PULLBACK OF THE LIFTING OF ELLIPTIC CUSP FORMS AND MIYAWAKI'S CONJECTURE

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To the memory of Prof. Isao Miyawaki

Abstract. We construct a lifting from Siegel cusp forms of degree r to Siegel cusps form of degree $r+2n$. For $r=n=1$, our result is a partial solution of a conjecture made by Miyawaki in 1992. In particular, we can calculate the standard L-function of a cusp form of degree 3 and weight 12, which is in accordance with Miyawaki's conjecture. We will give a conjecture on the Petersson inner product of the lifting in terms of certain L-values.

Introduction

Let $f(\tau) \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. In [18], we have constructed a lifting to a Siegel cusp form of even degree. Let $F(Z)$ be a lifting of $f(\tau)$. In this paper, we shall consider the pullback of $F(Z)$ to a block diagonal subset.

Let us recall the theory of pullback of an Eisenstein series to a block diagonal subset (cf. [3], [16]). Let $M_k(\mathrm{Sp}_n(\mathbb{Z}))$ (resp. $S_k(\mathrm{Sp}_n(\mathbb{Z}))$) be the space of Siegel modular forms (resp. Siegel cusp forms) of degree n and weight k. Assume that $g(Z) \in S_{2l}(\mathrm{Sp}_r(\mathbb{Z}))$ is a Hecke eigenform whose standard L-function is $L(s, g, \mathrm{st})$. For $m \geq r$, let $E_{2l}^{(m+r)}(Z)$ be the Siegel Eisenstein series of degree $m+r$ and weight $2l$. Assume, for simplicity, $E_{2l}^{(m+r)}(Z)$ is absolutely convergent. Put $g^c(Z) = \overline{g(-\bar{Z})}$. Note that $g^c(Z)$ is the cusp form obtained by taking complex conjugates of Fourier coefficients. Then

$$\int_{\mathrm{Sp}_r(\mathbb{Z})\backslash\mathfrak{h}_r} E_{2l}^{(m+r)}\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) \overline{g^c(W)} (\det \mathrm{Im} W)^{2l-r-1} dW$$

The author thanks his seminar members K. Hiraga, A. Ichino, K. Konno, T. Konno, H. Matumoto, H. Saito, T. Takahashi, and T. Yamazaki. He also thanks Prof. Ibukiyama for his suggestions. The author expresses his hearty gratitude to Prof. H. Yoshida and Prof. H. Saito for their continuous encouragement. Earlier version of this paper was prepared during his stay at the Institute for Advanced Study. He wishes to thank the hospitality of the Institute for Advanced Study.

is equal to the Klingen Eisenstein series $E^{(m)}(g;Z) \in M_{2l}(\mathrm{Sp}_m(\mathbb{Z}))$, up to multiplication by some L-values and elementary factors. In this theory, the unwinding method of the Eisenstein series played an important role.

Now let $h(\tau) \in S^+_{k+(1/2)}(\Gamma_0(4))$ be a Hecke eigenform in the Kohnen plus subspace $S^+_{k+(1/2)}(\Gamma_0(4))$ corresponding to the normalized Hecke eigenform $f(\tau)$. Put $L(s,f) = \sum_{N=1}^{\infty} a(N)N^{-s}$.

Let n be a non-negative integer such that $n + r \equiv k \bmod 2$. In [18], we have constructed a Hecke eigenform $F(Z) \in S_{k+n+r}(\mathrm{Sp}_{2n+2r}(\mathbb{Z}))$ whose standard L-function is equal to

$$\zeta(s) \prod_{i=1}^{2n+2r} L(s+k+n+r-i, f).$$

Note that $F(Z)$ is determined by $h(\tau)$. We shall call $F(Z)$ a Duke-Imamoglu lift of $f(\tau)$ (or $h(\tau)$) to degree $2n + 2r$. Assume that $2l = k + n + r$ and $g \in S_{k+n+r}(\mathrm{Sp}_r(\mathbb{Z}))$.

Now we consider the function $\mathcal{F}_{h,g}(Z)$ defined by the integral

$$\mathcal{F}_{h,g}(Z) = \int_{\mathrm{Sp}_r(\mathbb{Z})\backslash\mathfrak{h}_r} F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right) \overline{g^c(W)} (\det \mathrm{Im} W)^{k+n-1} dW,$$

for $Z \in \mathfrak{h}_{2n+r}$. Note that $\mathcal{F}_{h,g}$ is always cusp form, as $F(Z)$ is a cusp form. Then our main theorem is as follows.

Theorem 1.1. *Assume that $\mathcal{F}_{h,g}(Z)$ is not identically zero. Then the cusp form $\mathcal{F}_{h,g}(Z)$ is a Hecke eigenform whose standard L-function is equal to*

$$L(s, \mathcal{F}_{h,g}, \mathrm{st}) = L(s, g, \mathrm{st}) \prod_{i=1}^{2n} L(s+k+n-i, f).$$

As the usual unwinding method does not work for the cusp form $F(Z)$, we will make use of local representation theory instead.

It is an interesting problem to determine when $\mathcal{F}_{h,g} \neq 0$. We will give a conjecture for the Petersson inner product $\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle$. Let $L(s, \mathrm{st}(g) \boxtimes f)$ be the "tensor product" L-function of $L(s, g, \mathrm{st})$ and $L(s, f)$. Let $\Lambda(s, \mathrm{st}(g) \boxtimes f)$ be the product of $L(s, \mathrm{st}(g) \boxtimes f)$ and its gamma factor. We also define $\tilde{\Lambda}(s, f, \mathrm{Ad})$ (resp. $\tilde{\xi}(s)$) as the product of the adjoint L-function $L(s, f, \mathrm{Ad})$ (resp. Riemann zeta function) and some gamma function, which is slightly modified from the usual gamma factor. Then our conjecture is as follows.

