STRAIGHTENING MAPS FOR POLYNOMIALS WITH DISCONNECTED JULIA SETS

HIROYUKI INOU AND YIMIN WANG*

Abstract. In this paper, we prove that a polynomial f can be determined by the conformal conjugacy class of its restriction $f|_{D_f(\infty)}$ together with the hybrid class of the restriction of f to non-trivial periodic components of the filled Julia set $K(f)$. The proof is based on quasiconformal surgery.

1. Introduction

Fix an integer $d \geq 2$. We consider the set $\mathcal{P}_d$ of all the monic centered polynomials of degree d. For each $f \in \mathcal{P}_d$, the *basin of infinity of* f is defined by

$$D_f(\infty) = \{z \in \mathbb{C} \mid f^n(z) \to \infty \text{ as } n \to \infty\}.$$

Its complement $K(f)$ is called *the filled Julia set of* f. Set

$$\widetilde{\mathcal{B}_d} := \left\{ \left(\varphi \circ f|_{D_f(\infty)}\right) \circ \varphi^{-1} \;\middle|\; \begin{array}{l} \varphi\colon D_f(\infty) \to \varphi(D_f(\infty)) \text{ is conformal and} \\ \lim\limits_{z\to\infty} \frac{\varphi(z)}{z} = 1,\ f \in \mathcal{P}_d \end{array} \right\}.$$

Let us consider the projection

$$\pi\colon \mathcal{P}_d \to \widetilde{\mathcal{B}_d},\ f \mapsto f|_{D_f(\infty)}.$$

Note that the fiber over $(z^d, \mathbb{C}\setminus\overline{\mathbb{D}})$ is precisely *the connectedness locus* $\mathcal{C}_d$ consisting of maps in $\mathcal{P}_d$ with connected Julia sets. Clearly, any map $f \in \mathcal{C}_d$ is determined by the dynamics of f on its filled Julia set $K(f)$ [5, 7]. In contrast, it was proved in [19, 20] that each fiber of π containing a map $f \in \mathcal{P}_d$ with a Cantor Julia set is singleton.

One of the main topics in dynamical systems is to decompose a complicated system into several simplified models that could extract the main features of it. Intuitively, one might anticipate that the fibers of the map $\pi : \mathcal{P}_d \to \widetilde{\mathcal{B}_d}$ can be identified with the product of connectedness loci $\mathcal{C}_{d_i}$ of degrees $d_i \leq d$, each of which is connected [1, 4, 11]. More precisely, one may expect that a polynomial $f \in \mathcal{P}_d$ should be determined by the conformal conjugacy class of its restriction $f|_{D_f(\infty)}$ together with the hybrid classes of the restriction of f to non-trivial periodic components of the filled Julia set $K(f)$. Evidently, it holds for $d = 2$. The aim of this paper is to provide a rigorous proof for such an intuition by employing quasiconformal surgery. As a byproduct, we show that the fibers of π are connected. It is worth mentioning that DeMarco and Pilgrim [2] proved that the fibers of π' are

Date: July 29, 2025.

2020 *Mathematics Subject Classification.* Primary: 37F20 ; Secondaries: 37F31.

Key words and phrases. Polynomial dynamics, Tuning, Quasiconformal surgery, Renormalization.

*Corresponding author: Yimin Wang, yiminwang@shnu.edu.cn.

connected, utilizing a method that is entirely distinct. Here, $\pi'\colon \mathcal{M}_d \to \mathcal{B}_d$ denotes the projection from the Moduli space to the set $\mathcal{B}_d$ of all the conformal conjugacy classes of the maps $f|_{D_f(\infty)}$. More precisely, DeMarco and Pilgrim's strategy is to prove that each fiber of π' can be characterized as a nested intersection of connected sets [2]. In contrast, our approach employs the technique of quasiconformal surgery.

Let $f_0 \in \mathcal{P}_d$. We say a component of $K(f_0)$ is *critical* if it contains a critical point of f_0. Qiu-Yin [16], and independently Kozlovski-van Strien[9], proved that a component K of $K(f_0)$ is nontrivial if and only if the orbit of K under f_0 eventually lies in the orbit of a periodic component that is critical. Let us define a *mapping schema* $S_0 = (|S_0|, \sigma, \delta)$ *induced by* f_0 as follows. First let $|S_0|$ be the set of nontrivial components of $K(f_0)$ that is critical. The result of Qiu-Yin and Kozlovski-van Strien implies that for every $v \in |S_0|$ there exists a first moment $r(v) \geq 1$ such that $f_0^{r(v)}(v) \in |S_0|$. Now let $\sigma\colon |S_0| \to |S_0|$ be defined by

$$\sigma(v) = f_0^{r(v)}(v)$$

and let $\delta\colon |S_0| \to \{2, 3, \cdots,\}$ be defined by

$$\delta(v) = \deg(f_0^{r(v)}\colon v \to \sigma(v)).$$

In Section 2, we introduce the set of of monic and centered generalized polynomials $\mathcal{P}(S_0)$ over the schema S_0 and let $\mathcal{C}(S_0) \subset \mathcal{P}(S_0)$ be the set of generalized polynomials over the schema S_0 with fiber-wise connected Julia sets. We further define a map

$$F_0\colon \bigcup_{v\in|S_0|} v \to \bigcup_{v\in|S_0|} v$$

by $F_0|_v = f_0^{r(v)}|_v$. Note that F_0 can be extended to a GPL map over the schema S_0 with filled Julia set $\bigcup_{v\in|S_0|} v$. Let $\mathcal{F}_0$ denote the fiber $\pi^{-1}(\pi(f_0))$ of f_0 under π. Then we show that every $f \in \mathcal{F}_0$ naturally induces a GPL map over the schema S_0 with a fiber-wise connected Julia set. Such a GPL map is hybrid conjugate to a unique map $\chi(f)$ in $\mathcal{C}(S_0)$ (respecting the external markings). This yields a well defined map

$$\chi\colon \mathcal{F}_0 \to \mathcal{C}(S_0),\ f \mapsto \chi(f)$$

which we will call *a straightening map*. Our purpose is to prove :

Main Theorem. *Let $f_0 \in \mathcal{P}_d$ be a polynomial whose Julia set is not a Cantor set. Then the straightening map*

$$\chi\colon \mathcal{F}_0 \to \mathcal{C}(S_0),\ f \mapsto \chi(f).$$

is a bijection.

Moreover, the fiber $\mathcal{F}_0$ is compact and connected.

The proof is based on quasiconformal surgery. This theorem is an analogue of the main results in [7, 18]. In other words, every map in the fiber $\mathcal{F}_0$ can be tuned by a generalized polynomial $g \in \mathcal{C}(S_0)$. Let us mention that χ is discontinuous in general [6].

The paper is organized as follows. In Section 2, we revisit the definitions of GPL maps over mapping schemata and proceed to establish the straightening map denoted by χ. Advance to Section 3, where we apply quasiconformal surgery to substantiate the surjectivity attribute of the straightening map. Section 4 delves into the proof of rigidity for the maps in the fibers of π, a result which, in turn, corroborates the injectivity of the straightening map χ. Within this section, we also address the issue of connectedness of the fibers of π.

Acknowledgement. The authors would like to express their gratitude to Mitsuhiro Shishikura, Weixiao Shen and Yongcheng Yin for valuable suggestions. The authors thank the referee for useful comments. The first author is supported by JSPS KAKENHI Grant Numbers JP18K03367 and JK22K03373. The second author is supported by Natural Science Foundation of Shanghai (Grant No. 25ZR1402412) and Science and Technology Innovation Plan of Shanghai (Grant No. 23JC1403200).

2. Yoccoz puzzles and GPL maps

In this section, we first recall some definitions such as Yoccoz puzzles and GPL maps. As for prerequisites, the reader is expected to be familiar with polynomial dynamics. We refer the readers to [5, 13]. Throughout this paper, we fix an $f_0 \in \mathcal{P}_d$ whose Julia set is not a Cantor set.

The aim of this section is to define the straightening map

$$\chi\colon \mathcal{F}_0 \to \mathcal{C}(S_0),\ f \mapsto \chi(f).$$

2.1. **Yoccoz puzzles.** Let $f \in \mathcal{P}_d$ and let $\mathrm{Crit}(f)$ denote the critical points of f. The *Green function of f* is given by

$$G_f(z)\colon \mathbb{C} \to [0, +\infty),\ z \mapsto \lim_{n\to\infty} \frac{1}{d^n} \log^+ |f^n(z)|,$$

where $\log^+ x = \max\{\log x, 0\}$.

