PULLBACK OF THE LIFTING OF ELLIPTIC CUSP FORMS AND MIYAWAKI’S CONJECTURE

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To the memory of Prof. Isao Miyawaki

Abstract. We construct a lifting from Siegel cusp forms of degree $r$ to Siegel cusps form of degree $r + 2n$. For $r = n = 1$, our result is a partial solution of a conjecture made by Miyawaki in 1992. In particular, we can calculate the standard $L$-function of a cusp form of degree 3 and weight 12, which is in accordance with Miyawaki’s conjecture. We will give a conjecture on the Petersson inner product of the lifting in terms of certain $L$-values.

Introduction

Let $f(\tau) \in S_{2k}(\text{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. In [18], we have constructed a lifting to a Siegel cusp form of even degree. Let $F(Z)$ be a lifting of $f(\tau)$. In this paper, we shall consider the pullback of $F(Z)$ to a block diagonal subset.

Let us recall the theory of pullback of an Eisenstein series to a block diagonal subset (cf. [3], [16]). Let $M_k(\text{Sp}_n(\mathbb{Z}))$ (resp. $S_k(\text{Sp}_n(\mathbb{Z}))$) be the space of Siegel modular forms (resp. Siegel cusp forms) of degree $n$ and weight $k$. Assume that $g(Z) \in S_{2l}(\text{Sp}_r(\mathbb{Z}))$ is a Hecke eigenform whose standard $L$-function is $L(s, g, \text{st})$. For $m \geq r$, let $E^{(m+r)}_{2l}(Z)$ be the Siegel Eisenstein series of degree $m + r$ and weight $2l$. Assume, for simplicity, $E^{(m+r)}_{2l}(Z)$ is absolutely convergent. Put $g^c(Z) = g(-Z)$. Note that $g^c(Z)$ is the cusp form obtained by taking complex conjugates of Fourier coefficients. Then

$$
\int_{\text{Sp}_r(\mathbb{Z}) \backslash H_r} E^{(m+r)}_{2l} \left( \begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix} \right) \frac{g^c(W)(\det \text{Im} W)^{2l-r-1}}{dW}
$$

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is equal to the Klingen Eisenstein series $E^{(m)}(g; Z) \in M_{2l}(\text{Sp}_m(Z))$, up to multiplication by some $L$-values and elementary factors. In this theory, the unwinding method of the Eisenstein series played an important role.

Now let $h(\tau) \in S^+_{k+(1/2)}(\Gamma_0(4))$ be a Hecke eigenform in the Kohnen plus subspace $S^+_{k+(1/2)}(\Gamma_0(4))$ corresponding to the normalized Hecke eigenform $f(\tau)$. Put $L(s, f) = \sum_{N=1}^{\infty} a(N) N^{-s}$.

Let $n$ be a non-negative integer such that $n + r \equiv k \mod 2$. In [18], we have constructed a Hecke eigenform $F(Z) \in S_{k+n+r}(\text{Sp}_{2n+2r}(Z))$ whose standard $L$-function is equal to

$$\zeta(s) \prod_{i=1}^{2n+2r} L(s + k + n + r - i, f).$$

Note that $F(Z)$ is determined by $h(\tau)$. We shall call $F(Z)$ a Duke-Imamoglu lift of $f(\tau)$ (or $h(\tau)$) to degree $2n + 2r$. Assume that $2l = k + n + r$ and $g \in S_{k+n+r}(\text{Sp}_r(Z))$.

Now we consider the function $F_{h,g}(Z)$ defined by the integral

$$F_{h,g}(Z) = \int_{\text{Sp}_r(Z) \setminus h_2} F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) \overline{g^c(W)(\det \text{Im} W)^{k+n-1}} dW,$$

for $Z \in h_{2n+r}$. Note that $F_{h,g}$ is always cusp form, as $F(Z)$ is a cusp form. Then our main theorem is as follows.

**Theorem 1.1.** Assume that $F_{h,g}(Z)$ is not identically zero. Then the cusp form $F_{h,g}(Z)$ is a Hecke eigenform whose standard $L$-function is equal to

$$L(s, F_{h,g}, \text{st}) = L(s, g, \text{st}) \prod_{i=1}^{2n} L(s + k + n - i, f).$$

As the usual unwinding method does not work for the cusp form $F(Z)$, we will make use of local representation theory instead.

It is an interesting problem to determine when $F_{h,g} \neq 0$. We will give a conjecture for the Petersson inner product $\langle F_{h,g}, F_{h,g} \rangle$. Let $L(s, \text{st}(g) \boxtimes f)$ be the “tensor product” $L$-function of $L(s, g, \text{st})$ and $L(s, f)$. Let $\Lambda(s, \text{st}(g) \boxtimes f)$ be the product of $L(s, \text{st}(g) \boxtimes f)$ and its gamma factor. We also define $\Lambda(s, f, \text{Ad})$ (resp. $\zeta(s)$) as the product of the adjoint $L$-function $L(s, f, \text{Ad})$ (resp. Riemann zeta function) and some gamma function, which is slightly modified from the usual gamma factor. Then our conjecture is as follows.
Conjecture 5.1. Assume that \( n < k \). Then there exists an integer \( \alpha = \alpha(r, n, k) \) depending only on \( r, n, \) and \( k \) such that

\[
\Lambda(k + n, st(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i - 1, f, \text{Ad}) \tilde{\xi}(2i) = 2^{\alpha} \langle f, f \rangle \langle h, h \rangle \langle F_h, g \rangle \langle g, g \rangle.
\]

In particular, \( \mathcal{F}_{h, g} \) is non-zero if and only if \( \Lambda(k + n, st(g) \boxtimes f) \neq 0 \).

This paper is organized as follows. In §1, we formulate our main theorem. In §2, we discuss the relation to Miyawaki’s conjecture [27]. In §3, we develop some local representation theory. Using this representation theoretic argument, we prove our main theorem in §4. In §5, we formulate the conjecture and discuss some examples. We shall show that, if the conjecture is true, then the roles of \( g \) and \( \mathcal{F}_{h, g} \) can be interchanged. Note that this phenomenon does not have an analogue for the Eisenstein case for \( n > 0 \), since the Klingen Eisenstein series \( E^{(r + 2n)}(g, Z) \) is not an cusp form unless \( n = 0 \). The exceptional case \( n = 0 \) is discussed in §6. We shall show that an analogue of the conjecture for the Eisenstein case holds in that case.

In §7, we recall the result of Nebe and Venkov [30]. They determined Hecke eigenvectors in the space of theta functions associated to 24 Niemeier lattices. Using our theory, we can determine standard \( L \)-functions of 20 eigenvectors. In Appendix, we attach some computer calculation for an evidence for Conjecture 5.1.

Notation

If \( R \) is a ring, the symplectic group \( \text{Sp}_m(R) \) is defined by

\[
\text{Sp}_m(R) = \left\{ g \in \text{GL}_m(R) \left| g \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} g^{t} = \begin{pmatrix} 0_m & -1_m \\ 1_m & 0_m \end{pmatrix} \right. \right\}.
\]

We denote the Siegel upper-half plane of degree \( m \) by \( \mathfrak{h}_m \). For \( 2k = 12, 16, 18, 20, 22, \) or 26, the normalized Hecke eigenform of weight \( 2k \) is denoted by \( \phi_{2k}(\tau) \). Note that \( \phi_{12}(\tau) = \Delta(\tau) \). The space of Siegel modular forms with degree \( m \) and weight \( k \) is denoted by \( M_k(\text{Sp}_m(\mathbb{Z})) \) or by \( M_k^{(m)} \). The subspace of cusp forms of degree \( m \) and weight \( k \) is denoted by \( M_k(\text{Sp}_m(\mathbb{Z})) \) or by \( M_k^{(m)} \). For \( g \in M_k(\text{Sp}_m(\mathbb{Z})) \), we put \( g^c(Z) = \overline{g(-Z)} \). The Petersson inner product is denoted by \( \langle , \rangle \). When \( f, g, \) or \( h \) are Hecke eigenform, \( \mathbb{Q}(f), \mathbb{Q}(g) \) etc. are the field generated by Hecke eigenvalues. The (multi)set \( \{ \beta_1, \beta_1^{-1}, \beta_2, \beta_2^{-1}, \ldots, \beta_n, \beta_n^{-1} \} \) is sometimes denoted by \( \{ \beta_1^\pm, \beta_2^\pm, \ldots, \beta_n^\pm \} \).
1. Statement of the main theorem

As in Introduction, let

\[ f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\text{SL}_2(\mathbb{Z})) \]

and

\[
L(s, f) = \prod_p \left(1 - a(p)p^{-s} + p^{2k-1-2s}\right)^{-1} \\
= \prod_p \left[\left(1 - \alpha_p p^{k-s-(1/2)}\right)(1 - \alpha_p^{-1} p^{k-s-(1/2)})\right]^{-1}
\]

be a normalized Hecke eigenform and its \(L\)-function. In the Kohnen plus subspace \(S_{k+1/2}^+(\Gamma_0(4))\), there exists a Hecke eigenform \(h(\tau) = \sum_{N>0} c(N)q^N\) corresponding to \(f(\tau)\) by the Shimura correspondence. As is well-known, \(h(\tau)\) is unique up to a scalar. Let \(r\) and \(n\) be non-negative integers such that \(n + r \equiv k \mod 2\). By [18], there exists a Hecke eigenform \(F(Z) \in S_{k+n+r}(\text{Sp}_{2n+2r}(\mathbb{Z}))\), whose standard \(L\)-function is equal to

\[ \zeta(s) \prod_{i=1}^{2n+2r} L(s + k + n + r - i, f). \]

Moreover, if \(B\) is a positive definite half-integral symmetric matrix of size \(2r + 2n\) such that \((-1)^{r+n} \det(2B)\) is a fundamental discriminant, then the \(B\)-th Fourier coefficient of \(F\) is equal to \(c(\det(2B))\). Note that \(F(Z)\) is determined by \(h(\tau)\).

