalent map tangent to the identity at infinity, defined outside the ellipse E defined by (5.1).

Then ∞ is the corresponding parabolic fixed point for H and its attracting petal

$$\mathcal{P}_H = \Phi_{H,\mathrm{attr}}^{-1}(\{z \in \mathbb{C} \mid \operatorname{Re} z > 1\})$$

is contained in

$$\mathbb{V} = \mathbb{V}(u_0, \tfrac{2\pi}{3}) = \{z \in \mathbb{C} \mid z \neq u_0,\ |\arg(z - u_0)| < \tfrac{2\pi}{3}\},$$

where $\Phi_{H,\mathrm{attr}}$ is the attracting Fatou coordinate normalized so that $\Phi_{H,\mathrm{attr}}(cv_H) = 1$, $cv_H = 27$ is the critical value, and $u_0 = \frac{25}{\sqrt{3}}$ [10, Propositon 5.6].

For simplicity, we consider the upper renormalization. The *parabolic renormalization* of f is obtained by taking appropriate inverse images of the closures of the following domains

$$\begin{aligned} D_1 &= \Phi_{H,\mathrm{attr}}^{-1}(\{z \mid 1 < \operatorname{Re} z < 2,\ -2 < \operatorname{Im} z < 2\}), \\ D_1^\sharp &= \Phi_{H,\mathrm{attr}}^{-1}(\{z \mid 1 < \operatorname{Re} z < 2,\ \operatorname{Im} z > 2\}). \end{aligned}$$

Let $\mathcal{U}$ be the union of those inverse images (i.e., U in [10, §5]) and $\bigcup_{n=0}^{\infty} H^n(\overline{D_1 \cup D_1^\sharp})$. Then $\mathcal{U}$ is a simply connected domain in C^*.

Then the argument in [10, §5] shows that those inverse images can be regarded as subsets of a Riemann surface

$$(\mathbb{V} \sqcup X)/\sim,$$

where X is the Riemann surface defined in [10, §5] with the natural projection $\pi_X \colon X \to \mathbb{C}$, and $\sim$ is defined by

$$\mathbb{V} \ni z \sim_{\pm} w \in X \iff z = \pi_X(w), \text{ and } w \in X_{1+} \cup X_{2-}.$$

Moreover, there is a continuous branch of the argument on X taking values in $(-\pi, 3\pi)$. In particular, there exists an arc $\gamma : [0,1) \to \mathbb{C} \setminus \mathcal{U}$ such that

$$\gamma(0) = 0, \qquad \lim_{t \nearrow 1} \operatorname{Im} \gamma(t) \to -\infty,$$
$$\arg \gamma(t) \in (-2\pi, 4\pi).$$

Recall that f_j is a small perturbation of some map $h \in \mathcal{F}_1$. (Precisely speaking, we take α_* suffuciently small such that the following hold.) Consider the *pre-Fatou coordinate* $z = \tau_j(w)$ defined by

$$\tau_j(w) := \frac{\sigma_j}{1 - e^{-2\pi i \alpha_j w}}.$$

where $\alpha_j = \alpha(f_j)$.

Define $\tilde{S}_j$. Let

$$\tilde{\mathcal{U}_j} := \bigcup_{n=0}^{l_j} F_{j-1}^n(\tilde{S}_j).$$

Then by the construction and the continuity of Fatou coordinates, it follows that $\tilde{\mathcal{U}_j}$ is contained in a small perturbation of $\mathcal{U}$. inconsistent indices!

In particular, we still have an arc γ as above disjoint from $\tilde{\mathcal{U}_j}$.

□

Let Φ_k be the Fatou coordinate for f_k. It is defined on $\mathcal{P}_k := \mathcal{P}_{f_k}$, which contains S_{k+1}. Note that $V_k := \operatorname{Exp}^{s_k} \circ \Phi_{k-1}(S_k)$ is a punctured neighborhood of 0 and it contains σ_k for $k = 2, \ldots, N-1$. Let

$$\sigma_k^{k-1} := ((\operatorname{Exp}^{s_{k-1}} \circ \Phi_{k-1})|_{S_{k-1}})^{-1}(\sigma_k).$$

Then σ_k^{k-1} is a periodic orbit of f_{k-1}. As $\alpha(f_{k-1})$ tends to zero, the whole perdiodic orbit of σ_k^{k-1} tends to σ_{k-1}.