The level sets of G_f determine a foliation $\mathcal{F}$ of the basin $D_f(\infty)$ of infinity. Every leaf of $\mathcal{F}$ is a union of smooth simple closed curves unless it intersects the backward orbit of $\mathrm{Crit}(f)$ under f. Let $R_f = \max_{c\in\mathrm{Crit}(f)} G_f(c)$ be the *maximal escaping rate of f*. There exists a conformal map

$$\phi_f\colon \{z \in \mathbb{C} \mid G_f(z) > R_f\} \to \{z \in \mathbb{C} \mid |z| > e^{R_f}\}$$

such that $\phi_f(f(z)) = (\phi_f(z))^d$ whenever it makes sense. Such a conformal map is unique under the additional condition $\lim_{z\to\infty} \frac{\phi_f(z)}{z} = 1$ and is called the *Böttcher map of f*.

Fix a constant $1 \le r_0 = r_0(f) < d$ such that the level set $\{z \in \mathbb{C} \mid G_f(z) = r_0\}$ is a union of simple closed curves. Such an r_0 can be chosen as follows. Let C_f be the intersection of the grand orbit of $\mathrm{Crit}(f)$ and $\{1 \le G_f(z) < d\}$. Clearly, C_f is a finite and thus a discrete set. Put $b = \max_{z\in C_f} G_f(z)$. Set $r_0 = \frac{b+d}{2}$ and we are done.

Definition 2.1 (Puzzle pieces)**.** For each $n \in \mathbb{N}$, we let $\Gamma_n^f = \{G_f(z) = r_0 d^{-n}\}$. An *$f$-puzzle piece of depth n* is a bounded component of $\mathbb{C} \setminus \Gamma_n^f$.

It is fairly easy to see that each puzzle piece is a quasidisk with a smooth boundary under our assumptions. Moreover, the following Markov property holds: for any two distinct puzzle pieces, either they have disjoint closures or one is compactly contained in the other.

2.2. **GPL maps and external markings.** Recall that $f_0 \in \mathcal{P}_d$ has a disconnected Julia set that is not a Cantor set. Let us recall that the mapping schema $S_0 = (|S_0|, \sigma, \delta)$ induced by f_0 is a triple consisting of

- the set $|S_0|$ of all the nontrivial components of $K(f_0)$ that are critical,
- the first return map $\sigma : |S_0| \to |S_0|$,
- a degree map $\delta : |S_0| \to \{2, 3, \cdots\}$, $v \mapsto \deg(\sigma|_v)$.

A *generalized polynomial* P over S_0 is a map

$$P : |S_0| \times \mathbb{C} \to |S_0| \times \mathbb{C}$$

such that $P(v, z) = (\sigma(v), P_v(z))$ where $g_v(z)$ is a monic centered polynomial of degree $\delta(v)$. The *filled Julia set* $K(P)$ of P is the set of points in $|S_0| \times \mathbb{C}$ whose forward orbits are precompact. The boundary $\partial K(P)$ of the filled Julia set is called the *Julia set* $J(P)$ of P. The filled Julia set $K(P)$ is called *fiber-wise connected* if

$$K(P, v) := K(P) \cap (\{v\} \times \mathbb{C})$$

is connected for every $v \in |S_0|$. Let $\mathcal{P}(S_0)$ denote the set of all generalized polynomials over S_0. Let $\mathcal{C}(S_0) \subset \mathcal{P}(S_0)$ be the subset of all generalized polynomials with fiber-wise connected filled Julia sets over S_0.

We shall need external rays for $P \in \mathcal{C}(S_0)$ which can be defined similarly as in the case of a single polynomial. Indeed, for each $v \in |S_0|$ there exists a unique conformal $\varphi_{P,v} : \mathbb{C} \setminus K(P, v) \to \mathbb{C} \setminus \overline{\mathbb{D}}$ such that $\varphi_{P,v}(z)/z \to 1$ as $z \to \infty$ and that

$$\varphi_{P,\sigma(v)}(P_v(z)) = \varphi_{P,v}(z)^{\delta(v)}.$$

For $t \in \mathbb{R}/\mathbb{Z}$, the *external ray* $\mathcal{R}_P(v, t)$ is defined as $\varphi_{P,v}^{-1}(\{re^{2\pi i t} : r > 1\})$.

Let us now revisit the definition of *GPL maps* that were discussed in [7, 18]. We say $U \subset |S_0| \times \mathbb{C}$ is a *topological multi-disk* if each fiber

$$U_v := U \cap (\{v\} \times \mathbb{C})$$

is a simply connected domain. Let U and V be two topological multi-disk in $|S_0| \times \mathbb{C}$. We simply use the notation $U \Subset V$ to denote that U_v is compactly contained in V_v for each $v \in |S_0|$.

Definition 2.2. An GPL map g over the schema $|S_0|$ is a map

$$g : U \to V$$
$$(v, z) \mapsto (\sigma(v), g_v(z)),$$

with the following properties:

- $U \Subset V$ are two topological multi-disks in $|S_0| \times \mathbb{C}$;
- $g_v : U_v \to V_{\sigma(v)}$ is a proper map of degree $\delta(v)$ for each v.

The set $K(g) := \bigcap_{n=0}^{\infty} g^{-n}(U)$ is called the *filled-Julia set of* g.

Definition 2.3 (Access and external marking)**.** Let $g\colon U \to V$ be a GPL map with fiber-wise connected filled Julia set. A *path to* $K(g)$ is a continuous map $\gamma\colon [0, 1] \to V$ such that $\gamma((0, 1]) \subset V \setminus K(g)$ and $\gamma(0) \in J(g)$. We say two paths γ_0 and γ_1 to $K(g)$ are *homotopic* if there exists a continuous map $\tilde{\gamma}\colon [0, 1] \times [0, 1] \to V$ such that

(1) $t \mapsto \tilde{\gamma}(s, t)$ is a path to $K(g)$ for all $s \in [0, 1]$;
(2) $\tilde{\gamma}(0, t) = \gamma_0(t)$ and $\tilde{\gamma}(1, t) = \gamma_1(t)$ for all $t \in [0, 1]$;

(3) $\tilde{\gamma}(s,0) = \gamma_0(0)$ for all $s \in [0,1]$.

An *access to* $K(g)$ is a homotopy class of paths to $K(g)$.

An *external marking* for g is a collection $\Gamma = \{\Gamma_v\}_{v\in|S_0|}$, where each Γ_v is an access to $K(g)$ contained in $\{v\} \times \mathbb{C}$, which is forward invariant in the following sense. For every $v \in |S_0|$ and every representative γ of Γ_v, the connected component of $g(\gamma \cap U)$ which intersects $K(g)$ is a representative of $\Gamma_{\sigma(v)}$.

The *standard external marking* for a generalized polynomial P with fiber-wise connected Julia set is defined as the collection of $\{[\mathcal{R}_P(v,0)]\}_{v\in|S_0|}$, where $\mathcal{R}_P(v,0)$ is the external ray of P with 0-angle in the v-fiber.

Let g_1, g_2 be two GPL maps. We say that they are *quasiconformally conjugate* if there is a fiber-wise quasiconformal map $\varphi : |S_0| \times \mathbb{C} \to |S_0| \times \mathbb{C}$ such that $\varphi \circ g_1 = g_2 \circ \varphi$ near $K(g_1)$. We say that they are *hybrid equivalent* if they are quasiconformally conjugate and we can choose φ such that $\bar{\partial}\varphi = 0$ a.e. on $K(g_1)$. The Douady-Hubbard straightening theorem [5] extends in a straightforward way.

Theorem 2.4. [7, Theorem A] *Let g be a GPL map over S_0 with fiber-wise connected Julia set and let Γ be an external marking for g. There exists a unique $P \in \mathcal{C}(S_0)$ and a hybrid conjugacy between g and P which sends the external marking Γ for g to the standard external marking for P.*

2.3. **Renormalization via mapping schema.** The following lemma asserts that critical components of $K(f_0)$ can be separated by puzzle pieces.