Conjecture 5.1. Assume that $n < k$. Then there exists an integer $\alpha = \alpha(r, n, k)$ depending only on r, n, and k such that

$$\Lambda(k+n, \mathrm{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i-1, f, \mathrm{Ad}) \tilde{\xi}(2i) = 2^{\alpha} \frac{\langle f, f \rangle}{\langle h, h \rangle} \frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g} \rangle}{\langle g, g \rangle}.$$

In particular, $\mathcal{F}_{h,g}$ is non-zero if and only if $\Lambda(k+n, \mathrm{st}(g) \boxtimes f) \neq 0$.

This paper is organized as follows. In §1, we formulate our main theorem. In §2, we discuss the relation to Miyawaki's conjecture [27]. In §3, we develop some local representation theory. Using this representation theoretic argument, we prove our main theorem in §4. In §5, we formulate the conjecture and discuss some examples. We shall show that, if the conjecture is true, then the roles of g and $\mathcal{F}_{h,g}$ can be interchanged. Note that this phenomenon does not have an analogue for the Eisenstein case for $n > 0$, since the Klingen Eisenstein series $E^{(r+2n)}(g, Z)$ is not an cusp form unless $n = 0$. The exceptional case $n = 0$ is discussed in §6. We shall show that an analogue of the conjecture for the Eisenstein case holds in that case.

In §7, we recall the result of Nebe and Venkov [30]. They determined Hecke eigenvectors in the space of theta functions associated to 24 Niemeier lattices. Using our theory, we can determine standard L-functions of 20 eigenvectors. In Appendix, we attach some computer calculation for an evidence for Conjecture 5.1.

Notation

If R is a ring, the symplectic group $\mathrm{Sp}_m(R)$ is defined by

$$\mathrm{Sp}_m(R) = \left\{ g \in \mathrm{GL}_{2m}(R) \,\middle|\, g \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} {}^t g = \begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix} \right\}.$$

We denote the Siegel upper-half plane of degree m by $\mathfrak{h}_m$. For $2k = 12$, 16, 18, 20, 22, or 26, the normalized Hecke eigenform of weight $2k$ is denoted by $\phi_{2k}(\tau)$. Note that $\phi_{12}(\tau) = \Delta(\tau)$. The space of Siegel modular forms with degree m and weight k is denoted by $M_k(\mathrm{Sp}_m(\mathbb{Z}))$ or by $M_k^{(m)}$. The subspace of cusp forms of $M_k^{(m)}$ is denoted by $S_k(\mathrm{Sp}_m(\mathbb{Z}))$ or by $S_k^{(m)}$. For $g \in M_k(\mathrm{Sp}_r(\mathbb{Z}))$, we put $g^c(Z) = \overline{g(-\bar{Z})}$. The Petersson inner product is denoted by $\langle\ ,\ \rangle$. When f, g, or h are Hecke eigenform, $\mathbb{Q}(f)$, $\mathbb{Q}(h, g)$ etc. are the field generated by Hecke eigenvalues. The (multi)set $\{\beta_1, \beta_1^{-1}, \beta_2, \beta_2^{-1}, \ldots, \beta_n, \beta_n^{-1}\}$ is sometimes denoted by $\{\beta_1^{\pm 1}, \beta_2^{\pm 1}, \ldots, \beta_n^{\pm 1}\}$.

1. Statement of the main theorem

As in Introduction, let

$$f(\tau)=\sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$$

and

$$\begin{aligned} L(s,f) &= \prod_p (1-a(p)p^{-s}+p^{2k-1-2s})^{-1} \\ &= \prod_p \left[(1-\alpha_p p^{k-s-(1/2)})(1-\alpha_p^{-1}p^{k-s-(1/2)})\right]^{-1} \end{aligned}$$

be a normalized Hecke eigenform and its L-function. In the Kohnen plus subspace $S^+_{k+(1/2)}(\Gamma_0(4))$, there exists a Hecke eigenform

$$h(\tau)=\sum_{\substack{N>0\\ (-1)^k N\equiv 0,1(4)}} c(N)q^N$$

corresponding to $f(\tau)$ by the Shimura correspondence. As is well-known, $h(\tau)$ is unique up to a scalar. Let r and n be non-negative integers such that $n+r\equiv k \bmod 2$. By [18], there exists a Hecke eigenform $F(Z)\in S_{k+n+r}(\mathrm{Sp}_{2n+2r}(\mathbb{Z}))$, whose standard L-function is equal to

$$\zeta(s)\prod_{i=1}^{2n+2r} L(s+k+n+r-i,f).$$

Moreover, if B is a positive definite half-integral symmetric matrix of size $2r+2n$ such that $(-1)^{r+n}\det(2B)$ is a fundamental discriminant, then the B-th Fourier coefficient of F is equal to $c(\det(2B))$. Note that $F(Z)$ is determined by $h(\tau)$.

Let $g(Z)\in S_{k+n+r}(\mathrm{Sp}_r(\mathbb{Z}))$ be a Hecke eigenform, whose standard L-function is

$$L(s,g,\mathrm{st})=\prod_p \left[(1-p^{-s})\prod_{i=1}^{r}(1-\beta_i p^{-s})(1-\beta_i^{-1}p^{-s})\right]^{-1}.$$

We shall call $\{\beta_1^{\pm 1},\dots \beta_r^{\pm 1}\}$ the Satake parameter in this paper. We put

$$\mathcal{F}_{h,g}(Z)=\int_{\mathrm{Sp}_r(\mathbb{Z})\backslash\mathfrak{h}_r} F\left(\begin{pmatrix} Z & 0\\ 0 & W\end{pmatrix}\right)\overline{g^c(W)}(\det \mathrm{Im} W)^{k+n-1}dW,$$

Then we have $\mathcal{F}_{h,g}\in S_{k+n+r}(\mathrm{Sp}_{2n+r}(\mathbb{Z}))$. Now our main theorem is as follows.