Let \(g(Z) \in S_{k+n+r}(\text{Sp}_r(\mathbb{Z}))\) be a Hecke eigenform, whose standard \(L\)-function is

\[ L(s, g, \text{st}) = \prod_p \left[\left(1 - p^{-s}\right) \prod_{i=1}^{r} (1 - \beta_i p^{-s})(1 - \beta_i^{-1} p^{-s})\right]^{-1}. \]

We shall call \(\{\beta_1^{\pm 1}, \ldots, \beta_r^{\pm 1}\}\) the Satake parameter in this paper. We put

\[ \mathcal{F}_{h,g}(Z) = \int_{\text{Sp}_r(\mathbb{Z}) \backslash \text{Bun}_r} F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) g^c(W)(\det \text{Im} W)^{k+n-1} dW, \]

Then we have \(\mathcal{F}_{h,g} \in S_{k+n+r}(\text{Sp}_{2n+2r}(\mathbb{Z})).\) Now our main theorem is as follows.
**Theorem 1.1.** Assume that $\mathcal{F}_{h,g}(Z)$ is not identically zero. Then the cusp form $\mathcal{F}_{h,g}(Z)$ is a Hecke eigenform whose standard $L$-function is equal to

$$L(s, \mathcal{F}_{h,g}, \text{st}) = L(s, g, \text{st}) \prod_{i=1}^{2n} L(s + k + n - i, f).$$

**Remark 1.1.** When $r = 1$, the $L$-function $L(s, g, \text{st})$ is an Euler product of degree 3, and should not be confused with $L(s, g)$. To avoid possible confusion, we denote $L(s, g, \text{Ad})$ rather than $L(s, g, \text{st})$ for $r = 1$. Note also that the meaning of the Satake parameter for $f \in S_{2k}(\text{Sp}_1(\mathbb{Z}))$ is different from the usual one. In our convention, the Satake parameter of $f$ is $\{\alpha_p^{\pm 2}\}$.

**Remark 1.2.** We can interpret our theorem in terms of the Arthur conjecture. As in [18], we denote the hypothetical Langlands group by $\mathcal{L}$. Let $\tau$ be the cuspidal automorphic representation of $\text{GL}_2(\mathbb{C})$ generated by $f$, and $\rho_{\tau} : \mathcal{L} \rightarrow \text{SL}_2(\mathbb{C})$ the associated homomorphism.

Let $\rho_{g} : \mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_{2r+1}(\mathbb{C}) = \text{LSp}_r$ be the Arthur parameter for the cuspidal automorphic representation generated by $g(Z)$. Then the Arthur parameter for $\mathcal{F}_{h,g}$ should be given by the composition

$$\mathcal{L} \times \text{SL}_2(\mathbb{C}) \rightarrow \text{SO}_{4n}(\mathbb{C}) \times \text{SO}_{2r+1}(\mathbb{C}) \hookrightarrow \text{SO}_{4n+2r+1}(\mathbb{C}) = \text{LSp}_{2n+r}.$$ 

Here the first homomorphism is given by $(\rho_r \boxtimes \text{Sym}_{2n-1}) \times \rho_{g}$. (cf. [18]).

2. **Miyawaki’s conjecture**

It is known that $\dim_{\mathbb{C}} S_{12}(\text{Sp}_3(\mathbb{Z})) = 1$. Let $\Phi_{12}^{(3)}(Z) \in S_{12}(\text{Sp}_3(\mathbb{Z}))$ be a non-zero cusp form. Miyawaki [27] calculated some Hecke eigenvalues of the cusp form $\Phi_{12}^{(3)}(Z)$. Based on the numerical calculation, he made the following conjectures.

**Conjecture 2.1** (Miyawaki). The standard $L$-function of $\Phi_{12}^{(3)}(Z)$ is given by

$$L(s, \Phi_{12}^{(3)}, \text{st}) = L(s, \Delta, \text{Ad})L(s + 10, \phi_{20})L(s + 9, \phi_{20}).$$

More generally,

**Conjecture 2.2** (Miyawaki). Given normalized Hecke eigenforms $f \in S_{2k-4}(\text{SL}_2(\mathbb{Z}))$ and $g \in S_{k}(\text{SL}_2(\mathbb{Z}))$, there should be a Hecke eigenform $F_{f,g} \in S_{k}(\text{Sp}_3(\mathbb{Z}))$ whose standard $L$-function is equal to

$$L(s, g, \text{Ad})L(s + k - 2, f)L(s + k - 3, f).$$
In fact, Miyawaki formulated Conjecture 2.2 in terms of linear maps
\[ S_{2k-4}(\text{SL}_2(\mathbb{Z})) \otimes S_k(\text{SL}_2(\mathbb{Z})) \rightarrow S_k(\text{Sp}_3(\mathbb{Z})). \]
It seems there is no such a canonical map, but our construction defines a canonical map
\[ S_{k-(3/2)}^{+}(\Gamma_0(4)) \otimes S_k(\text{SL}_2(\mathbb{Z})) \rightarrow S_k(\text{Sp}_3(\mathbb{Z})) \]
induced by the bilinear map \( h \times g \mapsto F_{h,g} \). Note that Kohnen [20] defined a canonical linear map
\[ S^{+}_{k+(1/2)}(\Gamma_0(4)) \rightarrow S_k(\text{Sp}_{2n+2r}(\mathbb{Z})) \]
which coincides with the Duke-Imamoglu lifting when \( h \) is a Hecke eigenform. If \( F_{h,g} \) is non-zero for each Hecke eigenform \( h \) and \( g \), then Theorem 1.1 solves the Conjecture 2.2.

The author would like to propose to call \( G \) the Miyawaki lift of \( g(\mathbb{Z}) \in S_{k+r+n}(\text{Sp}_r(\mathbb{Z})) \) with respect to the Duke-Imamoglu lift \( F(\tau) \in S_{k+r+n}(\text{Sp}_{2r+2n}(\mathbb{Z})) \) of \( f(\tau) \), if \( G = cF_{h,g} \) for some \( c \neq 0 \).

In §7, we will show that \( \Phi_{14}^{(3)} \) is in fact the Miyawaki lifting of \( \Delta \) with respect to the Duke-Imamoglu lift of \( \phi_{20} \) to degree 4. In particular, Conjecture 2.1 is true.

Remark 2.1. In [27], Miyawaki also considered the spin \( L \)-functions, which we do not consider here. He also considered the spin and standard \( L \)-functions of \( \Phi_{14}^{(3)} \in S_{14}(\text{Sp}_3(\mathbb{Z})) \) and its generalization. He conjectured that the standard \( L \)-function of the cusp form \( \Phi_{14}^{(3)}(\mathbb{Z}) \) is equal to
\[ L(s, \Delta, \text{Ad})L(s + 13, \phi_{26})L(s + 12, \phi_{26}). \]
It seems that one needs an analogue of the lifting [18] such that the infinite part of the automorphic representation generated by the lifting is a cohomological induction from non-compact unitary group, to solve this conjecture. In fact, the Arthur conjecture suggests that there exists an irreducible discrete automorphic representation \( \pi \) of \( \text{Sp}_4(\mathbb{A}_Q) \) satisfying the following (i) and (ii):

(i) The standard \( L \)-function of \( \pi \) is \( \zeta(s) \prod_{i=11}^{14} L(s + i, \phi_{26}) \).
(ii) The infinite component of \( \pi \) is a cohomological induction from the non-compact unitary group \( U(3,1) \).

The infinite component of \( \pi \) is a non-tempered unitary representation with minimal \( K \)-type \( (14, 14, 14, -12) \). Taking a convolution with \( \Delta(\tau) \), one would get \( \Phi_{14}^{(3)}(\mathbb{Z}) \). It is very likely that \( \pi \) is generated by certain residue of the Eisenstein series associated to parabolic subgroup \( P_{2,2} \) with Levi factor \( \text{GL}_2 \times \text{Sp}_2 \).
3. Unramified principal series of $p$-adic groups

In this section, we shall prove some results on unramified principal series of symplectic groups over a $p$-adic field.