Lemma 2.5. *Let $|T|$ be the set of all the trivial critical component of $K(f_0)$. Set $S' = |T| \cup \bigcup_{n\geq 0} f_0^n(|S_0|)$. There exists a positive integer $N = N(f_0)$ such that any two different components $K_1, K_2 \in S'$ of $K(f_0)$ do not lie in the same puzzle piece of depth N.*

Proof. Note that S' contains finitely many elements. The lemma follows easily from the fact [9, 16] that

$$K = \bigcap_{n\in\mathbb{Z}_+} \overline{Y_n(K)} \text{ for all } K,$$

where K is a component of $K(f_0)$ in S' and $Y_n(K)$ is puzzle piece of depth n that contains K. □

From now on, we fix an integer $N > \max\{r(v) \mid v \in |S_0|\}$ given by Lemma 2.5. For any $v \in |S_0|$, set $V_v = \{v\} \times Y_N(v)$, where $Y_N(v)$ is the puzzle piece of depth N that contains v. Let $U_v = \{v\} \times Y_{N+r(v)}(v)$, where $r(v)$ is the smallest positive integer so that $f_0^{r(v)}(v) = \sigma(v)$ and $Y_{N+r(v)}(v)$ is the puzzle piece of depth $N + r(v)$ that contains v.

Lemma 2.6. *The map*

$$F_0 : \bigcup_{v\in|S_0|} U_v \to \bigcup_{v\in|S_0|} V_v,$$

$$(v,z) \mapsto (\sigma(v), f_0^{r(v)}(z))$$

is a GPL map over S_0 with filled Julia set $K(F_0) = \bigcup_{v\in|S_0|} \{v\} \times v$.

Proof. Set $U = \bigcup_{v\in|S_0|} U_v$ and $V = \bigcup_{v\in|S_0|} V_v$. We begin by proving that $F_0 : U \to V$ is a GPL map. Clearly, $U \Subset V$. It suffices to show that $F_0 : U_v \to V_{\sigma(v)}$ has degree $\delta(v)$ for all $v \in |S_0|$. Note that $f_0^j(Y_{N+r(v)}(v))$ does not contain a nontrivial critical component of $K(f_0)$ for all $1 \le j \le r(v) - 1$ since $r(v)$ is the first moment so that $f_0^{r(v)}(v) \in |S_0|$. According to Lemma 2.5, for each $0 \le j \le r(v)$,

$$f_0^j(Y_{N+r(v)}(v)) = Y_{N+r(v)-j}(f_0^j(v)) \subset Y_N(f_0^j(v))$$

does not contain any trivial critical component of $K(f_0)$. Thus

$$\deg(F_0|_{U_v}) = \deg(f_0|_{Y_{N+r(v)}(v)}) = \delta(v).$$

Finally note that

$$\bigcup_{v\in|S_0|} \{v\} \times v \subset K(F_0) \subset \bigcup_{v\in|S_0|} \bigcap_{n\in\mathbb{N}} \{v\} \times Y_n(v) = \bigcup_{v\in|S_0|} \{v\} \times v,$$

and the proof is complete. □

A standard argument of hyperbolic geometry shows that there exists a finite invariant set $\{\alpha_v\}_{v\in|S_0|}$ in the Julia set such that α_v lies in U_v, $F_0(\alpha_v) = \alpha_{\sigma(v)}$ for all $v \in |S_0|$ and the combinatorial rotation number of α_v is zero if v is periodic under σ.

Lemma 2.7. *There exists a family $\{\mathcal{R}_v\}_{v\in|S_0|}$ of smooth curves parameterized by $r \in (0, +\infty)$ with the following properties.*

- *For every $v \in |S_0|$, $\lim_{r\to 0} \mathcal{R}_v(r) = \alpha_v$.*
- *For every $v \in |S_0|$, $f_0^{r(v)}\mathcal{R}_v = \mathcal{R}_{\sigma(v)}$.*

Proof. It follows from [15, Main Theorem] that for every periodic α_v there exists at least one smooth external ray $\widetilde{\mathcal{R}_v}$ of f_0 landing at α_v. Evidently, such external rays $\widetilde{\mathcal{R}_v}$ can be chosen so that $f_0^{r(v)}\widetilde{\mathcal{R}_v} = \widetilde{\mathcal{R}_{\sigma(v)}}$ for all periodic $v \in |S_0|$. For every periodic $v \in |S_0|$, let θ_v be the external angle of $\widetilde{\mathcal{R}_v}$. We define $\mathcal{R}_v$ to be a perturbation of $\widetilde{\mathcal{R}_v}$ in the following way. Fix an $\epsilon > 0$ small enough so that the following holds. For every periodic $v \in |S_0|$,

- $\mathcal{R}_v(r) = \phi_{f_0}^{-1}(e^{r+i(2\pi i\theta_v+\epsilon r)})$ for $r > R_{f_0}$, where ϕ_{f_0} is the Böttcher coordinate of f_0 and R_{f_0} is the maximal escaping rate of f_0.
- $\mathcal{R}_v([R_f, +\infty))$ does not meet the post-critical set of f_0.
- We extend $\mathcal{R}_v$ by $f_0^{r(v)}\mathcal{R}_v(r) = \mathcal{R}_{\sigma(v)}(d^{r(v)}r)$ for all $r \in (0, +\infty)$.
- $\lim_{r\to 0} \mathcal{R}_v(r) = \alpha_v$.

Finally we define $\mathcal{R}_v$ for every preperiodic $v \in |S_0|$ by induction. Assume that $\mathcal{R}_{\sigma(v)}$ has been defined. We define $\mathcal{R}_v$ as a connected component of preimage of $f_0^{-r(v)}(\mathcal{R}_{\sigma(v)})$ such that $\lim_{r\to 0} \mathcal{R}_v(r) = \alpha_v$.

□

Remark 2.8. The result of Petersen and Zakeri[15] is only for periodic points. For an eventually periodic point, the external rays may bifurcate. See figure 1 for example.

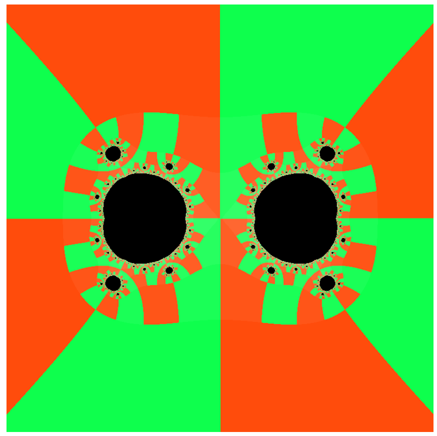

FIGURE 1. The Julia set of $(z^2 - .64)^2 + .8$. The β-fixed point (the landing point of the 0-ray) has a preimage in the critical preperiodic component having only broken (bifurcated) rays.

Such a family $\{\mathcal{R}_v\}_{v \in |S_0|}$ given by the lemma above induces an external marking $\Gamma_0 = \{[\{v\} \times \mathcal{R}_v]\}_{v \in |S_0|}$ for F_0.

The remainder of this section will be devoted to the construction of the straightening map

$$\chi\colon \mathcal{F}_0 \to \mathcal{C}(S_0),\ f \mapsto \chi(f).$$

We begin with the constructions of GPL maps induced by maps in the fiber $\mathcal{F}_0$.

Lemma 2.9. *For any $n \in \mathbb{N}$ and $f \in \mathcal{F}_0$, there exists a quasiconformal homeomorphism $\phi_{f,n} : \mathbb{C} \to \mathbb{C}$ with the following properties.*

- $\phi_{f,n} : \{z \mid G_{f_0}(z) > r_0 d^{-n}\} \to \{z \mid G_f(z) > r_0 d^{-n}\}$ *is a conformal map.*
- $\phi_{f,n} \circ f_0 = f \circ \phi_{f,n}$ *on* $\{z \mid G_{f_0}(z) \geq r_0 d^{-n}\}$.