Theorem 1.1. *Assume that $\mathcal{F}_{h,g}(Z)$ is not identically zero. Then the cusp form $\mathcal{F}_{h,g}(Z)$ is a Hecke eigenform whose standard L-function is equal to*

$$L(s, \mathcal{F}_{h,g}, \mathrm{st}) = L(s, g, \mathrm{st}) \prod_{i=1}^{2n} L(s + k + n - i, f).$$

Remark 1.1. When $r = 1$, the L-function $L(s, g, \mathrm{st})$ is an Euler product of degree 3, and should not be confused with $L(s, g)$. To avoid possible confusion, we denote $L(s, g, \mathrm{Ad})$ rather than $L(s, g, \mathrm{st})$ for $r = 1$. Note also that the meaning of the Satake parameter for $f \in S_{2k}(\mathrm{Sp}_1(\mathbb{Z}))$ is different from the usual one. In our convention, the Satake parameter of f is $\{\alpha_p^{\pm 2}\}$.

Remark 1.2. We can interpret our theorem in terms of the Arthur conjecture. As in [18], we denote the hypothetical Langlands group by $\mathcal{L}$. Let τ be the cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{C})$ generated by f, and $\rho_\tau : \mathcal{L} \to \mathrm{SL}_2(\mathbb{C})$ the associated homomorphism.

Let $\rho_g : \mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{2r+1}(\mathbb{C}) = {}^L\mathrm{Sp}_r$ be the Arthur parameter for the cuspidal automorphic representation generated by $g(Z)$. Then the Arthur parameter for $\mathcal{F}_{h,g}$ should be given by the composition

$$\mathcal{L} \times \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}_{4n}(\mathbb{C}) \times \mathrm{SO}_{2r+1}(\mathbb{C}) \hookrightarrow \mathrm{SO}_{4n+2r+1}(\mathbb{C}) = {}^L\mathrm{Sp}_{2n+r}.$$

Here the first homomorphism is given by $(\rho_\tau \boxtimes \mathrm{Sym}_{2n-1}) \times \rho_g$. (cf. [18]).

2. Miyawaki's conjecture

It is known that $\dim_{\mathbb{C}} S_{12}(\mathrm{Sp}_3(\mathbb{Z})) = 1$. Let $\Phi_{12}^{(3)}(Z) \in S_{12}(\mathrm{Sp}_3(\mathbb{Z}))$ be a non-zero cusp form. Miyawaki [27] calculated some Hecke eigenvalues of the cusp form $\Phi_{12}^{(3)}(Z)$. Based on the numerical calculation, he made the following conjectures.

Conjecture 2.1 (Miyawaki)**.** The standard L-function of $\Phi_{12}^{(3)}(Z)$ is given by

$$L(s, \Phi_{12}^{(3)}, \mathrm{st}) = L(s, \Delta, \mathrm{Ad}) L(s + 10, \phi_{20}) L(s + 9, \phi_{20}).$$

More generally,

Conjecture 2.2 (Miyawaki)**.** Given normalized Hecke eigenforms $f \in S_{2k-4}(\mathrm{SL}_2(\mathbb{Z}))$ and $g \in S_k(\mathrm{SL}_2(\mathbb{Z}))$, there should be a Hecke eigenform $F_{f,g} \in S_k(\mathrm{Sp}_3(\mathbb{Z}))$ whose standard L-function is equal to

$$L(s, g, \mathrm{Ad}) L(s + k - 2, f) L(s + k - 3, f).$$

In fact, Miyawaki formulated Conjecture 2.2 in terms of linear maps

$$S_{2k-4}(\mathrm{SL}_2(\mathbb{Z})) \otimes S_k(\mathrm{SL}_2(\mathbb{Z})) \to S_k(\mathrm{Sp}_3(\mathbb{Z})).$$

It seems there is no such a canonical map, but our construction defines a canonical map

$$S^+_{k-(3/2)}(\Gamma_0(4)) \otimes S_k(\mathrm{SL}_2(\mathbb{Z})) \to S_k(\mathrm{Sp}_3(\mathbb{Z}))$$

induced by the bilinear map $h \times g \mapsto \mathcal{F}_{h,g}$. Note that Kohnen [20] defined a canonical linear map

$$\begin{array}{ccc} S^+_{k+(1/2)}(\Gamma_0(4)) & \to & S_k(\mathrm{Sp}_{2n+2r}(\mathbb{Z})) \\ h & \mapsto & F \end{array}$$

which coincides with the Duke-Imamoglu lifting when h is a Hecke eigenform. If $\mathcal{F}_{h,g}$ is non-zero for each Hecke eigenform h and g, then Theorem 1.1 solves the Conjecture 2.2.

The author would like to propose to call G the Miyawaki lift of $g(Z) \in S_{k+r+n}(\mathrm{Sp}_r(\mathbb{Z}))$ with respect to the Duke-Imamoglu lift $F(Z) \in S_{k+r+n}(\mathrm{Sp}_{2r+2n}(\mathbb{Z}))$ of $f(\tau)$, if $G = c\mathcal{F}_{h,g}$ for some $c \neq 0$.

In §7, we will show that $\Phi^{(3)}_{12}$ is in fact the Miyawaki lifting of Δ with respect to the Duke-Imamoglu lift of ϕ_{20} to degree 4. In particular, Conjecture 2.1 is true.