In this section, $F$ denotes a non-archimedean local field of characteristic 0. The symbols $\varpi$ and $q$ denote a prime element and the order of the residue field of $F$, respectively. An algebraic group and its group of $F$-rational points are denoted by the same symbol.

When $G$ is a locally compact group, $\delta_G$ is the modulus character of $G$. If $(\rho, V_\rho)$ and $(\rho', V_{\rho'})$ are smooth representation of a totally disconnected locally compact group $G$, then $\mathcal{B}_G(\rho, \rho')$ is the space of bilinear form $B$ on $V_\rho \times V_{\rho'}$ such that $B(\rho(g)v, \rho'(g)v') = B(v, v')$ for any $v \in V_\rho$, $v' \in V_{\rho'}$, and $g \in G$. Note that if $\rho'$ is admissible, then $\mathcal{B}_G(\rho, \rho') \cong \text{Hom}_G(\rho, \tilde{\rho}')$.

When $\rho$ is a smooth representation of a closed subgroup $H$ of a totally disconnected locally compact group $G$, we denote the normalized induced representation (resp. normalized compactly induced representation) by $\text{Ind}_H^G\rho$ (resp. $\text{c-Ind}_H^G\rho$).

Fix integers $m$ and $r$ such that $m \geq r \geq 0$. We put $G_1 = \text{Sp}_r$, $G_2 = \text{Sp}_m$, and $H = \text{Sp}_{m+r}$. We denote the Siegel parabolic subgroup of $H$ by $P_H$. $G_1 \times G_2$ can be embedded into $H$ by

$$(A_1 \ B_1) \times (A_2 \ B_2) \mapsto \begin{pmatrix} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{pmatrix}.$$ 

We think of $G_1 \times G_2$ as a subgroup of $H$.

For $i = 0, 1, \ldots, r$, put

$$\eta_i = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & -1_i & 0 \\ 0 & 1_{r-i} & 0 & 0 & 0 & 0 & 0 \\ 1_i & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1_{m-i} & 0 & 0 & 0 \\ 1_i & 0 & 1_i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1_{r-i} & 0 & 0 \\ 0 & 0 & 0 & 1_i & 0 & -1_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1_{m-i} \end{pmatrix}.$$ 

Here the size of the blocks are $i, r-i, i, m-i, i, r-i, i$, and $m-i$.

The following lemma is well-known (cf. [4], [16]).

**Lemma 3.1.** The set $\{\eta_0, \eta_1, \ldots, \eta_r\}$ forms a set of representatives for the double cosets $P_H \backslash H / (G_1 \times G_2)$.

\[\square\]
For $i = 0, 1, \ldots, r$, put $Q_i = (\eta_i^{-1} P_H \eta_i) \cap (G_1 \times G_2)$. Then, by direct calculation, we have

$$Q_i = \left\{ \begin{pmatrix} \alpha & 0 & \beta & \ast \\ \ast & A & \ast & \ast \\ \gamma & 0 & \delta & \ast \\ 0 & 0 & 0 & D \end{pmatrix} \times \begin{pmatrix} \alpha & 0 & -\beta & \ast \\ \ast & A' & \ast & \ast \\ -\gamma & 0 & \delta & \ast \\ 0 & 0 & 0 & D' \end{pmatrix} \in G_1 \times G_2 \right\} \cap (G_1 \times G_2)$$

$$(\alpha \; \beta \; \gamma \; \delta) \in \mathrm{Sp}_i, \; A = 'D^{-1} \in \mathrm{GL}_{r-i}, \; A' = 'D'^{-1} \in \mathrm{GL}_{m-i}, \right\}.$$

We define the parabolic subgroups $P_i^{(1)} \subset G_1$ and $P_i^{(2)} \subset G_2$ by

$$P_i^{(1)} = \left\{ \begin{pmatrix} \alpha & 0 & \beta & \ast \\ \ast & A & \ast & \ast \\ \gamma & 0 & \delta & \ast \\ 0 & 0 & 0 & D \end{pmatrix} \in G_1 \right\}, \quad (\alpha \; \beta \; \gamma \; \delta) \in \mathrm{Sp}_i, \; A = 'D^{-1} \in \mathrm{GL}_{r-i},$$

$$P_i^{(2)} = \left\{ \begin{pmatrix} \alpha & 0 & \beta & \ast \\ \ast & A' & \ast & \ast \\ \gamma & 0 & \delta & \ast \\ 0 & 0 & 0 & D' \end{pmatrix} \in G_2 \right\}, \quad (\alpha \; \beta \; \gamma \; \delta) \in \mathrm{Sp}_i, \; A' = 'D'^{-1} \in \mathrm{GL}_{m-i},$$

**Lemma 3.2.** Let $\iota$ be the automorphism of $\mathrm{Sp}_i$ given by

$$(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}) \mapsto (\begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}).$$

For any irreducible admissible representation $\pi$ of $\mathrm{Sp}_i$, we have $\pi \circ \iota \simeq \tilde{\pi}$.

**Proof.** A proof of this lemma can be found in [28], Chapter 4-II.

**Lemma 3.3.** Let $G$ be a unimodular totally disconnected locally compact group, and $\rho$ and $\rho'$ irreducible admissible representations of $G$. If $B_{G \times G}(\mathrm{c-Ind}^G_{\Delta G}1, \rho \boxtimes \rho') \neq 0$, then $\rho' \simeq \tilde{\rho}$. Here, $\Delta G$ is the diagonal subgroup of $G \times G$.

**Proof.** This lemma seems well-known, but for the sake of completeness, we give a proof. Note that $\mathrm{c-Ind}^G_{\Delta G}1 \simeq C^\infty_0(G)$ by restriction to the second factor. For each compact open subgroup $K$ of $G$, we put

$$e_K = \text{Volume}(K)^{-1} \times (\text{characteristic function of } K).$$

We define an injection

$$\varphi : B_{G \times G}(C^\infty_0(G), \rho_1 \boxtimes \rho_2) \to B_G(\rho, \rho').$$
as follows. Given $U \in \mathcal{B}_{G \times G}(C^\infty_0(G), \rho \boxtimes \rho')$, $w \in \rho$, and $w' \in \rho'$, we put
\[
\varphi(U)(w, w') = U(e_K, w \boxtimes w')
\]
for sufficiently small open compact subgroup $K$. It is easy to check that this definition does not depend on the choice of $K$ and that $\varphi$ is an injective map. Hence the lemma. 

Let $\pi_1$ (resp. $\pi_2$) be an irreducible unramified principal series representation of $G_1$ (resp. $G_2$). Then there exist unramified quasi-characters $\lambda_1, \lambda_2, \ldots, \lambda_r$ (resp. $\lambda'_1, \lambda'_2, \ldots, \lambda'_m$) such that $\pi_1$ (resp. $\pi_2$) is the unique unramified constituent of the induced representation

\[
\text{Ind}_{B_{G_1}}^{G_1} \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_r
\]
(resp. $\text{Ind}_{B_{G_2}}^{G_2} \lambda'_1 \boxtimes \lambda'_2 \boxtimes \cdots \boxtimes \lambda'_m$).

Here, $B_{G_1}$ (resp. $B_{G_2}$) is a Borel subgroup of $G_1$ (resp. $G_2$). Put $\beta_i = \lambda_i(\varpi)$ $(i = 1, 2, \ldots, r)$ and $\beta'_j = \lambda'_j(\varpi)$ $(j = 1, 2, \ldots, m)$. By definition, the set of the Satake parameters of $\pi_1$ and $\pi_2$ are $\{\beta_1^{\pm 1}, \beta_2^{\pm 1}, \ldots, \beta_r^{\pm 1}\}$ and $\{\beta'_1^{\pm 1}, \beta'_2^{\pm 1}, \ldots, \beta'_m^{\pm 1}\}$, respectively.

Note that the standard Levi subgroup of $PH$ is isomorphic to $GL_{m+r}$.

A one-dimensional representation of $GL_{m+r}$ is of the form $\omega \cdot \det$ for some quasi-character $\omega : F^\times \to \mathbb{C}^\times$. The induced representation $\text{Ind}_{PH}^H(\omega \cdot \det)$ is called a degenerate principal series.