Proof. First note that there exists a conformal map $\Phi : D_{f_0}(\infty) \to D_f(\infty)$ such that

$$\Phi \circ f_0 = f \circ \Phi \text{ on } D_{f_0}(\infty) \tag{2.1}$$

since f and f_0 lie in the same fiber of π. Set $W = \{z \mid G_{f_0}(z) > r_0 d^{-n}\}$ and $W' = \{z \mid G_f(z) > r_0 d^{-n}\}$. It follows from (2.1) that Φ maps W conformally onto W'. Since ∂W and $\partial W'$ are unions of disjoint smooth curves, Φ can be extended smoothly from $\overline{W}$ onto $\overline{W'}$. Let D be an arbitrary component of $\mathbb{C} \setminus \overline{W}$ and let D' denote the bounded component $\mathbb{C} \setminus \Phi(\partial D)$. Then we extend $\Phi\colon \partial D \to \partial D'$ to a diffeomorphism $h_D\colon D \to D'$ in an arbitrarily smooth way. Finally we define $\phi_{f,n} = \Phi$ on $\overline{W}$ and $\phi_{f,n} = h_D$ on D, where D runs over all the component of $\mathbb{C} \setminus \overline{W}$. It is clear that $\phi_{f,n} : \mathbb{C} \to \mathbb{C}$ is a quasiconformal map satisfying all the desired conditions. □

Set $N' = \max_{v \in |S_0|} N + r(v)$. For any $f \in \pi^{-1}(\pi(f_0))$, let $\psi = \phi_{f,N'}$ be given by Lemma 2.9. For any $v \in |S_0|$, let $U'_v = \{v\} \times \psi(Y_{N+r(v)}(v))$ and $V'_v = \{v\} \times \psi(Y_N(v))$.

Lemma 2.10. *The map*

$$F: \bigcup_{v\in|S_0|} U'_v \to \bigcup_{v\in|S_0|} V'_v,$$

$$(v,z) \mapsto (\sigma(v), f^{r(v)}(z))$$

is a GPL map over S_0 *with fiber-wise connected filled Julia set.*

Proof. Let us first check that F is well defined. To this end, we need to show that

$$f^{r(v)}(\psi(Y_{N+r(v)}(v))) = \psi(Y_N(v))$$

for any $v \in |S_0|$. Note that $\psi(Y_{N+r(v)}(v))$ is an f-puzzle piece of depth $N + r(v)$, which yields $f^{r(v)}(\psi(Y_{N+r(v)}(v)))$ is an f-puzzle piece of depth N. Since $\psi \circ f_0^{r(v)} = f^{r(v)} \circ \psi$ on $\partial Y_{N+r(v)}(v)$, we conclude that the two bounded Jordan disks $f^{r(v)}(\psi(Y_{N+r(v)}(v)))$ and $\psi(Y_N(v))$ share a common boundary. In fact,

$$\partial(f^{r(v)}(\psi(Y_{N+r(v)}(v)))) = \partial(\psi(f_0^{r(v)}(Y_{N+r(v)}(v)))) = \partial(\psi(f(Y_N(v))).$$

Therefore $f^{r(v)}(\psi(Y_{N+r(v)}(v))) = \psi(f(Y_N(v))$.

We now proceed to show that F is a GPL map. Obviously, $\bigcup_{v\in|S_0|} U'_v \Subset \bigcup_{v\in|S_0|} V'_v$. It remains to prove that for any $v \in |S_0|$,

$$f^{r(v)}: \psi(Y_{N+r(v)}(v)) \to \psi(Y_N(v))$$

is a holomorphic proper map of degree $\delta(v)$. This follows easily from that fact that the boundary map

$$f^{r(v)}: \partial\psi(Y_{N+r(v)}(v)) \to \partial\psi(Y_N(v))$$

is a $\delta(v)$-to-one covering. Indeed,

$$f_0^{r(v)}: \partial Y_{N+r(v)}(v) \to \partial Y_N(v)$$

is a $\delta(v)$-to-one covering and ψ is a conjugacy between $f_0^{r(v)}|_{\partial Y_{N+r(v)}(v)}$ and $f^{r(v)}|_{\partial\psi(Y_{N+r(v)}(v))}$.

We are left with the task of checking the filled Julia set $K(F)$ is fiber-wise connected. Conversely, suppose that $K(F)$ is not fiber-wise connected. Then there exists a moment $s \geq 1$ such that $F^{-s}(\bigcup_{v\in|S_0|} U'_v)$ is not a topological multi-disk. More precisely, there exists $v_0 \in |S_0|$ and two disjoint f-puzzle piece $Y, \widetilde{Y}$ such that both $\{v_0\}\times Y$ and $\{v_0\}\times\widetilde{Y}$ are different components of $(F^{-s}(\bigcup_{v\in|S_0|} U'_v))\cap(\{v_0\}\times\mathbb{C})$. Recall that there exists a conformal map $\Phi\colon D_{f_0}(\infty) \to D_f(\infty)$ such that $\Phi\circ f_0 = f\circ\Phi$. Note that ∂Y and $\partial\widetilde{Y}$ lie in $D_f(\infty)$. We check at once that $\{v_0\}\times\Phi^{-1}(\partial Y)$ and $\{v_0\}\times\Phi^{-1}(\partial\widetilde{Y})$ are the boundaries of two disjoint components of $F_0^{-s}(\bigcup_{v\in|S_0|} U_v)$. This contradicts that $K(F_0) = \bigcup_{v\in|S_0|}\{v\}\times v$ is fiber-wise connected. □

Now the straightening map

$$\chi\colon \mathcal{F}_0 \to \mathcal{C}(S_0),\ f \mapsto \chi(f)$$

can be defined as follows. For any $f \in \mathcal{F}_0$, let F be as given by Lemma 2.10. Recall that there exists a unique conformal map $\Phi\colon D_{f_0}(\infty) \to D_f(\infty)$ such that $\Phi'(\infty) = 1$ and $\Phi\circ f_0 = f\circ\Phi$. For every $v \in |S_0|$, let $\mathcal{R}'_v := \Phi(\mathcal{R}_v)$. Note that

$\Gamma = \{[\{v\} \times \mathcal{R}'_v]\}_{v \in |S_0|}$ is an external marking for the GPL map F. Now let $\chi(f)$ be the unique map in $\mathcal{C}(S_0)$ that is hybrid equivalent to F (respecting the external markings) given by Theorem 2.4.

3. Straightening maps are surjective

Recall that $f_0 \in \mathcal{P}_d$ is a map whose Julia set is not a Cantor set. The straightening map

$$\chi \colon \mathcal{F}_0 \to \mathcal{C}(S_0),\ f \mapsto \chi(f)$$

induced by f_0 has been defined in the last section. The objective of this section is to establish the following:

Theorem 3.1. *The straightening map χ is surjective.*

Keep in mind that N is an integer given by Lemma 2.5. For any $v \in |S_0|$, set $V_v = \{v\} \times Y_N(v)$, where $Y_N(v)$ is the puzzle piece of depth N that contains v. Let $U_v = \{v\} \times Y_{N+r(v)}(v)$, where $r(v)$ is the smallest positive integer so that $f_0^{r(v)}(v) = \sigma(v)$ and $Y_{N+r(v)}(v)$ is the puzzle piece of depth $N + r(v)$ that contains v. By Lemma 2.6, the map

$$F_0 : \bigcup_{v \in |S_0|} U_v \to \bigcup_{v \in |S_0|} V_v,$$
$$(v, z) \mapsto (\sigma(v), f_0^{r(v)}(z))$$

is a GPL map over S_0 with filled Julia set $K(F_0) = \bigcup_{v \in |S_0|} \{v\} \times v$. Bear in mind that a collection of perturbed external rays $\{\mathcal{R}_v\}_{v \in |S_0|}$ induces an external marking for F_0.

Convention. Let

$$\Pi \colon |S_0| \times \mathbb{C} \to \mathbb{C}$$
$$(v, z) \mapsto z$$

be the projection to the second coordinate.

The proof of Theorem 3.1 is based on the following lemma.

Lemma 3.2. *For any $g \in \mathcal{C}(S_0)$, there is a quasiregular map $\widehat{f} \colon \mathbb{C} \to \mathbb{C}$ with the following properties.*

(1) $\widehat{f} = f_0$ *on* $\bigcup_{v \in |S_0|} \mathbb{C} \setminus \Pi(U_v)$.

(2) *The map*

$$\widehat{F} : \bigcup_{v \in |S_0|} U_v \to \bigcup_{v \in |S_0|} V_v,$$
$$(v, z) \mapsto (\sigma(v), \widehat{f}^{r(v)}(z))$$

is a quasiregular map. Moreover,

$$\widehat{F} : \bigcup_{v \in |S_0|} W_v \to \bigcup_{v \in |S_0|} U_v$$

is holomorphic and it is a GPL map over S_0 hybrid equivalent to g, where $W_v = \widehat{F}^{-1}(U_{\sigma(v)})$ for every $v \in |S_0|$.