Remark 2.1. In [27], Miyawaki also considered the spin L-functions, which we do not consider here. He also considered the spin and standard L-functions of $\Phi^{(3)}_{14} \in S_{14}(\mathrm{Sp}_3(\mathbb{Z}))$ and its generalization. He conjectured that the standard L-function of the cusp form $\Phi^{(3)}_{14}(Z)$ is equal to

$$L(s, \Delta, \mathrm{Ad})L(s+13, \phi_{26})L(s+12, \phi_{26}).$$

It seems that one needs an analogue of the lifting [18] such that the infinite part of the automorphic representation generated by the lifting is a cohomological induction from non-compact unitary group, to solve this conjecture. In fact, the Arthur conjecture suggests that there exists an irreducible discrete automorphic representation π of $\mathrm{Sp}_4(\mathbb{A}_\mathbb{Q})$ satisfying the following (i) and (ii);

(i) The standard L-function of π is $\zeta(s)\prod_{i=11}^{14} L(s+i, \phi_{26})$.
(ii) The infinite component of π is a cohomological induction from the non-compact unitary group $U(3,1)$.

The infinite component of π is a non-tempered unitary representation with minimal K-type $(14, 14, 14, -12)$. Taking a convolution with $\Delta(\tau)$, one would get $\Phi^{(3)}_{14}(Z)$. It is very likely that π is generated by certain residue of the Eisenstein series associated to parabolic subgroup $P_{2,2}$ with Levi factor $\mathrm{GL}_2 \times \mathrm{Sp}_2$.

3. Unramified principal series of p-adic groups

In this section, we shall prove some results on unramified principal series of symplectic groups over a p-adic field.

In this section, F denotes a non-archimedean local field of characteristic 0. The symbols ϖ and q denote a prime element and the order of the residue field of F, respectively. An algebraic group and its group of F-rational points are denoted by the same symbol.

When G is a locally compact group, δ_G is the modulus character of G. If (ρ, V_ρ) and $(\rho', V_{\rho'})$ are smooth representation of a totally disconnected locally compact group G, then $\mathcal{B}_G(\rho, \rho')$ is the space of bilinear form B on $V_\rho \times V_{\rho'}$ such that $B(\rho(g)v, \rho'(g)v') = B(v, v')$ for any $v \in V_\rho$, $v' \in V_{\rho'}$, and $g \in G$. Note that if ρ' is admissible, then $\mathcal{B}_G(\rho, \rho') \simeq \mathrm{Hom}_G(\rho, \tilde{\rho}')$.

When ρ is a smooth representation of a closed subgroup H of a totally disconnected locally compact group G, we denote the normalized induced representation (resp. normalized compactly induced representation) by $\mathrm{Ind}_H^G \rho$ (resp. c-$\mathrm{Ind}_H^G \rho$).

Fix integers m and r such that $m \geq r \geq 0$. We put $G_1 = \mathrm{Sp}_r$, $G_2 = \mathrm{Sp}_m$, and $H = \mathrm{Sp}_{m+r}$. We denote the Siegel parabolic subgroup of H by P_H. $G_1 \times G_2$ can be embedded into H by

$$\begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \times \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix} \mapsto \left(\begin{array}{cc|cc} A_1 & 0 & B_1 & 0 \\ 0 & A_2 & 0 & B_2 \\ \hline C_1 & 0 & D_1 & 0 \\ 0 & C_2 & 0 & D_2 \end{array}\right).$$

We think of $G_1 \times G_2$ as a subgroup of H.

For $i = 0, 1, \ldots, r$, put

$$\eta_i = \left(\begin{array}{cccc|cccc} 0 & 0 & 0 & 0 & 0 & 0 & -\mathbf{1}_i & 0 \\ 0 & \mathbf{1}_{r-i} & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1}_i & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \mathbf{1}_{m-i} & 0 & 0 & 0 & 0 \\ \hline \mathbf{1}_i & 0 & \mathbf{1}_i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{r-i} & 0 & 0 \\ 0 & 0 & 0 & 0 & \mathbf{1}_i & 0 & -\mathbf{1}_i & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{m-i} \end{array}\right).$$

Here the size of the blocks are i, $r-i$, i, $m-i$, i, $r-i$, i, and $m-i$.

The following lemma is well-known (cf. [4], [16]).

Lemma 3.1. *The set $\{\eta_0, \eta_1, \ldots, \eta_r\}$ forms a set of representatives for the double cosets $P_H \backslash H/(G_1 \times G_2)$.* $\square$

For $i = 0, 1, \ldots, r$, put $Q_i = (\eta_i^{-1} P_H \eta_i) \cap (G_1 \times G_2)$. Then, by direct calculation, we have

$$Q_i = \left\{ \left(\begin{array}{cc|cc} \alpha & 0 & \beta & * \\ * & A & * & * \\ \hline \gamma & 0 & \delta & * \\ 0 & 0 & 0 & D \end{array} \right) \times \left(\begin{array}{cc|cc} \alpha & 0 & -\beta & * \\ * & A' & * & * \\ \hline -\gamma & 0 & \delta & * \\ 0 & 0 & 0 & D' \end{array} \right) \in G_1 \times G_2 \,\middle|\, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_i,\ A = {}^tD^{-1} \in \mathrm{GL}_{r-i},\ A' = {}^tD'^{-1} \in \mathrm{GL}_{m-i}, \right\}.$$

We define the parabolic subgroups $P_i^{(1)} \subset G_1$ and $P_i^{(2)} \subset G_2$ by

$$P_i^{(1)} = \left\{ \left(\begin{array}{cc|cc} \alpha & 0 & \beta & * \\ * & A & * & * \\ \hline \gamma & 0 & \delta & * \\ 0 & 0 & 0 & D \end{array} \right) \in G_1 \,\middle|\, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_i,\ A = {}^tD^{-1} \in \mathrm{GL}_{r-i}, \right\},$$

$$P_i^{(2)} = \left\{ \left(\begin{array}{cc|cc} \alpha & 0 & \beta & * \\ * & A' & * & * \\ \hline \gamma & 0 & \delta & * \\ 0 & 0 & 0 & D' \end{array} \right) \in G_2 \,\middle|\, \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{Sp}_i,\ A' = {}^tD'^{-1} \in \mathrm{GL}_{m-i}, \right\}.$$