**Proposition 3.1.** Let $\omega : F^\times \to \mathbb{C}^\times$ be an unramified quasi-character. Put $\alpha = \omega(\varpi)$. If

\[
\mathcal{B}_{G_1 \times G_2}(\text{Ind}_{PH}^H(\omega^{-1} \cdot \det)|_{G_1 \times G_2}, \pi_1 \boxtimes \pi_2) \neq \{0\},
\]

then as a multiset, $\{\beta_1^{\pm 1}, \beta_2^{\pm 1}, \ldots, \beta_r^{\pm 1}\}$ is equal to

\[
\{\beta_1^{\pm 1}, \beta_2^{\pm 1}, \ldots, \beta_r^{\pm 1}\} \cup \{(\alpha^{\pm 1}q^{(m-r-1)/2}, \alpha^{\pm 1}q^{(m-r-3)/2}, \ldots, \alpha^{\pm 1}q^{-(m-r-1)/2}\}.
\]

**Proof.** We proceed as in Rallis [32] Chapter II. Let $X_i$ $(i = 0, \ldots, r)$ be the subspace of $\text{Ind}_{PH}^H(\omega^{-1} \cdot \det)$ that consists of the elements whose supports are contained in

\[
\bigcup_{j=i}^r PH \, \eta_i(G_1 \times G_2).
\]

We put $X_{r+1} = \{0\}$. Then

\[
\{0\} = X_{r+1} \subset X_r \subset \cdots \subset X_1 \subset X_0 = \text{Ind}_{PH}^H(\omega^{-1} \cdot \det)
\]
are $G_1 \times G_2$ invariant subspaces, and
\[ X_i/X_{i+1} \simeq \text{c-Ind}_{Q_i}^{G_i} \omega_i \delta_{P_i}^{-1/2}. \]
Here $\delta_{P_i}$ (resp. $\delta_{Q_i}$) is the modulus character of $P_i$ (resp. $Q_i$), and $\omega_i$ is the character of $Q_i$ defined by
\[ \omega_i(t) = (\omega^{-1} \det(\eta_i t \eta_i^{-1}) \delta_{P_i}^{1/2}(\eta_i t \eta_i^{-1}). \]
It is easy to see
\[ \omega_i(t) = \omega^{-1}(\det A \det A') \det A \det A'|^{(m+r+1)/2}, \]
\[ \delta_{Q_i}^{1/2}(t) = |\det A|^{(r+i+1)/2} \det A'|^{(m+i+1)/2} \]
for
\[ t = \begin{pmatrix} \alpha & 0 & \beta & * \\ * & A & * & * \\ \gamma & 0 & \delta & * \\ 0 & 0 & 0 & D \end{pmatrix} \times \begin{pmatrix} \alpha & 0 & -\beta & * \\ * & A' & * & * \\ -\gamma & 0 & \delta & * \\ 0 & 0 & 0 & D' \end{pmatrix} \in Q_i. \]

The Jacquet modules $r_{\nu_i}^{G_1} \pi_1$ and $r_{\nu_i}^{G_2} \pi_2$ are representations of $\text{Sp}_1 \times \text{GL}_{r-i}$ and $\text{Sp}_2 \times \text{GL}_{m-i}$, respectively. By Lemma 3.2 and Lemma 3.3, the Jacquet modules $r_{\nu_i}^{G_1} \pi_1$ and $r_{\nu_i}^{G_2} \pi_2$ have irreducible subquotients of the form
\[ \rho^{(1)} \boxtimes (\omega \cdot \det)|^{-(m-i)/2} \]
and
\[ \rho^{(2)} \boxtimes (\omega \cdot \det)|^{-(r-i)/2}, \]
respectively, such that $\rho^{(1)} \simeq \rho^{(2)}$ for some $i$ ($0 \leq i \leq r$).
Let $\{\beta^{\nu_1}_{\pm 1}, \beta^{\nu_2}_{\pm 1}, \ldots, \beta^{\nu_i}_{\pm 1}\}$ be the set of Satake parameters of $\rho^{(1)} \simeq \rho^{(2)}$. Then the set of Satake parameters of $\pi_1$ is
\[ \{\beta^{\nu_1}_{\pm 1}, \beta^{\nu_2}_{\pm 1}, \ldots, \beta^{\nu_i}_{\pm 1}\} \]
\[ \cup \{\alpha q^{(m-r+1)/2}\pm 1, \alpha q^{(m-r+3)/2}\pm 1, \ldots, \alpha q^{(m-r+2i-1)/2}\pm 1\}. \]
On the other hand, the set of Satake parameters of $\pi_2$ is
\[ \{\beta^{\nu_1}_{\pm 1}, \beta^{\nu_2}_{\pm 1}, \ldots, \beta^{\nu_i}_{\pm 1}\} \]
\[ \cup \{\alpha q^{(r-m+1)/2}\pm 1, \alpha q^{(r-m+3)/2}\pm 1, \ldots, \alpha q^{(r-m+2i-1)/2}\pm 1\} \]
\[ = \{\beta^{\nu_1}_{1}, \beta^{\nu_2}_{1}, \ldots, \beta^{\nu_i}_{1}\} \]
\[ \cup \{\alpha^{-1} q^{(m-r-1)/2}, \alpha^{-1} q^{(m-r-3)/2}, \ldots, \alpha^{-1} q^{-(m-r-1)/2}\}. \]
Hence the proposition. \qed
4. Proof of Theorem 1.1

Now we go back to the situation of §2. As in the last section, $G_1 = \text{Sp}_r$, $G_2 = \text{Sp}_m$, and $H = \text{Sp}_{m+r}$. Let $\omega_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be the unramified character determined by $\omega_p(p) = \alpha_p$.

The $p$-component of the irreducible cuspidal automorphic representation of $H(\mathbb{A})$ generated by $F(Z)$ is the degenerate principal series

$$\text{Ind}_{P_H(\mathbb{Q}_p)}^{H(\mathbb{Q}_p)}(\omega_p \cdot \text{det}),$$

since the Satake parameter is

$$\{(\alpha_p^{-m+r+1/2})^{\pm 1}, (\alpha_p^{-m+r-1/2})^{\pm 1}, \ldots, (\alpha_p^{m+r+1/2})^{\pm 1}\}.$$

Let $H(G_i(\mathbb{A}_f))$ $(i = 1, 2)$ be the Hecke algebra for the finite adele group $G_i(\mathbb{A}_f)$. Then $H(G_1(\mathbb{A}_f)) \cdot g$ (resp. $H(G_2(\mathbb{A}_f)) \cdot F_{h,g}$) is the finite part of the cuspidal automorphic representation of $G_1(\mathbb{A})$ (resp. $G_2(\mathbb{A})$) generated by $g$ (resp. $F_{h,g}$). $H(G_1(\mathbb{A}_f)) \cdot g$ is an irreducible representation of $G_1(\mathbb{A}_f)$. Let $\pi_1$ be the $p$-component of $H(G_1(\mathbb{A}_f)) \cdot g$. Then $\pi_1$ is an unramified principal series with Satake parameter $\{\beta_{p,1}^{\pm 1}, \ldots, \beta_{p,r}^{\pm 1}\}$.

On the other hand, since $F_{h,g}(Z)$ is a cusp form, the representation $H(G_2(\mathbb{A}_f)) \cdot F_{h,g}$ of $G_2(\mathbb{A}_f)$ is unitary and of finite length. Let $\pi_2$ be the $p$-component of some irreducible direct summand of $H(G_2(\mathbb{A}_f)) \cdot F_{h,g}$.

Then $\pi_2$ is also an unramified principal series. Observe that

$$\begin{align*}
\int_{S_{p,n+r}(\mathbb{Z})} \int_{S_{p,r}(\mathbb{Z})} & F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) g^c(W) F_{h,g}(Z) \\
\times (\text{det } \text{Im} Z)^{k-n-1} (\text{det } \text{Im} W)^{k+n-1} dW dZ \\
= & (F_{h,g}, F_{h,g}) \neq 0.
\end{align*}$$

It follows that

$$\mathcal{B}_{G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p)}(\text{Ind}_{P_H(\mathbb{Q}_p)}^{H(\mathbb{Q}_p)}(\omega_p^{-1} \cdot \text{det}|_{G_1(\mathbb{Q}_p) \times G_2(\mathbb{Q}_p)}, \pi_1 \boxtimes \pi_2) \neq \{0\}.$$  

By Proposition 3.1, any irreducible component of $H(G_2(\mathbb{A}_f)) \cdot F_{h,g}$ has Satake parameter

$$\{(\beta_{p,1}^{\pm 1}, \ldots, \beta_{p,r}^{\pm 1}, (\alpha_p p^{n-(1/2)})^{\pm 1}, \ldots, (\alpha_p p^{-(n+1/2)})^{\pm 1}\}.$$  

In particular, $H(G_2(\mathbb{A}_f)) \cdot F_{h,g}$ is isotypic. Since it is generated by the class 1 vector $F_{h,g}$, it is irreducible. It follows that $F_{h,g}$ is a Hecke eigenform and its standard $L$-function is equal to

$$L(s, F_{h,g}, \text{st}) = L(s, g, \text{st}) \prod_{i=1}^{2n} L(s + k + n - i, f).$$
5. A conjecture on the Petersson inner product

It is an interesting problem to determine when $\mathcal{F}_{h,g} \not\equiv 0$. Here we are going to give a conjecture on the Petersson inner product of $\mathcal{F}_{h,g}$.