(3) *The* non-regular part $\mathcal{NR} = \{z \in \mathbb{C} : \bar\partial \widehat{f}(z) \neq 0\}$ *is contained in* $\bigcup_{v\in|S_0|} \Pi(U_v \setminus W_v)$. *Moreover, for any* $z \in \mathbb{C}$*, the orbit of* z *under* $\widehat{f}$ *pass through* $\mathcal{NR}$ *at most once.*

(4) *There is a collection* $\{\gamma_v\}_{v\in|S_0|}$ *of paths characterized by the following attributes.*
   - *for every* $v \in |S_0|$, $\widehat{f}^{r(v)}(\gamma_v) = \gamma_{\sigma(v)}$;
   - *for every* $v \in |S_0|$, $\gamma_v \setminus \Pi(U_v) = \mathcal{R}_v$;
   - $\{\gamma_v\}_{v\in|S_0|}$ *induces an external marking* $\widehat{\Gamma}$ *for* $\widehat{F}$*. Furthermore, there is a hybrid conjugacy between* $\widehat{F}$ *and* g *sending* $\widehat{\Gamma}$ *to the standard external marking for* g*.*

We will postpone the proof of this lemma to Section 3.1 and instead utilize it to demonstrate Theorem 3.1.

Proof of Theorem 3.1. For any $g \in \mathcal{C}(S_0)$, let $\widehat{f}$ be given by Lemma 3.2. By the third property of $\widehat{f}$ and [17, Lemma 1], there exists a quasiconformal map $\Phi\colon \mathbb{C} \to \mathbb{C}$ with the following properties.

- Φ is holomorphic near infinity, $\Phi'(\infty) = 1$.
- $\bar\partial\Phi = 0$ a.e. on $K(\widehat{F})$.
- $f := \Phi \circ \widehat{f} \circ \Phi^{-1}$ is a monic centered polynomial.

Our aim is to show that $f \in \chi^{-1}(g)$.

By the properties (2) and (3) of $\widehat{f}$, the map

$$F\colon \bigcup_{v\in|S_0|} \{v\} \times \Phi(\Pi(W_v)) \to \bigcup_{v\in|S_0|} \{v\} \times \Phi(\Pi(U_v))$$

$$(v, z) \mapsto (\sigma(v), f^{r(v)}(z))$$

is a GPL map over S_0 hybrid equivalent to g.

Let us first check that $f \in \pi^{-1}(\pi(f_0))$. In other words, $f|_{D_f(\infty)}$ and $f_0|_{D_{f_0}(\infty)}$ are conformally conjugate. We use the notation P_n to denote the complement of the union of puzzle pieces of depth n of f_0. It suffices to prove that there exists a sequence of homeomorphisms $\phi_n\colon \mathbb{C} \to \mathbb{C}$ so that for each n sufficiently large,

- ϕ_n is holomorphic on P_n.
- $f \circ \phi_n = \phi_n \circ f_0$ on P_n.
- $\phi_{n+1} = \phi_n$ on P_n.

Indeed, once such a sequence ϕ_n is constructed, the map ϕ, defined by $\phi|_{P_n} = \phi_n$, becomes a desired conformal conjugacy.

Recall that N is a sufficiently large integer given by Lemma 2.5. Note that $f \circ \Phi = \Phi \circ f_0$ on P_N by the first property of $\widehat{f}$. Let $\phi_N = \Phi$ and assume that ϕ_n has been constructed for some $n \geq N$. We now proceed to construct ϕ_{n+1} as follows. For each $Y \subset \mathbb{C}$, denote $Y' = \phi_n(Y)$. It suffices to construct, for each puzzle piece Y of depth n, a homeomorphism ϕ_{n+1} so that ϕ_{n+1} is holomorphic and $f \circ \phi_{n+1} = \phi_{n+1} \circ f_0$ on $Y \cap P_{n+1}$.

Case 1. Y does not contain any critical points of f_0. Then $f_0 : Y \to f(Y)$ is conformal, and so is $f : Y' \to f(Y')$. This follows from the fact that $\phi_n \circ f_0 = f \circ \phi_n$ on ∂Y. The map ϕ_{n+1} can be defined easily by

$$\phi_{n+1} = (f|_{f(Y')})^{-1} \circ \phi_n \circ f_0|_Y.$$

Case 2. Y contains a critical point c in v_0 for some $v_0 \in |S_0|$. Let K_0 denote the filled Julia set of F in the fiber $\{v_0\} \times \mathbb{C}$. Note that $\Pi(K_0) \subset Y' = \phi_n(Y)$. Let B (resp . $\tilde{B}$) be the puzzle pieces of depth of $n+1$ of f_0 (resp. f) that contains v_0 (resp. $\Pi(K_0)$). Then both the two maps

$$f_0^{r(v_0)}\colon Y \setminus B \to f_0^{r(v_0)}(Y) \setminus f_0^{r(v_0)}(B)$$

and

$$f^{r(v_0)}\colon Y' \setminus \tilde{B} \to f^{r(v_0)}(Y') \setminus f^{r(v_0)}(\tilde{B})$$

are coverings of degree $\delta(v_0)$. Thus there is a homeomorphism ϕ_{n+1} such that

$$f_0^{r(v_0)} \circ \phi_n = \phi_{n+1} \circ f^{r(v_0)} \text{ on } Y \setminus B$$

and

$$\phi_{n+1} = \phi_n \text{ on } \partial Y.$$

Extending ϕ_{n+1} in an arbitrary way to a homeomorphism from B onto $\tilde{B}$, and we obtained our desired map ϕ_{n+1}.

Case 3. Y contains a critical point $c \notin \bigcup_{v\in|S_0|} v$. Since N is a sufficiently large integer given by Lemma 2.5, c is the unique critical point of f_0 in Y. Then $f_0 : Y \to f(Y)$ is a branched covering, and so is $f : Y' \to f(Y')$. Let Y_N be the puzzle piece of depth N of f_0 that contains c. Note that $\tilde{c} = \Phi(c)$ is the unique critical point in $Y'_N = \Phi(Y_N) = \phi_n(Y_N)$ since Φ conjugates f_0 to f outside $\bigcup_{v\in|S_0|} \Pi(U_v)$. So $\tilde{c}$ is the unique critical point of f in Y'. Let X (resp. $\tilde{X}$) be the puzzle piece of depth $n+1$ of f_0 (resp. f) that contains c (resp. $\tilde{c}$). Then both the two maps

$$f_0\colon Y \setminus X \to f_0(Y) \setminus f_0(X)$$

and

$$f\colon Y' \setminus \tilde{X} \to f(Y') \setminus f(\tilde{X})$$

are coverings of the same degree. The subsequent proof mirrors that of Case 2.

It remains to prove that there exists a hybrid conjugacy between F and g which respects the external markings. Keep in mind that we have constructed a conformal conjugacy between $f_0|_{D_{f_0}(\infty)}$ and $f|_{D_f(\infty)}$ such that $\phi = \Phi$ near infinity. Together with the property (4), we conclude that $\mathcal{R}'_v := \Phi(\gamma_v)$ is an external ray of f with the same external angle as $\mathcal{R}_v$ for all $v \in |S_0|$. Let Ψ be a hybrid conjugacy between $\widehat{F}$ and g, as given by the property (4), which sends the external marking $\widehat{\Gamma} = \{[\{v\} \times \gamma_v]\}_{v\in|S_0|}$ for $\widehat{F}$ to the standard external marking for g. It follows immediately that

$$\iota\colon \bigcup_{v\in|S_0|} \{v\} \times \Phi(\Pi(U_v)) \to \bigcup_{v\in|S_0|} \{v\} \times \mathbb{C}$$
$$(v, z) \mapsto \Psi((v, \Phi^{-1}(z)))$$

is a hybrid conjugacy between F and g sending the external marking $\Gamma = \{[\{v\} \times \mathcal{R}'_v]\}_{v\in|S_0|}$ for F to the standard external marking for g. □

3.1. **Quasiconformal surgery.** For each $v \in |S_0|$, set $m = \max_{v\in|S_0|} \operatorname{mod}(V_v \setminus \overline{U_v})$. Choose an annulus A_v lying in $V_v \setminus \overline{U_v}$ with the following properties.

- $\Pi((V_v \setminus \overline{U_v}) \setminus A_v)$ is contained in the basin of infinity of f_0.
- The core geodesic of A_v is homotopic to that of $V_v \setminus \overline{U_v}$ in the annulus $(V_v \setminus \overline{U_v})$.