Lemma 3.2. *Let ι be the automorphism of Sp_i given by $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \mapsto \begin{pmatrix} \alpha & -\beta \\ -\gamma & \delta \end{pmatrix}$. For any irreducible admissible representation π of Sp_i, we have $\pi \circ \iota \simeq \tilde{\pi}$.*

Proof. A proof of this lemma can be found in [28], Chapter 4-II. □

Lemma 3.3. *Let G be a unimodular totally disconnected locally compact group, and ρ and ρ' irreducible admissible representations of G. If $\mathcal{B}_{G\times G}(\text{c-Ind}_{\Delta \mathrm{G}}^{\mathrm{G}\times\mathrm{G}} 1, \rho \boxtimes \rho') \neq 0$, then $\rho' \simeq \tilde{\rho}$. Here, ΔG is the diagonal subgroup of $G \times G$.*

Proof. This lemma seems well-known, but for the sake of completeness, we give a proof. Note that $\text{c-Ind}_{\Delta \mathrm{G}}^{\mathrm{G}\times\mathrm{G}} 1 \simeq \mathrm{C}_0^\infty(\mathrm{G})$ by restriction to the second factor. For each compact open subgroup K of G, we put

$$e_K = \mathrm{Volume}(K)^{-1} \times (\text{ characteristic function of } K).$$

We define an injection

$$\varphi : \mathcal{B}_{G\times G}(C_0^\infty(G), \rho_1 \boxtimes \rho_2) \to \mathcal{B}_G(\rho, \rho')$$

as follows. Given $U \in \mathcal{B}_{G\times G}(C_0^\infty(G), \rho \boxtimes \rho')$, $w \in \rho$, and $w' \in \rho'$, we put

$$\varphi(U)(w, w') = U(e_K, w \boxtimes w')$$

for sufficiently small open compact subgroup K. It is easy to check that this definition does not depend on the choice of K and that φ is an injective map. Hence the lemma. $\square$

Let π_1 (resp. π_2) be an irreducible unramified principal series representation of G_1 (resp. G_2). Then there exist unramified quasi-characters $\lambda_1,\ \lambda_2, \dots, \lambda_r$ (resp. $\lambda'_1,\ \lambda'_2, \dots, \lambda'_m$) such that π_1 (resp. π_2) is the unique unramified constituent of the induced representation

$$\mathrm{Ind}_{B_{G_1}}^{G_1} \lambda_1 \boxtimes \lambda_2 \boxtimes \cdots \boxtimes \lambda_r$$
$$(\text{resp. } \mathrm{Ind}_{B_{G_2}}^{G_2} \lambda'_1 \boxtimes \lambda'_2 \boxtimes \cdots \boxtimes \lambda'_m).$$

Here, B_{G_1} (resp. B_{G_2}) is a Borel subgroup of G_1 (resp. G_2). Put $\beta_i = \lambda_i(\varpi)$ $(i = 1, 2, \dots, r)$ and $\beta'_j = \lambda'_j(\varpi)$ $(j = 1, 2, \dots, m)$. By definition, the set of the Satake parameters of π_1 and π_2 are $\{\beta_1^{\pm 1}, \beta_2^{\pm 1}, \cdots, \beta_r^{\pm 1}\}$ and $\{{\beta'}_1^{\pm 1}, {\beta'}_2^{\pm 1}, \cdots, {\beta'}_m^{\pm 1}\}$, respectively.

Note that the standard Levi subgroup of P_H is isomorphic to GL_{m+r}. A one-dimensional representation of GL_{m+r} is of the form $\omega \circ \det$ for some quasi-character $\omega : F^\times \to \mathbb{C}^\times$. The induced representation $\mathrm{Ind}_{P_H}^H(\omega \circ \det)$ is called a degenerate principal series.

Proposition 3.1. *Let $\omega : F^\times \to \mathbb{C}^\times$ be an unramified quasi-character. Put $\alpha = \omega(\varpi)$. If*

$$\mathcal{B}_{G_1\times G_2}(\mathrm{Ind}_{P_H}^H(\omega^{-1} \circ \det)|_{G_1\times G_2}, \pi_1 \boxtimes \pi_2) \neq \{0\},$$

then as a multiset, $\{{\beta'}_1^{\pm 1}, {\beta'}_2^{\pm 1}, \dots, {\beta'}_m^{\pm 1}\}$ is equal to

$$\{{\beta_1}^{\pm 1}, {\beta_2}^{\pm 1}, \dots, {\beta_r}^{\pm 1}\}$$
$$\cup \{(\alpha^{\pm 1} q^{(m-r-1)/2}, \alpha^{\pm 1} q^{(m-r-3)/2}, \dots, \alpha^{\pm 1} q^{-(m-r-1)/2}\}.$$

Proof. We proceed as in Rallis [32] Chapter II. Let X_i $(i = 0, \dots, r)$ be the subspace of $\mathrm{Ind}_{P_H}^H(\omega^{-1} \circ \det)$ that consists of the elements whose supports are contained in

$$\bigcup_{j=i}^{r} P_H\, \eta_i(G_1 \times G_2).$$

We put $X_{r+1} = \{0\}$. Then

$$\{0\} = X_{r+1} \subset X_r \subset \cdots \subset X_1 \subset X_0 = \mathrm{Ind}_{P_H}^H(\omega^{-1} \circ \det)$$

are $G_1 \times G_2$ invariant subspaces, and

$$X_i/X_{i+1} \simeq \text{c-Ind}_{Q_i}^{G_1\times G_2}\omega_i\delta_{Q_i}^{-1/2}.$$

Here δ_{P_H} (resp. δ_{Q_i}) is the modulus character of P_H (resp. Q_i), and ω_i is the character of Q_i defined by

$$\omega_i(t) = (\omega^{-1} \circ \det)(\eta_i t\eta_i^{-1})\delta_{P_H}^{1/2}(\eta_i t\eta_i^{-1}).$$