Let $L(s, \text{st}(g) \boxtimes f)$ be the $L$-function defined by

$$L(s, \text{st}(g) \boxtimes f) = \prod_p \det(1 - 2 - A_p \cdot p^{-s})^{-1},$$

where

$$L(s, f) = \prod_p \det(1 - A_p \cdot p^{-s})^{-1}, \quad A_p \in \text{GL}_2(\mathbb{C}),$$

$$L(s, g, \text{st}) = \prod_p \det(1 - B_p \cdot p^{-s})^{-1}, \quad B_p \in \text{GL}_{2r+1}(\mathbb{C}).$$

The gamma factor of $L(s, \text{st}(g) \boxtimes f)$ is given by

$$L_\infty(s, \text{st}(g) \boxtimes f) = \Gamma_C(s) \prod_{i=1}^r \Gamma_C(s + n - k + i) \Gamma_C(s + n + k + i - 1).$$

Here, $\Gamma_C(s) = 2(2\pi)^{-s} \Gamma(s)$.

We put $\Lambda(s, \text{st}(g) \boxtimes f) = L_\infty(s, \text{st}(g) \boxtimes f) L(s, \text{st}(g) \boxtimes f)$. Then the functional equation should be

$$\Lambda(2k - s, \text{st}(g) \boxtimes f) = (-1)^{k+r} \Lambda(s, \text{st}(g) \boxtimes f)$$

We also need the adjoint $L$-function $L(s, f, \text{Ad})$ of $f$. We put

$$\xi(s) = \Gamma_\mathbb{R}(s) \zeta(s),$$

$$\Lambda(s, f, \text{Ad}) = \Gamma_\mathbb{R}(s + 1) \Gamma_C(s + 2k - 1) L(s, f, \text{Ad}).$$

Here, $\Gamma_\mathbb{R}(s) = \pi^{-s/2} \Gamma(s/2)$. Then the following functional equations hold.

$$\xi(1 - s) = \xi(s),$$

$$\Lambda(1 - s, f, \text{Ad}) = \Lambda(s, f, \text{Ad}).$$

We modify $\xi(s)$ and $\Lambda(s, f, \text{Ad})$ as follows.

$$\tilde{\xi}(s) = \Gamma_\mathbb{R}(s + 1) \xi(s) = \Gamma_C(s) \zeta(s),$$

$$\tilde{\Lambda}(s, f, \text{Ad}) = \Gamma_\mathbb{R}(s) \Lambda(s, f, \text{Ad}) = \Gamma_C(s) \Gamma_C(s + 2k - 1) L(s, f, \text{Ad}).$$

If $i$ is a positive integer, $\tilde{\xi}(2i) = |B_{2i}|/2i \in \mathbb{Q}^\times$. It is well-known that $\tilde{\Lambda}(2i - 1, f, \text{Ad})/\langle f, f \rangle \in \mathbb{Q}(f)^\times$ for $1 \leq i < k$. 

**Conjecture 5.1.** Assume that \( n < k \). Then there exists an integer \( \alpha = \alpha(r, n, k) \) depending only on \( r, n, \) and \( k \) such that

\[
\Lambda(k + n, \text{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i - 1, f, \text{Ad}) \tilde{\xi}(2i) = 2^{\alpha} \frac{\langle f, f \rangle \langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g} \rangle}{\langle h, h \rangle \langle g, g \rangle}.
\]

In particular, \( \mathcal{F}_{h,g} \) is non-zero if and only if \( \Lambda(k + n, \text{st}(g) \boxtimes f) \neq 0 \).

In the case \( r = n = 1 \), the left hand side does not vanish. Therefore our conjecture implies Miyawaki’s conjecture 2.2.

When \( \mathcal{F}_{h,g} \neq 0 \), one can rewrite the right hand side in a more symmetric way. Namely, choose any non-zero \( G \in \mathbb{C} \cdot \mathcal{F}_{h,g} \). Then

\[
\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g} \rangle = \frac{|\langle F|_{h_r \times h_{r+2n}}, g^c \times G \rangle|^2}{\langle G, G \rangle \langle g, g \rangle}.
\]

Here \( \langle F|_{h_r \times h_{r+2n}}, g^c \times G \rangle \) is a Petersson inner product on \((\text{Sp}_r(Z) \backslash h_r) \times (\text{Sp}_{r+2n}(Z) \backslash h_{r+2n})\). Therefore the conjecture takes the form

\[
\text{(C)} \quad \Lambda(k + n, \text{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i - 1, f, \text{Ad}) \tilde{\xi}(2i)
\]

\[
= 2^{\alpha} \frac{\langle f, f \rangle |\langle F|_{h_r \times h_{r+2n}}, g^c \times G \rangle|^2}{\langle h, h \rangle \langle g, g \rangle \langle G, G \rangle}.
\]

**Remark 5.1.** By some computer calculation (cf. Appendix), it seems the values of \( \alpha = \alpha(r, n, k) \) are

(a) \( \alpha(0, n, k) = 2kn + 2n - k - 1 \),

(b) \( \alpha(r, 0, k) = r^2 + 2kr + r - k - 1 \),

(c) \( \alpha(r, n, k) = r^2 + 2kr + 2kn + 2rn + 2n + r - k - 2 \)

for \( r, n > 0 \). As for the case \( n = 0 \), we will give some evidence for (C) in the next section.

**Remark 5.2.** Note that \( s = k + n \) is a critical point for \( \Lambda(s, \text{st}(g) \boxtimes f) \) in the sense of Deligne [9]. In particular, the left hand side of (C) should be finite. Deligne’s conjecture [9] implies the ratio RHS/LHS should belong to the field \( \mathbb{Q}(f, g) \) under the assumption \( n < k \). (cf. Yoshida [36]). When \( r = 0 \), see Choie and Kohnen [7], Lanphier [26].

**Example 5.1.** When \( r = n = 0 \), we have \( F(Z) = c(1) \). In this case, our conjecture is a special case of the result of Kohmen-Zagier [23]

\[
\Lambda(k, f) = 2^{1-k} \frac{\langle f, f \rangle}{\langle h, h \rangle} |c(1)|^2.
\]

It follows that our conjecture holds for \( n = r = 0 \) with \( \alpha(0, 0, k) = 1 - k \).
Example 5.2. When $r = 0$, $n = 1$, our conjecture is compatible with the Petersson inner product formula for the Saito-Kurokawa lift

$$\Lambda(k + 1, f) = 3 \cdot 2^{-k+3} \frac{\langle F, F \rangle}{\langle h, h \rangle}$$

proved by Kohnen [21] and Kohnen and Skoruppa [22]. See also Krieg [24], Oda [31], and Furusawa [15]. This is equivalent with

$$\Lambda(k + 1, f) \tilde{\Lambda}(1, f, \text{Ad}) \tilde{\xi}(2) = 2^{k+1} \frac{\langle f, f \rangle}{\langle h, h \rangle} \langle F, F \rangle,$$

since $\tilde{\Lambda}(1, f, \text{Ad}) = 2^{2k} \langle f, f \rangle$. It follows that our conjecture holds for $(r, n) = (0, 1)$ with $\alpha(0, 1, k) = k + 1$.

So far, we have assumed $n \geq 0$. We now consider the case $n < 0$. We shall show that if Conjecture 5.1 is true, the roles of $g$ and $G$ can be interchanged.

**Proposition 5.1.** Assume that Conjecture 5.1 is true and $\mathcal{F}_{h, g} \neq 0$. Then $\mathcal{F}_{h, G} \in \mathbb{C} \cdot g$ for any $G \in \mathbb{C} \cdot \mathcal{F}_{h, g}$. Here, $\mathcal{F}_{h, G}$ is the Miyawaki lifting of $G \in S_{k+r+n}(\text{Sp}_{r+2n}(\mathbb{Z}))$ to $S_{k+r+n}(\text{Sp}_r(\mathbb{Z}))$ with respect to $F \in S_{k+r+n}(\text{Sp}_{2r+2n}(\mathbb{Z}))$.

**Proof.** Choose an orthonormal basis $\{g_i\}_{i \in I}$ of $S_{k+r+n}(\text{Sp}_r(\mathbb{Z}))$ which consists of Hecke eigenforms. We may assume $g \in \{g_i\}_{i \in I}$. The pullback $F|_{h \times h + 2n}$ can be expressed as

$$F|_{h \times h + 2n} = \sum_{i \in I} g_i^c \times G_i, \quad G_i = \mathcal{F}_{h, g_i}.$$

It is enough to show that $\langle G_i, G_j \rangle = 0$ for $i \neq j$. By Theorem 1.1, we may assume $g_i$ and $g_j$ have the same Hecke eigenvalues.