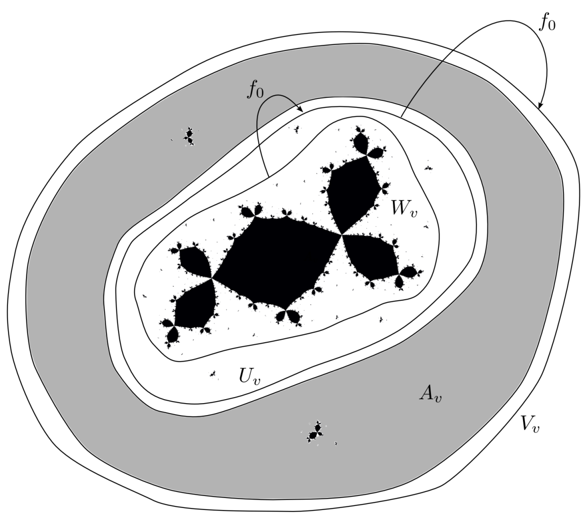

FIGURE 2. The domains U_v, V_v and W_v, and the annulus A_v (for the case $\sigma(v) = v$ for simplicity). Note that A_v contains $K(f_0) \cap (V_v \setminus \overline{U_v})$.

Indeed, this can be done easily since $\partial\Pi(U_v)$ and $\partial\Pi(V_v)$ are contained in the basin of infinity of f_0 (see Figure 2). Note that each ray $\mathcal{R}_v$ ($v \in |S_0|$) intersects ∂V_v (resp. ∂U_v) transversally at a single point x_v (resp. a_v).

We conclude this section by providing a proof of Lemma 3.2.

Proof of Lemma 3.2. For any $g \in \mathcal{C}(S_0)$, we first fix a GPL restriction

$$g\colon \bigcup_{v\in|S_0|} W'_v \to \bigcup_{v\in|S_0|} U'_v$$

of g such that

- $\min_{v\in|S_0|} \operatorname{mod}(U'_v \setminus \overline{W'_v}) \geq m$.
- Each external ray $\mathcal{R}_g(v, 0)$ intersects $\partial U'_v$ (resp. $\partial W'_v$) transversally at a single point a'_v and b'_v.

Indeed, such a GPL restriction can be chosen easily by requiring the boundaries of U'_v to be equipotential curves of g with sufficient large potential. For every $v \in |S_0|$, let $\varphi_v\colon U_v \to U'_v$ be a conformal homeomorphism such that $\varphi_v(v, a_v) = a'_v$. Set $W_v = \varphi_v^{-1}(W'_v)$ and $b_v = \varphi_v^{-1}(b'_v)$. Let $\widetilde{A}_v$ be an annulus with smooth boundaries contained compactly in $U_v \setminus \overline{W_v}$ such that

- $\delta(v) \operatorname{mod}(\widetilde{A}_v) = \operatorname{mod}(A_{\sigma(v)})$.

- The core geodesic of $\widetilde{A}_v$ is homotopic to that of $U_v \setminus \overline{W_v}$ in the annulus $(U_v \setminus \overline{W_v})$.

The existence of $\widetilde{A}_v$ is guaranteed by the fact that

$$\operatorname{mod}(U'_v \setminus \overline{W'_v}) \geq m \geq \operatorname{mod}(V_{\sigma(v)} \setminus \overline{U_{\sigma(v)}}) > \operatorname{mod} A_{\sigma(v)} > \frac{\operatorname{mod} A_{\sigma(v)}}{\delta(v)}.$$

We first define a quasiregular map $\widehat{F} \colon \bigcup_{v\in|S_0|} U_v \to \bigcup_{v\in|S_0|} V_v$ as follows. For every $v \in |S_0|$, we define $\widehat{F}(v,z) := \varphi^{-1}_{\sigma(v)} \circ g \circ \varphi_v(v,z)$ for $(v,z) \in W_v$. Clearly, $\widehat{F} \colon \bigcup_{v\in|S_0|} W_v \to \bigcup_{v\in|S_0|} U_v$ is conformally conjugate to $g \colon \bigcup_{v\in|S_0|} W'_v \to \bigcup_{v\in|S_0|} U'_v$. Moreover, a conformal conjugacy, between $\widehat{F}$ and g,

$$\begin{aligned} \hat{\imath} \colon \bigcup_{v\in|S_0|} U_v &\to \bigcup_{v\in|S_0|} U'_v \\ (v,z) &\mapsto \varphi_v(v,z) \end{aligned}$$

sends the external marking $\widehat{\Gamma} = \{[\varphi_v^{-1}(\mathcal{R}_g(v,0))]\}$ for $\widehat{F}$ to the standard external marking for g. Let $\widehat{F}|_{\widetilde{A}_v}$ be defined as an arbitrary holomorphic covering of degree $\delta(v)$ from $\widetilde{A}_v$ onto $A_{\sigma(v)}$. Let $\widehat{F}(v,z) := (\sigma(v), f_0^{r(v)}(z))$ for $(v,z) \in \partial U_v$. Up to this point, we have defined $\widehat{F}$ on $\bigcup_{v\in|S_0|} U_v \setminus (W_v \cup \widetilde{A}_v)$. Finally, we interpolate $\widehat{F}$ quasiregularly so that the path $\widehat{F}(\varphi_v^{-1}(\mathcal{R}_g(v,0) \cap (U'_v \setminus W'_v)))$ is homotopic to $\mathcal{R}_{\sigma(v)} \cap (V_{\sigma(v)} \setminus \overline{U_{\sigma(v)}})$ rel $\{(\sigma(v), x_{\sigma(v)}), (\sigma(v), a_{\sigma(v)})\}$ in the annulus $V_{\sigma(v)} \setminus \overline{U_{\sigma(v)}}$.

Now we proceed to construct a quasiregular map $\widehat{f} \colon \mathbb{C} \to \mathbb{C}$ so that

$$\widehat{F}(v,z) = (\sigma(v), \widehat{f}^{r(v)}(z))$$

for all $v \in |S_0|$ and $(v,z) \in U_z$. Note that for every $v \in |S_0|$,

$$f_0^{r(v)-1} \colon f_0(\Pi(U_v)) \to \Pi(V_{\sigma(v)})$$

is conformal. Let

$$\widehat{f}(z) = \begin{cases} f_0(z) & \text{if } z \in \bigcup_{v\in|S_0|} \mathbb{C} \setminus \Pi(U_v), \\ \left(f_0^{r(v)-1}|_{f_0(\Pi(U_v))}\right)^{-1} \circ \Pi \circ \widehat{F}(v,z) & \text{if } z \in \Pi(U_v),\ v \in |S_0|. \end{cases}$$

Clearly, $\widehat{f}$ satisfies properties (1) and (2) in the statement of Lemma 3.2.

It remains to check $\widehat{f}$ satisfies properties (3) and (4). Note that the non-regular part $\mathcal{NR} = \{z \in \mathbb{C} : \bar{\partial}\widehat{f}(z) \neq 0\}$ of $\widehat{f}$ is contained in $\bigcup_{v\in|S_0|} \Pi\left((U_v \setminus W_v) \setminus \widetilde{A}_v\right)$. Moreover, all the points in $\bigcup_{v\in|S_0|} \Pi\left((U_v \setminus W_v) \setminus \widetilde{A}_v\right)$ can never return back to this (wandering) set since

$$\bigcup_{v\in|S_0|} \widehat{f}^{r(v)}\left(\Pi((U_v \setminus W_v) \setminus \widetilde{A}_v)\right) = \bigcup_{v\in\sigma(|S_0|)} \left(\Pi\left((V_v \setminus U_v) \setminus A_v\right)\right)$$

is contained in the basin of infinity of f_0. Thus the property (3) holds. For every $v \in |S_0|$, we have to construct γ_v satisfying the property (4). Define γ_v to be

$\mathcal{R}_v \setminus \Pi(U_v)$ on $\mathbb{C} \setminus \Pi(U_v)$. Clearly,

$$\widehat{f}^{r(v)}(\gamma_v \setminus \Pi(U_v)) = f_0^{r(v)}(\mathcal{R}_v \setminus \Pi(U_v)) = \mathcal{R}_{\sigma(v)} \setminus \Pi(V_{\sigma(v)}) = \gamma_{\sigma(v)} \setminus \Pi(V_{\sigma(v)}).$$

Then we extend $\{\gamma_v\}_{v\in|S_0|}$ so that $\widehat{f}^{r(v)}(\gamma_v) = \gamma_{\sigma(v)}$ for all $v \in |S_0|$. We finish the proof by declaring that $\{[\{v\} \times \gamma_v]\}$ is the same external marking for $\widehat{F}$ as $\widehat{\Gamma} = \{[\varphi_v^{-1}(\mathcal{R}_g(v, 0))]\}$. Indeed, this follows from the fact that the two paths

$$\mathcal{R}_{\sigma(v)} \cap (V_{\sigma(v)} \setminus \overline{U_{\sigma(v)}}) = \gamma_{\sigma(v)} \setminus \overline{U_{\sigma(v)}}) \text{ and } \widehat{F}(\varphi_v^{-1}(\mathcal{R}_g(v, 0) \cap (U'_v \setminus W'_v)))$$

are homotopic rel $\{(\sigma(v), x_{\sigma(v)}), (\sigma(v), a_{\sigma(v)})\}$ in the annulus $V_{\sigma(v)} \setminus \overline{U_{\sigma(v)}}$.