It is easy to see

$$\begin{aligned}\omega_i(t) &= \omega^{-1}(\det A \det A')|\det A \det A'|^{(m+r+1)/2},\\ \delta_{Q_i}^{1/2}(t) &= |\det A|^{(r+i+1)/2}|\det A'|^{(m+i+1)/2}\end{aligned}$$

for

$$t = \left(\begin{array}{cc|cc} \alpha & 0 & \beta & * \\ * & A & * & * \\ \hline \gamma & 0 & \delta & * \\ 0 & 0 & 0 & D \end{array}\right) \times \left(\begin{array}{cc|cc} \alpha & 0 & -\beta & * \\ * & A' & * & * \\ \hline -\gamma & 0 & \delta & * \\ 0 & 0 & 0 & D' \end{array}\right) \in Q_i.$$

The Jacquet modules $r_{P_i^{(1)}}^{G_1}\pi_1$ and $r_{P_i^{(2)}}^{G_2}\pi_2$ are representations of $\mathrm{Sp}_i \times \mathrm{GL}_{r-i}$ and $\mathrm{Sp}_i \times \mathrm{GL}_{m-i}$, respectively. By Lemma 3.2 and Lemma 3.3, the Jacquet modules $r_{P_i^{(1)}}^{G_1}\pi_1$ and $r_{P_i^{(2)}}^{G_2}\pi_2$ have irreducible subquotients of the form

$$\rho^{(1)} \boxtimes (\omega \circ \det)|\det|^{-(m-i)/2}$$

and

$$\rho^{(2)} \boxtimes (\omega \circ \det)|\det|^{-(r-i)/2},$$

respectively, such that $\rho^{(1)} \simeq \rho^{(2)}$ for some i $(0 \le i \le r)$.

Let $\{\beta''^{\pm1}_1, \beta''^{\pm1}_2, \dots, \beta''^{\pm1}_i\}$ be the set of Satake parameters of $\rho^{(1)} \simeq \rho^{(2)}$. Then the set of Satake parameters of π_1 is

$$\begin{aligned}&\{\beta''^{\pm1}_1, \beta''^{\pm1}_2, \dots, \beta''^{\pm1}_i\}\\ &\quad\cup\{(\alpha q^{(m-r+1)/2})^{\pm1}, (\alpha q^{(m-r+3)/2})^{\pm1}, \dots, (\alpha q^{(m+r-2i-1)/2})^{\pm1}\}.\end{aligned}$$

On the other hand, the set of Satake parameters of π_2 is

$$\begin{aligned}&\{\beta''^{\pm1}_1, \beta''^{\pm1}_2, \dots, \beta''^{\pm1}_i\}\\ &\quad\cup\{(\alpha q^{(r-m+1)/2})^{\pm1}, (\alpha q^{(r-m+3)/2})^{\pm1}, \dots, (\alpha q^{(m+r-2i-1)/2})^{\pm1}\}\\ =&\{\beta''^{\pm1}_1, \beta''^{\pm1}_2, \dots, \beta''^{\pm1}_i\}\\ &\quad\cup\{(\alpha q^{(m-r+1)/2})^{\pm1}, (\alpha q^{(m-r+3)/2})^{\pm1}, \dots, (\alpha q^{(m+r-2i-1)/2})^{\pm1}\}\\ &\quad\cup\{\alpha^{\pm1}q^{(m-r-1)/2}, \alpha^{\pm1}q^{(m-r-3)/2}, \dots, \alpha^{\pm1}q^{-(m-r-1)/2}\}.\end{aligned}$$

Hence the proposition.

□

4. Proof of Theorem 1.1

Now we go back to the situation of §2. As in the last section, $G_1 = \mathrm{Sp}_r$, $G_2 = \mathrm{Sp}_m$, and $H = \mathrm{Sp}_{m+r}$. Let $\omega_p : \mathbb{Q}_p^\times \to \mathbb{C}^\times$ be the unramified character determined by $\omega_p(p) = \alpha_p$,

The p-component of the irreducible cuspidal automorphic representation of $H(\mathbb{A})$ generated by $F(Z)$ is the degenerate principal series

$$\mathrm{Ind}_{P_H(\mathbb{Q}_p)}^{H(\mathbb{Q}_p)}(\omega_p \circ \det),$$

since the Satake parameter is

$$\{(\alpha_p p^{-(m+r+1)/2})^{\pm 1}, (\alpha_p p^{-(m+r-1)/2})^{\pm 1}, \dots , (\alpha_p p^{(m+r+1)/2})^{\pm 1}\}.$$

Let $\mathcal{H}(G_i(\mathbb{A}_\mathrm{f}))$ $(i = 1, 2)$ be the Hecke algebra for the finite adele group $G_i(\mathbb{A}_\mathrm{f})$. Then $\mathcal{H}(G_1(\mathbb{A}_\mathrm{f}))\cdot g$ (resp. $\mathcal{H}(G_2(\mathbb{A}_\mathrm{f}))\cdot\mathcal{F}_{h,g}$) is the finite part of the cuspidal automorphic representation of $G_1(\mathbb{A})$ (resp. $G_2(\mathbb{A})$) generated by g (resp. $\mathcal{F}_{h,g}$). $\mathcal{H}(G_1(\mathbb{A}_\mathrm{f}))\cdot g$ is an irreducible representation of $G_1(\mathbb{A}_\mathrm{f})$. Let π_1 be the p-component of $\mathcal{H}(G_1(\mathbb{A}_\mathrm{f}))\cdot g$. Then π_1 is an unramified principal series with Satake parameter $\{\beta_{p,1}^{\pm 1}, \dots , \beta_{p,r}^{\pm 1}\}$. On the other hand, since $\mathcal{F}_{h,g}(Z)$ is a cusp form, the representation $\mathcal{H}(G_2(\mathbb{A}_\mathrm{f}))\cdot\mathcal{F}_{h,g}$ of $G_2(\mathbb{A}_\mathrm{f})$ is unitary and of finite length. Let π_2 be the p-component of some irreducible direct summand of $\mathcal{H}(G_2(\mathbb{A}_\mathrm{f}))\cdot\mathcal{F}_{h,g}$. Then π_2 is also an unramified principal series. Observe that