Let $V$ be the subspace of $S_{k+r+n}(\text{Sp}_r(\mathbb{Z}))$ generated by all Hecke eigenforms with the same Hecke eigenvalues as $g$. We define $V' \subset S_{k+r+n}(\text{Sp}_{2r+2n}(\mathbb{Z}))$ similarly. Then our assumption implies the map $g \mapsto \mathcal{F}_{h, g}$ is an isometry from $V$ onto an subspace of $V'$ up to scalar multiplication. It follows that $G_i$ and $G_j$ are orthogonal for $i \neq j$. \[\square\]

**Proposition 5.2.**

$$\left[ \Lambda(s + k - n, \text{st}(G) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(s - 2i + 1, f, \text{Ad})^{-1} \tilde{\xi}(s - 2i + 2)^{-1} \right]_{s=0} = \Lambda(k + n, \text{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i - 1, f, \text{Ad}) \tilde{\xi}(2i).$$
Proof. By Theorem 1.1, $\Lambda(s + k - n, st(G) \boxtimes f)$ is the product of

$$\prod_{i=1}^{2n} \Lambda(s + 2k - i, f \times f)$$

and

$$\Lambda(s + k - n, st(g) \boxtimes f) = (-1)^{k+r} \Lambda(-s + k + n, st(g) \boxtimes f).$$

Since $\Lambda(s + 2k - 1, f \times f) = \Lambda(s, f, Ad)\xi(s)$, we have

$$\prod_{i=1}^{2n} \Lambda(s + 2k - i, f \times f) \prod_{i=1}^{n} \tilde{\Lambda}(s - 2i + 1, f, Ad)^{-1} \tilde{\xi}(s - 2i + 2)^{-1}$$

$$= \prod_{i=1}^{n} \Gamma_R(s - 2i + 1)^{-1} \Gamma_R(s - 2i + 3)^{-1}$$

$$\times \prod_{i=1}^{n} \Lambda(-s + 2i - 1, f, Ad)\xi(-s + 2i).$$

Now using $\Gamma_R(s + 1)\Gamma_R(-s + 1) = \sin(\pi s/2)$, we have

$$\prod_{i=1}^{n} \Gamma_R(-2i + 1)^{-1} \Gamma_R(-2i + 3)^{-1} = (-1)^{n} \prod_{i=1}^{n} \Gamma_R(2i - 1)\Gamma_R(2i + 1).$$

Hence the proposition. \qed

Remark 5.3. The polynomial which shows up in the right hand side of Remark 5.1 (c) is not invariant under $(r, n) \mapsto (r + 2n, -n)$.

6. Some evidence for the case $n = 0$

In this section, we discuss the case when $n = 0$. In this case we conjecture $\alpha(r, 0, k) = r^2 + 2rk + r - k - 1$.

By Kohnen-Zagier [23],

(KZ) \quad |c(|D|)|^2 \frac{\langle f, f \rangle}{\langle h, h \rangle} = 2^{k-1} |D|^{-1/2} \Lambda(k, f, \chi_D),

for any fundamental discriminant $D$ such that $(-1)^k D > 0$. Here,

$$\Lambda(s, f, \chi_D) = |D|^s \Gamma(s) L(s, f, \chi_D).$$

It follows that if $c(|D|) \neq 0$, our conjecture is equivalent to the following:

$$(C') \quad \Lambda(k, st(g) \boxtimes f) = 2^{r+2k-2} \frac{\Lambda(k, f, \chi_D)}{\sqrt{|D| |c(|D|)|^2}} \frac{\langle F_{h,g}, F_{h,g} \rangle}{\langle g, g \rangle}.$$
When \( f = E_{2k} \) is the Eisenstein series, the equation (C) does not make sense, but (C') makes sense. As \( L(s, E_{2k}) = \zeta(2) \zeta(s-2k+1) \), we think of \( L(s, \text{st}(g) \boxtimes E_{2k} \boxtimes E_{2k}) \) as \( L(s, g, \text{st})L(s-2k+1, g, \text{st}) \), while the gamma factor is the same as \( L_\infty(s, \text{st}(g) \boxtimes f) \). Let \( h(\tau) \) be the Cohen Eisenstein series \( \mathcal{H}_{k+1/2} \in M_+^{(2)}(\Gamma_0(4)) \) and \( F = \mathcal{E}_{k+r}^{(2r)} = 2^{-r} \mathcal{A}_{r,k} \cdot E_k^{(2r)} \) the normalized Eisenstein series, where

\[
\mathcal{A}_{r,k} = \zeta(1-k-r) \prod_{i=1}^{r} \zeta(1-2k-2r+2i).
\]

introduced in [18]. \( F = \mathcal{E}_{k+r}^{(2r)} \) can be thought of as the Duke-Imamoglu lift of \( \mathcal{H}(\tau) \).

**Proposition 6.1.** If \( f = E_{2k} \), \( h = \mathcal{H}_{k+1/2} \), and \( F = \mathcal{E}_{k+r}^{(2r)} \), then the equation (C') holds.

**Proof.** This is essentially a result of Böcherer [3]. When \( f = E_{2k} \), \( h = \mathcal{H}_{k+1/2} \), we have

\[
c(|D|) = L(1-k, \chi_D) = (-1)^{k(k-1)/2} |D|^{k-(1/2)} 2(2\pi)^{-k-1/2} k(k_g)^k L(k, \chi_D),
\]

and so

\[
\frac{\Lambda(k, f, \chi_D)}{\sqrt{|D|c(|D|)^2}} = (-1)^{k(k-1)/2}.
\]

By the functional equation (cf. [3]) of \( L(s, g, \text{st}) \), we have

\[
L(1-k, g, \text{st}) = (-1)^{k(k-1)/2} 2(2\pi)^{-2r-k-1} k(k_g)^k \prod_{i=1}^{r} \frac{\Gamma(2k+i-1)}{\Gamma(i)} \cdot L(k, g, \text{st}).
\]

Therefore, we have to prove

\[
\frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g} \rangle}{\langle g, g \rangle} = 2^{-2r^2+2r-6r-2k+4} \pi^{-2r^2+r-4r-2k} \times \Gamma(k)^2 \prod_{i=1}^{r} \Gamma(2k+i-1)^2 \cdot L(k, g, \text{st})^2.
\]

On the other hand, by the result of Böcherer [3], we have \( \mathcal{F}_{h,g} = \mathcal{B}_{r} \cdot g \), where

\[
\mathcal{B}_r = (-1)^{r(k+r)/2} 2(-r^2+r-2r+2k+2)^{2} \pi^{r^2+r} \gamma(2k+r-1) \frac{\Gamma_r(k+r)}{\Gamma_r(k)} \times \zeta(k+r)^{-1} \prod_{i=1}^{r} \zeta(2k+2r-2i)^{-1} L(k, g, \text{st}) \cdot \mathcal{A}_{r,k}.
\]

Here \( \Gamma_r(s) = \prod_{i=1}^{r} \Gamma(s - ((i-1)/2)) \).
By the functional equation of the Riemann zeta function and the definition of $\mathcal{E}_{k+r}$, we have
\[
\frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle}{\langle g, g \rangle} = 2^{2r-2+3r-6(r-4k+4)\pi^{-r^2+2r-4k-2}} \frac{\Gamma_r(k + r/2)}{\Gamma_r(k + r)^2} \times \Gamma(k + r)^2 \prod_{i=1}^{r} \Gamma(2k + 2r - 2i)^2 L(k, g, st)^2.
\]

Now, the next lemma proves Proposition 6.1.

**Lemma 6.1.**
\[
\frac{\Gamma_r(s + r - 1)}{\Gamma_r(s + r)} = 2^{(r^2 - r)/2} \frac{\Gamma(s + r - 1)}{\Gamma(s + r)} \prod_{i=1}^{r} \frac{\Gamma(2s + 2r - 2i)}{\Gamma(2s + i - 1)}.
\]

**Proof.** Put
\[
A_r(s) = 2^{(r^2 - r)/2} \frac{\Gamma_r(s + r - 1)}{\Gamma_r(s + r)} \prod_{i=1}^{r} \frac{\Gamma(2s + 2r - 2i)}{\Gamma(2s + i - 1)}.
\]

Then obviously $A_1(s) = 1$.
\[
A_{r+1}(s) = 2^{-r} \frac{\Gamma(s + r + 1)}{\Gamma(s + r + 1/2)} \Gamma(s + r + 1) \Gamma(2s + 2r)
\]
\[
= 2^{-r} \frac{\Gamma(s + r + 1/2)}{\Gamma(s + r)} \Gamma(2s + 2r).
\]

By the duplication formula for the gamma function, we have
\[
\Gamma(s + r + 1/2) \Gamma(s + r) = \sqrt{\pi} 2^{1-r-2s} \Gamma(2s + r),
\]
\[
\Gamma(s + r + 1/2) \Gamma(s + r) = \sqrt{\pi} 2^{1-2r-2s} \Gamma(2s + 2r).
\]

Hence $A_{r+1}(s) = A_r(s)$.

We restate Proposition 6.1 in the following form.

**Proposition 6.2.** Assume that $k + r \equiv 2 \mod 2$ and $g \in S_{k+r}(\text{Sp}_r(\mathbb{Z}))$.

Then
\[
\frac{\langle E_{k+r}^{(2r)} |_{\mathfrak{h}^* \times \mathfrak{h}^*}, g^c \times g \rangle}{\langle g, g \rangle} = 2^{-(r^2 - r + 2rk - 2)/2} |A_r(k)|^{-1} \Lambda(s, g, st).
\]

Here $\Lambda(s, g, st) = \Gamma(s + k + r - i) L(s, g, st)$.
7. Theta functions associated with Niemeier lattices

In this section, we write \( M_k^{(n)} = M_k(\text{Sp}_n(\mathbb{Z})) \) and \( S_k^{(n)} = S_k(\text{Sp}_n(\mathbb{Z})) \), for simplicity.