□

4. Rigidity of maps in a fiber of π

Let $\chi\colon \mathcal{F}_0 \to \mathcal{C}(S_0)$ be the straightening map defined in section 2. Up to this point, we have showed that χ is a surjection. In this section, we finish the proof the main theorem by proving the following result.

Theorem 4.1. *Let f and $\widetilde{f}$ be maps in the fiber $\mathcal{F}_0$. If $\chi(f) = \chi(\widetilde{f})$, then $f = \widetilde{f}$.*

Proof. Let us first prove that f and $\widetilde{f}$ are quasiconformally conjugate. Note that there is a conformal conjugacy $\Phi\colon \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus K(\widetilde{f})$, between f and $\widetilde{f}$, off the filled Julia set $K(f)$. Moreover, $\Phi'(\infty) = 1$. Keep in mind that there is a GPL map

$$F_0 : \bigcup_{v\in|S_0|} U_v \to \bigcup_{v\in|S_0|} V_v,$$

$$(v, z) \mapsto (\sigma(v), f_0^{r(v)}(z))$$

over S_0 with filled Julia set $\bigcup_{v\in|S_0|} \{v\} \times v$. Here, $V_v = \{v\} \times Y_N(v)$ is the puzzle piece of depth N of f_0 that contains v for all $v \in |S_0|$ and N is given by Lemma 2.5. Let $\Psi\colon \mathbb{C} \setminus K(f) \to \mathbb{C} \setminus K(f_0)$ (resp. $\widetilde{\Psi}\colon \mathbb{C} \setminus K(\widetilde{f}) \to \mathbb{C} \setminus K(f_0)$) be the conformal conjugacy, between f (resp. $\widetilde{f}$) and f_0, off the filled Julia set $K(f)$ (resp. $K(\widetilde{f})$).

Convention. For any Jordan domain Y with $\partial Y \subset \mathbb{C} \setminus K(f_0)$, we use Y' (resp. $\widetilde{Y}$) to denote the Jordan domain bounded by $\Psi(\partial Y)$. (resp. $\widetilde{\Psi}(\partial Y)$).

Claim. *There exists a positive number M such that for every $v \in |S_0|$ and $n \geq N$, $\Phi\colon \partial Y'_n(v) \to \partial \widetilde{Y}_n(v)$ admits an M-quasiconformal extension Φ_n inside $Y'_n(v)$. Moreover, $\Phi_n \circ f = \widetilde{f} \circ \Phi_n$ on $\bigcap_{n\geq N} Y'_n(v)$.*

Once the Claim is proved, it is immediately implied by [14, Theorem 1.3] that Φ extends to a quasiconformal conjugacy between f and $\widetilde{f}$.

Now we turn to prove the Claim. The proof is based on induction and a pullback argument. Note that

$$F : \bigcup_{v\in|S_0|} U'_v \to \bigcup_{v\in|S_0|} V'_v,$$

$$(v, z) \mapsto (\sigma(v), f^{r(v)}(z))$$

and

$$\widetilde{F} : \bigcup_{v\in|S_0|} \widetilde{U}_v \to \bigcup_{v\in|S_0|} \widetilde{V}_v,$$

$$(v,z)\mapsto(\sigma(v),\widetilde{f}^{r(v)}(z))$$

are two hybrid equivalent GPL maps with fiber-wise connected Julia set. For every $v\in|S_0|$, let $K(v,F)$ and $K(v,\widetilde{F})$ denote the filled Julia set of F and $\widetilde{F}$ in the fiber $\{v\}\times\mathbb{C}$ respectively. Since $\chi(f)=\chi(\widetilde{f})$, there exists a hybrid conjugacy $H\colon \bigcup_{v\in|S_0|} U'_v\to\bigcup_{v\in|S_0|}\widetilde{U}_v$ between F and $\widetilde{F}$ respecting the external markings. Let $r=\max_{v\in|S_0|} r(v)$. Recall that N is an integer given by Lemma 2.5, which is bigger than r. We finish the proof of the Claim by constructing a sequence $\{\Phi_n\}_{n\geq N}$ of quasiconformal maps

$$\Phi_n\colon \bigcup_{v\in|S_0|} Y'_{N-r}(v)\to\bigcup_{v\in|S_0|}\widetilde{Y}_{N-r}(v)$$

with the following properties.

- $\Phi_n=\Phi$ on $\bigcup_{v\in|S_0|} D_f(\infty)\setminus Y'_n(v)$.
- $\Phi_n=\Pi\circ H(v,z)$ on $K(v,F)$ for all $v\in|S_0|$.
- $\mathrm{Dil}(\Phi_n)\leq\mathrm{Dil}(\Phi_N)$ for all $n\geq N$,

where $\mathrm{Dil}(\cdot)$ is the maximal dilatation. Define

$$\Phi_N(z)=\begin{cases}\Phi(z) & \text{if } z\in\bigcup_{v\in|S_0|} D_f(\infty)\setminus V'_v,\\ \Pi\circ H(v,z) & \text{if } z\in U'_v,\ v\in|S_0|.\end{cases}$$

Then we extend Φ_N to be a quasiconformal map from $\bigcup_{v\in|S_0|} Y'_{N-r}(v)$ onto $\bigcup_{v\in|S_0|}\widetilde{Y}_{N-r}(v)$ in an arbitrary way. Let $M=\mathrm{Dil}(\Phi_N)$. Now assume that we have constructed a quasiconformal map

$$\Phi_n\colon \bigcup_{v\in|S_0|} Y'_{N-r}(v)\to\bigcup_{v\in|S_0|}\widetilde{Y}_{N-r}(v)$$

with the desired properties. Let us construct a quasiconformal map

$$\Phi_{n+1}\colon \bigcup_{v\in|S_0|} Y'_{N-r}(v)\to\bigcup_{v\in|S_0|}\widetilde{Y}_{N-r}(v)$$

as follows. For every $v\in|S_0|$,

$$f^{r(v)}\colon Y'_N(v)\setminus K(v,F)\to Y'_{N-r(v)}(\sigma(v))\setminus K(\sigma(v),F)$$

and

$$\widetilde{f}^{r(v)}\colon \widetilde{Y}_N(v)\setminus K(v,\widetilde{F})\to\widetilde{Y}_{N-r(v)}(\sigma(v))\setminus K(\sigma(v),\widetilde{F})$$

are holomorphic coverings with the same degree $\delta(v)$. Thus we can lift

$$\Phi_n\colon Y'_{N-r(v)}(\sigma(v))\setminus K(\sigma(v),F)\to\widetilde{Y}_{N-r(v)}(\sigma(v))\setminus K(\sigma(v),\widetilde{F})$$

to

$$\Phi_{n+1}\colon Y'_N(v)\setminus K(v,F)\to\widetilde{Y}_N(v)\setminus K(v,\widetilde{F})$$

so that

- $\mathrm{Dil}(\Phi_{n+1}|_{Y'_N(v)\setminus K(v,F)})\leq\mathrm{Dil}(\Phi_n)\leq M$ for all $v\in|S_0|$.
- $\Phi_{n+1}=\Phi$ on $\bigcup_{v\in|S_0|} D_f(\infty)\cap(Y'_N(v)\setminus Y'_{n+1}(v))$.

Finally, let us extend Φ_{n+1} to an M-quasiconformal map from $\bigcup_{v\in|S_0|} Y'_{N-r}(v)$ onto $\bigcup_{v\in|S_0|} \widetilde{Y}_{N-r}(v)$ by setting $\Phi_{n+1}|_{Y'_{N-r}(v)\setminus Y'_N(v)} = \Phi_N$ and $\Phi_{n+1}|_{K(v,F)}(z) = \Pi \circ H(v,z)$ for all $v \in |S_0|$. Therefore, we have proved the Claim.