$$\begin{aligned}\int_{\mathrm{Sp}_{2n+r}(\mathbb{Z})\backslash\mathfrak{h}_{2n+r}} \int_{\mathrm{Sp}_r(\mathbb{Z})\backslash\mathfrak{h}_r} & \overline{F\left(\begin{pmatrix} Z & 0 \\ 0 & W \end{pmatrix}\right)} g^c(W)\mathcal{F}_{h,g}(Z) \\ & \times (\det \mathrm{Im} Z)^{k-n-1}(\det \mathrm{Im} W)^{k+n-1} dW dZ \\ = & \langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle \neq 0.\end{aligned}$$

It follows that

$$\mathcal{B}_{G_1(\mathbb{Q}_p)\times G_2(\mathbb{Q}_p)}(\mathrm{Ind}_{P_H(\mathbb{Q}_p)}^{H(\mathbb{Q}_p)}(\omega^{-1} \circ \det)|_{G_1(\mathbb{Q}_p)\times G_2(\mathbb{Q}_p)}, \tilde{\pi}_1 \boxtimes \pi_2) \neq \{0\}.$$

By Proposition 3.1, any irreducible component of $\mathcal{H}(G_2(\mathbb{A}_\mathrm{f}))\cdot\mathcal{F}_{h,g}$ has Satake parameter

$$\{\beta_{p,1}^{\pm 1}, \dots , \beta_{p,r}^{\pm 1}, (\alpha_p p^{n-(1/2)})^{\pm 1}, \dots , (\alpha_p p^{-n+(1/2)})^{\pm 1}\}.$$

In particular, $\mathcal{H}(G_2(\mathbb{A}_\mathrm{f}))\cdot\mathcal{F}_{h,g}$ is isotypic. Since it is generated by the class 1 vector $\mathcal{F}_{h,g}$, it is irreducible. It follows that $\mathcal{F}_{h,g}$ is a Hecke eigenform and its standard L-function is equal to

$$L(s, \mathcal{F}_{h,g}, \mathrm{st}) = L(s, g, \mathrm{st}) \prod_{i=1}^{2n} L(s + k + n - i, f).$$

5. A conjecture on the Petersson inner product

It is an interesting problem to determine when $\mathcal{F}_{h,g} \not\equiv 0$. Here we are going to give a conjecture on the Petersson inner product of $\mathcal{F}_{h,g}$.

Let $L(s, \mathrm{st}(g) \boxtimes f)$ be the L-function defined by

$$L(s, \mathrm{st}(g) \boxtimes f) = \prod_p \det(\mathbf{1}_{4r+2} - A_p \otimes B_p \cdot p^{-s})^{-1},$$

where

$$L(s, f) = \prod_p \det(\mathbf{1}_2 - A_p \cdot p^{-s})^{-1}, \quad A_p \in \mathrm{GL}_2(\mathbb{C}),$$

$$L(s, g, \mathrm{st}) = \prod_p \det(\mathbf{1}_{2r+1} - B_p \cdot p^{-s})^{-1}, \quad B_p \in \mathrm{GL}_{2r+1}(\mathbb{C}).$$

The gamma factor of $L(s, \mathrm{st}(g) \boxtimes f)$ is given by

$$L_\infty(s, \mathrm{st}(g) \boxtimes f) = \Gamma_{\mathbb{C}}(s) \prod_{i=1}^{r} \Gamma_{\mathbb{C}}(s+n-k+i)\Gamma_{\mathbb{C}}(s+n+k+i-1).$$

Here, $\Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s)$.

We put $\Lambda(s, \mathrm{st}(g) \boxtimes f) = L_\infty(s, \mathrm{st}(g) \boxtimes f) L(s, \mathrm{st}(g) \boxtimes f)$. Then the functional equation should be

$$\Lambda(2k - s, \mathrm{st}(g) \boxtimes f) = (-1)^{k+r} \Lambda(s, \mathrm{st}(g) \boxtimes f)$$

We also need the adjoint L-function $L(s, f, \mathrm{Ad})$ of f. We put

$$\begin{aligned} \xi(s) &= \Gamma_{\mathbb{R}}(s)\zeta(s), \\ \Lambda(s, f, \mathrm{Ad}) &= \Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+2k-1)L(s, f, \mathrm{Ad}). \end{aligned}$$

Here, $\Gamma_{\mathbb{R}}(s) = \pi^{-s/2}\Gamma(s/2)$. Then the following functional equations hold.

$$\begin{aligned} \xi(1-s) &= \xi(s), \\ \Lambda(1-s, f, \mathrm{Ad}) &= \Lambda(s, f, \mathrm{Ad}). \end{aligned}$$

We modify $\xi(s)$ and $\Lambda(s, f, \mathrm{Ad})$ as follows.

$$\begin{aligned} \tilde{\xi}(s) &= \Gamma_{\mathbb{R}}(s+1)\xi(s) = \Gamma_{\mathbb{C}}(s)\zeta(s), \\ \tilde{\Lambda}(s, f, \mathrm{Ad}) &= \Gamma_{\mathbb{R}}(s)\Lambda(s, f, \mathrm{Ad}) = \Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+2k-1)L(s, f, \mathrm{Ad}). \end{aligned}$$

If i is a positive integer, $\tilde{\xi}(2i) = |B_{2i}|/2i \in \mathbb{Q}^\times$. It is well-known that $\tilde{\Lambda}(2i-1, f, \mathrm{Ad})/\langle f, f\rangle \in \mathbb{Q}(f)^\times$ for $1 \le i < k$.