We recall the results of [30]. A Niemeier lattice is a positive definite even unimodular lattice of degree 24. The number of isomorphism classes of Niemeier lattices is 24. Let \( L_i \) \((1 \leq i \leq 24)\) be Niemeier lattices, not isomorphic to each other.

Let \( V \) be the vector space with basis \([L_i] \mid 1 \leq i \leq 24\) where \([L_i]\) is the isomorphism class of \( L_i \).

The theta function of degree \( n \) associated with \( L_i \) is denoted by \( \Theta_{L_i}^{(n)}(Z) \in M_{12}^{(n)} \). By extending linearly, we obtain a linear map

\[
\Theta^{(n)} : V \longrightarrow M_{12}^{(n)}
\]

\[
\sum_i c_i [L_i] \mapsto \sum_i c_i \Theta_{L_i}^{(n)}(Z).
\]

Let \( V_n = \text{Ker}(\Theta^{(n)}) \). Then \( \Theta^{(12)} \) is injective (cf. [13], [5]). If \( n' + n'' = n \), then the restriction of \( \Theta_{L_i}^{(n)}(Z) \) to \( \mathfrak{n}_{n'} \times \mathfrak{n}_{n''} \) is given by

\[
\Theta_{L_i}^{(n)} \left( \begin{pmatrix} Z' & 0 \\ 0 & Z'' \end{pmatrix} \right) = \Theta_{L_i}^{(n')}(Z') \Theta_{L_i}^{(n'')}(Z'').
\]

As an element of \( V \), we put \( e_i = [L_i] \). Following Nebe and Venkov, we define the Hermitian inner product \((\ , \ )\) on \( V \) by

\[
(e_i, e_j) = \begin{cases} (\#\text{Aut}(L_i)), & i = j, \\ 0, & i \neq j. \end{cases}
\]

and a multiplication on \( V \) by

\[
e_i \circ e_j = \begin{cases} (\#\text{Aut}(L_i))e_i, & i = j \\ 0, & i \neq j. \end{cases}
\]

Nebe and Venkov defined Hecke operators \( K_{p,i}, (1 \leq i \leq 12) \) and \( T(p) \) acting on \( V \) and calculated Hecke eigenvectors \( d_1, d_2, \ldots, d_{24} \).

We put

\[
d_i = \sum_j c_{ij} e_j,
\]

\[
e_i = \sum_j b_{ij} d_j.
\]

A table of coefficients \( c_{ij} \) \((i,j = 1, 2, \ldots, 24)\) can be found in [29]. Note that \( c_{ij}, b_{ij} \in \mathbb{Q} \). As both \( \{e_1, e_2, \ldots, e_{24}\} \) and \( \{d_1, d_2, \ldots, d_{24}\} \)
are orthogonal basis of $V$, we have

$$b_{ij} = \langle e_i, e_i \rangle \overline{c_{ji}} (d_j, d_j)^{-1} = (\#\text{Aut}(L_i)) (d_j, d_j)^{-1} c_{ji}.$$ 

Nebe and Venkov showed that the degree $n_i$ of $d_i$ is as follows:

<table>
<thead>
<tr>
<th>$n_1$</th>
<th>$n_2$</th>
<th>$n_3$</th>
<th>$n_4$</th>
<th>$n_5$</th>
<th>$n_6$</th>
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<th>$n_{11}$</th>
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<tr>
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<tr>
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<td>$n_{17}$</td>
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For the definition of the degree, see [30]. Note that they have shown that $n_i = \min \{n \mid \Theta^{(n)}(d_i) \neq 0\}$ in this case (See [30], Lemma 2.5). As for $n_{19}$ and $n_{21}$, they have shown that $7 \leq n_{19} \leq 9$, $8 \leq n_{21} \leq 10$, but we do not use $d_{19}$ or $d_{21}$.

Note that the Petersson inner product $\langle \Theta^{(n)}(d_i), \Theta^{(n)}(d_j) \rangle$ vanishes for $i \neq j$, since the Hecke eigenvalues are different. We put $F_i = \Theta^{(n)}(d_i) \in S^{(n)}_{12}$. Note that $F_i^c = F_i$ for $i = 1, 2, \ldots, 24$.

**Lemma 7.1.** Let $d_i$, $d_j$, and $d_k$ be Hecke eigenvectors of $V$. Then we have

$$\langle \Theta^{(n_i+n_j)}(d_k)|_{b_{n_i} \times b_{n_j}}, F_i \times F_j \rangle = \frac{\langle F_i, F_i \rangle \langle F_j, F_j \rangle}{(d_i, d_i) (d_j, d_j)} (d_k, d_i \circ d_j).$$

In particular, $(d_k, d_i \circ d_j) \neq 0$ if and only if the left hand side is not zero.

**Proof.** The left hand side is equal to

$$\sum_{m=1}^{24} c_{km} \langle \Theta^{(n_i+n_j)}_{L_{km}}|_{b_{n_i} \times b_{n_j}}, \Theta^{(n_i)}(d_i) \times \Theta^{(n_j)}(d_j) \rangle$$

$$= \sum_{m=1}^{24} c_{km} \langle \Theta^{(n_i)}_{L_{km}}, \Theta^{(n_i)}(d_i) \rangle \langle \Theta^{(n_j)}_{L_{km}}, \Theta^{(n_j)}(d_j) \rangle$$

$$= \sum_{m=1}^{24} c_{km} \langle \sum_{l=1}^{24} b_{ml} \Theta^{(n_i)}(d_i), \Theta^{(n_i)}(d_i) \rangle \langle \sum_{l=1}^{24} b_{ml} \Theta^{(n_j)}(d_i), \Theta^{(n_j)}(d_j) \rangle$$

$$= \langle F_i, F_i \rangle \langle F_j, F_j \rangle \sum_{m=1}^{24} c_{km} b_{ml} b_{mj}$$

$$= \frac{\langle F_i, F_i \rangle \langle F_j, F_j \rangle}{(d_i, d_i) (d_j, d_j)} \sum_{m=1}^{24} (\#\text{Aut}(L_m))^2 c_{km} c_{lm} c_{jm}.$$
On the other hand,
\[
(d_k, d_i \circ d_j) = (d_k, \left( \sum_{m=1}^{24} c_{im} e_m \right) \circ \left( \sum_{l=1}^{24} c_{jl} e_m \right))
\]
\[
= (d_k, \sum_{m=1}^{24} (\#\text{Aut}(L_m)) c_{im} c_{jm} e_m)
\]
\[
= \sum_{m=1}^{24} (\#\text{Aut}(L_m)) c_{im} c_{jm} (d_k, e_m)
\]
\[
= \sum_{m=1}^{24} (\#\text{Aut}(L_m))^2 c_{im} c_{jm} c_{km}.
\]

Hence the lemma. \hfill \Box

Nebe and Venkov [30] claimed that \(F_{11} \in S_{12}^{(6)}\), \(F_{13} \in S_{12}^{(8)}\), and \(F_{24} \in S_{12}^{(12)}\) are the Duke-Imamoglu lift of \(\phi_{18} \in S_{18}^{(1)}\), \(\phi_{16} \in S_{16}^{(1)}\), and \(\Delta \in S_{12}^{(1)}\), respectively. In fact this is easily verified by comparing the eigenvalue of \(T(2)\) (See [29]). Nebe and Venkov [30] have shown that

\((i, j) = (2, 23), (3, 22), (4, 20), (5, 17), (6, 18), (7, 14), (8, 16)\).

Proposition 3.1 implies that \(F_j\) is the Miyawaki lift of \(F_i\) with respect to \(F_{24} \in S_{12}^{(12)}\). Similarly, using the structure constants found in [29], one can prove that \(F_8 \in S_{12}^{(5)}\) and \(F_6 \in S_{12}^{(4)}\) are Miyawaki lift of \(F_2 \in S_{12}^{(1)}\) and \(F_3 \in S_{12}^{(2)}\), respectively. One can also prove that \(F_{12} \in S_{12}^{(7)}\), \(F_9 \in S_{12}^{(6)}\), and \(F_7 \in S_{12}^{(5)}\) are the Miyawaki lift of \(F_2 \in S_{12}^{(1)}\), \(F_3 \in S_{12}^{(2)}\), and \(F_4 \in S_{12}^{(3)}\) with respect to \(F_{13} \in S_{12}^{(8)}\), respectively. We summarize these as Table A and Table B.