Bear in mind that we have proved that there exists a quasiconformal conjugacy Ψ between f and $\widetilde{f}$ such that $\Psi = \Phi$ on $D_f(\infty) = \mathbb{C} \setminus K(f)$ and $\Psi = \Pi \circ H(v,z)$ for all $v \in |S_0|$ and $(v,z) \in K(v,F)$. Let $\mathcal{K} = \bigcup_{v\in|S_0|} \Pi(K(v,F))$. It is fairly easy to see $\bar{\partial}\Psi = 0$ a.e. on $\mathbb{C} \setminus (K(f) \setminus \bigcup_{k\geq 0} f^{-k}(\mathcal{K}))$. Then Ψ is a conformal conjugacy between f and $\widetilde{f}$ since there is no invariant line fields on $K(f) \setminus \bigcup_{k\geq 0} f^{-k}(\mathcal{K})$ (see [9, 19, 12] for example). It implies that Ψ is the identity since $\Psi'(\infty) = \Phi'(\infty) = 1$. Hence $f = \widetilde{f}$.

□

Now we are in a position to prove the main theorem.

Proof of the Main Theorem. Firstly, $\mathcal{F}_0 = \pi^{-1}(\pi(f_0))$ is compact. The proof is essentially the same as that of [8, Theorem 3.5]. It remains to prove that $\mathcal{F}_0$ is connected. We will demonstrate that if E is a non-empty subset of $\mathcal{C}(f_0)$ that is both open and closed, then $\chi(E)$ is a closed subset within $\mathcal{C}(S_0)$. Coupled with the connectivity of $\mathcal{C}(S_0)$ ([18, Theorem 7], see also [1, 11]) and the bijectivity of χ, this observation ensures that $\mathcal{F}_0$ is inherently connected. Indeed, if $\mathcal{F}_0$ is not connected, then there exists two disjoint nonempty closed subsets E_1 and E_2 of $\mathcal{F}_0$ such that $E_1 \cup E_2 = \mathcal{F}_0$. It follows that $\mathcal{C}(S_0)$ is the union of $\chi(E_1)$ and $\chi(E_2)$, where $\chi(E_1)$ and $\chi(E_2)$ are two disjoint nonempty closed subsets of $\mathcal{C}(S_0)$. This is because χ is a bijection. However, this is impossible since $\mathcal{C}(S_0)$ is connected.

We are left with the task of showing that $\chi(E)$ is closed in $\mathcal{C}(S_0)$. Let a sequence g_n in $\chi(E)$ converge to g. We have to show that $g \in \chi(E)$. Note that E is compact since $\mathcal{F}_0$ is compact. Hence, passing to a subsequence, we may assume that $f_n = \chi^{-1}(g_n)$ converges to f_∞ in E. Recall that

$$F_0 : \bigcup_{v\in|S_0|} U_v \to \bigcup_{v\in|S_0|} V_v,$$

$$(v,z) \mapsto (\sigma(v), f_0^{r(v)}(z))$$

is a GPL map over S_0. For each $n \geq 1$, let ϕ_n be the conformal conjugacy between f_0 and f_n defined on $D_{f_0}(\infty)$.

Convention. For all $n \in \mathbb{Z}_+ \cup \{\infty\}$ and any subset Y in the fiber $\{v\} \times \mathbb{C}$, we denote $\{v\} \times \phi_n(\Pi(Y))$ by Y^n.

Note that for every $n \in \mathbb{Z}_+ \cup \{\infty\}$,

$$F_n : \bigcup_{v\in|S_0|} U_v^n \to \bigcup_{v\in|S_0|} V_v^n,$$

$$(v,z) \mapsto (\sigma(v), f_n^{r(v)}(z))$$

is a GPL map over S_0 and it is hybrid equivalent to g_n for $n \in \mathbb{Z}_+$. Since $f_n \to f_\infty$, we have

$$\inf_n \min_{v \in |S_0|} \operatorname{mod}(V_v^n \setminus U_v^n) > 0.$$

As in[5, Section 7] (see also [13, Proposition 2.5]), we may choose hybrid conjugacies H_n between F_n and g_n, respecting the external markings, so that the maximal dilatation of H_n is uniformly bounded. Therefore, passing to a further subsequence, we conclude that there exists a quasiconformal conjugacy H_∞ between F_∞ and g respecting the external marking. Thus f_∞ is conjugate to $\chi^{-1}(g)$ via a quasiconformal map H which is conformal outside the filled Julia set of f_∞ and satisfies $H'(\infty) = 1$.

Then the Beltrami path connecting f_∞ and $\chi^{-1}(g)$ is a connected set contained in $\mathcal{C}(f_0)$ intersecting E, hence it is contained in E and thus $g = \chi(\chi^{-1}(g)) \in \chi(E)$. □

References

1. B. Branner and J. Hubbard. *The Iteration of Cubic Polynomials Part I: The Global Topology of Parameter Space*. Acta Math., **160** (1988), 143-206.
2. L. DeMarco and K. Pilgrim. *Polynomial Basins of Infinity*. Geom. Funct. Anal. **21** (2011), 920-950.
3. L. DeMarco and C. T. McMullen. *Trees and the Dynamics of Polynomials*. Annales Scientifiques de l'Ecole Normale Supérieure, 2008, 41(3): 337-383.
4. A. Douady and H. Hubbard, *Étude Dynamique des Polynômes Complexes*. Publications Mathématiques d'Orsay, Université de Paris-Sud, Département de Mathématiques, Orsay, 1984.
5. A. Douady and H. Hubbard, *On the Dynamics of Polynomial-like Mappings*, Ann. Sci. Ecole Norm. Sup. (4) **18** (1985), 287-343.
6. H. Inou, *Combinatorics and Topology of Straightening Maps II: Discontinuity*, arXiv:0903.4289v2.
7. H. Inou and J. Kiwi, *Combinatorics and Topology of Straightening Maps I: Compactness and Bijectivity*. Adv. Math. **231** (2012), 2666-2733.
8. H. Inou and Y. Wang. *Straightening Maps for Polynomials with One Bounded Critical Orbit*. The Journal of Geometric Analysis, **33** (2023), 10: 314.
9. O. Kozlovski and S. van Strien. *Local Connectivity and Quasi-Conformal Rigidity of Non-Renormalizable Polynomials*. Proceedings of the London Mathematical Society, **99** (2009), 275-296.
10. O. Kozlovski, W. Shen, and S. van Strien. *Rigidity for Real Polynomials*, Ann. Math., **165** (2007), 749-841.
11. P. Lavaurs. PhD Thesis at University of Paris Sud, Orsay, 1989.
12. G. Levin and S. van Strien, *Total Disconnectedness and Absence of Invariant Linefields for Real Polynomials*, Asteriques, 261 (2000), 161-172.
13. M. Lyubich. *Dynamics of Quadratic Polynomials I-II*, Acta Mathematica 178.2 (1997): 185-297.
14. W. Peng and L. Tan. *Quasi-conformal Rigidity of Multicritical Maps*. Transactions of the American Mathematical Society, **367** (2015), 1151-1182.
15. C. L. Petersen and S. Zakeri. *Periodic Points and Smooth Rays*. Conform. Geom. Dyn. **25**, 170–178 (2021).
16. W. Qiu and Y. Yin. *Proof of the Branner-Hubbard Conjecture on Cantor Julia Sets*. Science in China Series A: Mathematics, 2009, 52(1): 45-65.
17. M. Shishikura, *On the Quasiconformal Surgery of Rational Functions*. Ann. Sci. Ecole Norm. Sup. **20** (1987), 1-29.
18. W. Shen and Y. Wang. *Primitive Tuning via Quasiconformal Surgery*. Isr. J. Math. 245, 259–293 (2021).
19. Y. Yin and T. Zhai. *No Invariant Line Fields on Cantor Julia Sets*. arXiv preprint math/0609255, 2006.

20. Y. Zhai. *Rigidity for Rational Maps with Cantor Julia Sets*. Science in China Series A: Mathematics, **51** (2008), 79-92.

Department of Mathematics, Kyoto University
Email address: inou@math.kyoto-u.ac.jp

Department of Mathematics, Shanghai Normal University
Email address: yiminwang@shnu.edu.cn