Now recall some detail on Fatou coordinates (see [18] for more details). For each k, let

$$\tau_k(w) := \frac{\sigma_k}{1 - e^{-2\pi i \alpha_k w}}.$$

We consider new coordinate w, called the *pre-Fatou coordinate* for f_k, defined by $z = \tau_k(w)$ where $\alpha_k = \alpha(f_k)$. Fix a large constant $b > 0$ and let

$$\mathcal{Q}_\alpha := \left\{ w \in \mathbb{C} \mid |\arg(w - b)| < \frac{3}{2}\pi, \ \left| \arg\left(\frac{1}{\alpha} - b - w\right)\right| < \frac{3}{2}\pi \right\}.$$

If α_* is sufficiently small, we can take a lift F_k of f_k by τ_k such that it is univalent on $\mathcal{Q}_{\alpha_k}$ and

$$F_k(w) = w + 1 + o(1) \quad \text{as } \operatorname{Im} w \to +\infty.$$

Then there exists a Fatou coordinate $\tilde{\Phi}_k \colon W_{\alpha_k} \to \mathbb{C}$, i.e., a univalent map satisfying

$$\tilde{\Phi}_k(F_k(w)) = \Phi_k(w) + 1$$

where both sides are defined. Moreover, there exist constants $c_k^{\pm}$ such that

$$\Phi_f(w) = w + c_k^{\pm} + o(1) \tag{6.1}$$

as $w \to \infty$ satisfying $\theta_1' < \arg(\pm(w - w_0) < \theta_2'$ where $w_0 \in \mathbb{C}$ and $\frac{\pi}{4} < \theta_1' < \theta_2' < \frac{3\pi}{4}$.

Lemma 6.2. *If α_* is sufficiently small, there exists a holomorphic extension of an inverse branch η_k of $\mathrm{Exp}^{s_{k-1}} \circ \Phi_{k-1}$ defined on $\mathcal{P}_k$ such that*

$$\eta_k(A_{f,s_k} \cup C_f) \subset \mathcal{P}_{k-1}.$$

Need $\mathcal{Q}$ because of the slope?

Note that by Proposition 5.4, the petal $\mathcal{P}_k$ approach to 0 in logarithmic spiral when $\alpha(f_k) \neq \mathbb{R}$, and this implies that $\eta_k(\mathcal{P}_k)$ is not contained in $\mathcal{P}_{k-1}$.

Proof. □

7. Controlling critical orbits

Now consider the sequence of polynomials $(g_{\underline{q_n}})_{n \in \mathbb{N}}$ for a sequence $\underline{q} = (q_1, q_2, \dots)$ growing sufficiently fast in Theorem 3.12.

Recall that the first element g_{q_1} is affinely conjugate to

$$\hat{g}_{q_1} = e^{2\pi i/q_1} z(1+z)^2 = e^{2\pi i/q_1} P(z),$$

whose linear conjugate satisfies

$$e^{-2\pi i/q_1} \hat{g}_{q_1}(e^{2\pi i/q_i} z) \in \blacktriangleright\!\!\blacktriangleleft \ltimes \mathcal{F}_1.$$

Hence we can consider the near-parabolic renormalizations of g_{q_1} at the parabolic fixed point as those of $\hat{g}_{g_1}$. The top near-parabolic renormalization $\mathcal{R}_t(g_{q_1})$. Then by definition, $\mathcal{R}_t(g_{q_1})$ has a parabolic fixed point at the origin, i.e., $\mathcal{R}_t(g_{q_1}) \in \mathcal{F}_1$ by Theorem 5.5.

Since g_{q_1,q_2} is a perturbation of g_{q_1}, it is natural to expect that g_{q_1,q_2} is twice near-parabolic renormalizable, and so on.

By induction, assume $g_{\underline{q_{n-1}}}$ is $(n-1)$-times near-parabolic renormalizable such that only the first near-parabolic renormalization is top, but the other $(n-2)$ renormalizations are bottom.

Now fix the sequence $\underline{q}$. We see that

$$f_{n,k} := {\mathcal{R}_b}^{k-1} \circ \mathcal{R}_t(g_{\underline{q_n}}) \quad (1 \le k \le n)$$

are defined.

Let $n \ge 2$ and assume $f_{n-1,k}$ $(1 \le k \le n-1)$ are defined. By Lemma 3.11, $g_{\underline{q_n}}$ is a perturbation of $g_{\underline{q_{n-1}}}$. Hence if q_n is sufficiently large, $f_{n,k}$ $(1 \le k \le n)$ are defined, and $f_{n,n-1}$ lies in $\mathbb{D}_{\alpha_*} \ltimes \mathcal{F}_1$.

By Proposition 5.4,

In fact, by induction, we have the following:

Lemma 7.1. *If $\underline{q} = (q_1, q_2, \dots)$ grows sufficiently fast, then $g_{\underline{q_n}}$ is n-times near-parabolic renormalizable. Only the first near-parabolic renormalization is top, but the other $(n-1)$ renormalizations are bottom.*

Moreover,

(1) ${\mathcal{R}_b}^{n-1} \circ \mathcal{R}_t(g_{\underline{q_n}}) \in \mathcal{F}_1$.
(2) *Let* $s_{n-1} \in \{+, -\}$ *satisfy*

Proof. We have already shown for the case $n = 1$.

Let $n \geq 2$ and assume $g_{\underline{q_{n-1}}}$ satisfies the lemma. $\square$

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