8. Appendix

We briefly explain how to calculate both sides of (C) by computers. For the calculation of various \(L\)-values, we have used a very useful program due to Dokchitser [10]. The Petersson norm \(\langle f, f \rangle\) can be easily computed by \(\tilde{\Lambda}(1, f, \text{Ad}) = 2^{2k}\langle f, f \rangle\). Similarly, \(\langle h, h \rangle\) can be computed by Kohnen-Zagier formula (KZ). The Petersson norm of \(g\) or \(G\) can be computed by Proposition 6.2 and Lemma 7.1. Finally, \(\langle F|_{h_r \times h_{r+2n}, g \times G}\rangle\) is computed by Lemma 7.1. Note that the structure constants \((d_k, d_i \circ d_j)\) are already computed by Nebe [29].
We discuss the case $\phi_{20} \in S_{20}^{(1)}$, $g = \Delta \in S_{12}^{(1)}$, and $G \in S_{12}^{(3)}$. We put
\[
d_1' = d_1/1027637932586061520960267, \\
d_2' = -d_2/8104867379578640543040, \\
d_4' = d_4/846305351287603200, \\
d_5' = -d_5/21269424185860.
\]

We give a table of coefficients of $d_2$, $d_4$, and $d_5$ below (See Nebe [29]). The coefficients of $d_1$ can be found in [29] or [8], p. 413. Then $E_{12}'^{(2r)} = \Theta(2r)(d_1')$, $F_2' = \Theta(1)(d_2') = \Delta \in S_{12}^{(1)}$, and $F_4' = \Theta(3)(d_4') \in S_{12}^{(3)}$ is the Miyawaki’s cusp form [27]. Let $h = q - 56q^4 + 360q^5 - 13680q^8 + \cdots \in S_{21/2}^+(\Gamma(4))$. Then $F_5' = \Theta(4)(d_5') \in S_{12}^{(4)}$ is the Duke-Imamoglu lift of $h(r)$ to degree 4.

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</table>

We need the following computer calculations.

\[
(d_2', d_4') = 2^{31} \cdot 3^{10} \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 283^{-1} \cdot 617^{-1} \cdot 3617^{-1} \cdot 43867^{-1},
\]
\[
(d_5', d_4') = 2^{16} \cdot 3^{-1} \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 283 \cdot 617 \cdot 691^{-1} \cdot 3617^{-1},
\]
\[
(d_4', d_1' \circ d_4') = 2^{31} \cdot 3^{16} \cdot 5^2 \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23,
\]
\[
(d_4', d_2' \circ d_4') = -2^{24} \cdot 3^{12} \cdot 5^7 \cdot 7^2 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 691^{-1} \cdot 3617^{-2} \cdot 43867^{-1}.
\]
\[\langle \Delta, \Delta \rangle = 0.000001035362056804320922347816812225164593224907 \ldots \]
\[\langle \phi_20, \phi_{20} \rangle = 0.000008265541531659703164230062760258225715343908 \ldots \]
\[\langle \Delta, \Delta \rangle / \langle h, h \rangle = 0.09887227906528174118675236994533682997115288715 \ldots \]
\[\hat{\Lambda}(9, \Delta, \text{Ad}) = 0.13958431766868979132086560789461824236408711579 \ldots \]

We can now calculate the Petersson norm \(\langle F'_4, F'_4 \rangle\). By Proposition 6.2 and Lemma 7.1, we have
\[\langle F'_4, F'_4 \rangle = 2^{-29} \frac{(d'_4, d'_4)^2}{(d'_4, d'_4 \circ d'_4)} |A_{3,9}|^{-1} \hat{\Lambda}(9, \Delta, \text{Ad}) \Lambda(18, \phi_{20}) \Lambda(19, \phi_{20}) \]
\[\hat{\Delta} = 2^{19} \cdot 3^2 \cdot 7^{-1} \cdot \langle \Delta, \Delta \rangle, \]
\[\Lambda(18, \phi_{20}) \Lambda(19, \phi_{20}) = 2^{23} \cdot 3^2 \cdot 7^2 \cdot 17 \cdot 283^{-2} \cdot 617^{-1} \langle \phi_{20}, \phi_{20} \rangle, \]
\[\Lambda(11, \text{Ad}(\Delta) \otimes \phi_{20}) = 0.000000334470806144084988864020192110373963031495 \ldots \]
\[\hat{\Delta} = 2^{24} \cdot 3^2 \cdot 5^2 \langle \Delta, \Delta \rangle^2 \langle \phi_{20}, \phi_{20} \rangle \langle h, h \rangle^{-1}. \]

On the other hand, we have
\[\Lambda(11, \text{st}(g) \otimes f) \hat{\Delta}(1, f, \text{Ad}) \xi(2) = 2^{49} \cdot 3 \cdot 5^2 \langle \Delta, \Delta \rangle^3 \langle \phi_{20}, \phi_{20} \rangle^2 \langle h, h \rangle^{-1}. \]
Hence the equation (C) holds approximately in this case with \(\alpha = 34\). Other examples are shown in Table C.

We give another example \(n = k = 6, r = 0, g = 1, f = \Delta,\) and \(F = G = F_{24}\). Then by computer calculation,
\[\Lambda(12, \text{st}(g) \otimes f) \prod_{i=1}^{6} \hat{\Lambda}(2i-1, f, \text{Ad}) = 2^{273} \langle \Delta, \Delta \rangle^6 \Lambda(12, \Delta) \]
\[= 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23. \]
On the other hand, using Böcherer’s result [3], one can show
\[\frac{\langle f, f \rangle}{\langle h, h \rangle} \langle F, F \rangle = \frac{\langle \Delta, \Delta \rangle^6 \Lambda(12, \Delta)}{25 \cdot 3^3 \cdot 5^2 \cdot 7^2 \cdot 11 \cdot 13 \cdot 23}. \]
Therefore it seems (C) holds in this case as well. Notice that the assumption \(k > n\) is not satisfied in this case and that \(\Lambda(12, \Delta)\) is not a critical value in the sense of Deligne [9].
**Tabe A: Standard $L$-functions**

\[
L(s, F_3, \text{st}) = \zeta(s) \prod_{10 \leq i \leq 11} L(s + i, \phi_{22}),
\]

\[
L(s, F_4, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{9 \leq i \leq 10} L(s + i, \phi_{20}),
\]

\[
L(s, F_5, \text{st}) = \zeta(s) \prod_{8 \leq i \leq 11} L(s + i, \phi_{20}),
\]

\[
L(s, F_6, \text{st}) = \zeta(s) \prod_{10 \leq i \leq 11} L(s + i, \phi_{22}) \prod_{8 \leq i \leq 9} L(s + i, \phi_{18}),
\]

\[
L(s, F_7, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{9 \leq i \leq 10} L(s + i, \phi_{20}) \prod_{7 \leq i \leq 8} L(s + i, \phi_{16}),
\]

\[
L(s, F_8, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{7 \leq i \leq 10} L(s + i, \phi_{18}),
\]

\[
L(s, F_9, \text{st}) = \zeta(s) \prod_{10 \leq i \leq 11} L(s + i, \phi_{22}) \prod_{6 \leq i \leq 9} L(s + i, \phi_{16}),
\]

\[
L(s, F_{11}, \text{st}) = \zeta(s) \prod_{6 \leq i \leq 11} L(s + i, \phi_{18}),
\]

\[
L(s, F_{12}, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{5 \leq i \leq 10} L(s + i, \phi_{16}),
\]

\[
L(s, F_{14}, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{9 \leq i \leq 10} L(s + i, \phi_{20}) \prod_{7 \leq i \leq 8} L(s + i, \phi_{16}) \prod_{5 \leq i \leq 6} L(s + i, \Delta),
\]

\[
L(s, F_{16}, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{7 \leq i \leq 10} L(s + i, \phi_{18}) \prod_{5 \leq i \leq 6} L(s + i, \Delta),
\]

\[
L(s, F_{13}, \text{st}) = \zeta(s) \prod_{4 \leq i \leq 11} L(s + i, \phi_{16}),
\]

\[
L(s, F_{17}, \text{st}) = \zeta(s) \prod_{8 \leq i \leq 11} L(s + i, \phi_{20}) \prod_{4 \leq i \leq 7} L(s + i, \Delta),
\]

\[
L(s, F_{18}, \text{st}) = \zeta(s) \prod_{10 \leq i \leq 11} L(s + i, \phi_{22}) \prod_{8 \leq i \leq 9} L(s + i, \phi_{18}) \prod_{4 \leq i \leq 7} L(s + i, \Delta),
\]

\[
L(s, F_{20}, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{9 \leq i \leq 10} L(s + i, \phi_{20}) \prod_{3 \leq i \leq 8} L(s + i, \Delta),
\]

\[
L(s, F_{22}, \text{st}) = \zeta(s) \prod_{10 \leq i \leq 11} L(s + i, \phi_{22}) \prod_{2 \leq i \leq 9} L(s + i, \Delta),
\]

\[
L(s, F_{23}, \text{st}) = L(s, \Delta, \text{Ad}) \prod_{10}^{i=1} L(s + i, \Delta),
\]

\[
L(s, F_{24}, \text{st}) = \zeta(s) \prod_{1=0}^{11} L(s + i, \Delta).
\]
• Table B: Liftings

<table>
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<th>g</th>
<th>f</th>
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• Table C: The author has checked that the equation (C) holds up to at least 30 decimals in the following cases:

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References


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