Conjecture 5.1. Assume that $n < k$. Then there exists an integer $\alpha = \alpha(r, n, k)$ depending only on r, n, and k such that

$$\Lambda(k+n, \mathrm{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i-1, f, \mathrm{Ad})\tilde{\xi}(2i) = 2^{\alpha} \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle}{\langle g, g\rangle}.$$

In particular, $\mathcal{F}_{h,g}$ is non-zero if and only if $\Lambda(k+n, \mathrm{st}(g) \boxtimes f) \neq 0$.

In the case $r = n = 1$, the left hand side does not vanish. Therefore our conjecture implies Miyawaki's conjecture 2.2.

When $\mathcal{F}_{h,g} \neq 0$, one can rewrite the right hand side in a more symmetric way. Namely, choose any non-zero $G \in \mathbb{C} \cdot \mathcal{F}_{h,g}$. Then

$$\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle = \frac{|\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g^c \times G\rangle|^2}{\langle G, G\rangle}$$

Here $\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g^c \times G\rangle$ is a Petersson inner product on $(\mathrm{Sp}_r(\mathbb{Z})\backslash\mathfrak{h}_r) \times (\mathrm{Sp}_{r+2n}(\mathbb{Z})\backslash\mathfrak{h}_{r+2n})$. Therefore the conjecture takes the form

$$\text{(C)} \qquad \Lambda(k+n, \mathrm{st}(g) \boxtimes f) \prod_{i=1}^{n} \tilde{\Lambda}(2i-1, f, \mathrm{Ad})\tilde{\xi}(2i) = 2^{\alpha} \frac{\langle f, f\rangle}{\langle h, h\rangle} \frac{|\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g^c \times G\rangle|^2}{\langle g, g\rangle\langle G, G\rangle}.$$

Remark 5.1. By some computer calculation (cf. Appendix), it seems the values of $\alpha = \alpha(r, n, k)$ are

(a) $\alpha(0, n, k) = 2kn + 2n - k - 1,$

(b) $\alpha(r, 0, k) = r^2 + 2kr + r - k - 1,$

(c) $\alpha(r, n, k) = r^2 + 2kr + 2kn + 2rn + 2n + r - k - 2$

for r, $n > 0$. As for the case $n = 0$, we will give some evidence for (C) in the next section.

Remark 5.2. Note that $s = k+n$ is a critical point for $\Lambda(s, \mathrm{st}(g)\boxtimes f)$ in the sense of Deligne [9]. In particular, the left hand side of (C) should be finite. Deligne's conjecture [9] implies the ratio RHS/LHS should belong to the field $\mathbb{Q}(f, g)$ under the assumption $n < k$. (cf. Yoshida [36]). When $r = 0$, see Choie and Kohnen [7], Lanphier [26].

Example 5.1. When $r = n = 0$, we have $F(Z) = c(1)$. In this case, our conjecture is a special case of the result of Kohnen-Zagier [23]

$$\Lambda(k, f) = 2^{1-k} \frac{\langle f, f\rangle}{\langle h, h\rangle} |c(1)|^2.$$

It follows that our conjecture holds for $n = r = 0$ with $\alpha(0, 0, k) = 1-k$.

Example 5.2. When $r=0$, $n=1$, our conjecture is compatible with the Petersson inner product formula for the Saito-Kurokawa lift

$$\Lambda(k+1,f)=3\cdot 2^{-k+3}\frac{\langle F,F\rangle}{\langle h,h\rangle}$$

proved by Kohnen [21] and Kohnen and Skoruppa [22]. See also Krieg [24], Oda [31], and Furusawa [15]. This is equivalent with

$$\Lambda(k+1,f)\tilde{\Lambda}(1,f,\mathrm{Ad})\tilde{\xi}(2)=2^{k+1}\frac{\langle f,f\rangle}{\langle h,h\rangle}\langle F,F\rangle,$$

since $\tilde{\Lambda}(1,f,\mathrm{Ad})=2^{2k}\langle f,f\rangle$. It follows that our conjecture holds for $(r,n)=(0,1)$ with $\alpha(0,1,k)=k+1$.

So far, we have assumed $n\geq 0$. We now consider the case $n<0$. We shall show that if Conjecture 5.1 is true, the roles of g and G can be interchanged.

Proposition 5.1. *Assume that Conjecture 5.1 is true and $\mathcal{F}_{h,g}\neq 0$. Then $\mathcal{F}_{h,G}\in\mathbb{C}\cdot g$ for any $G\in\mathbb{C}\cdot\mathcal{F}_{h,g}$. Here, $\mathcal{F}_{h,G}$ is the Miyawaki lifting of $G\in S_{k+r+n}(\mathrm{Sp}_{r+2n}(\mathbb{Z}))$ to $S_{k+r+n}(\mathrm{Sp}_r(\mathbb{Z}))$ with respect to $F\in S_{k+r+n}(\mathrm{Sp}_{2r+2n}(\mathbb{Z}))$.*

Proof. Choose an orthonormal basis $\{g_i\}_{i\in I}$ of $S_{k+r+n}(\mathrm{Sp}_r(\mathbb{Z}))$ which consists of Hecke eigenforms. We may assume $g\in\{g_i\}_{i\in I}$. The pull-back $F|_{\mathfrak{h}_r\times\mathfrak{h}_{r+2n}}$ can be expressed as

$$F|_{\mathfrak{h}_r\times\mathfrak{h}_{r+2n}}=\sum_{i\in I}g_i^c\times G_i,\qquad G_i=\mathcal{F}_{h,g_i}.$$

It is enough to show that $\langle G_i,G_j\rangle=0$ for $i\neq j$. By Thoerem 1.1, we may assume g_i and g_j have the same Hecke eigenvalues.

Let V be the subspace of $S_{k+r+n}(\mathrm{Sp}_r(\mathbb{Z}))$ generated by all Hecke eigenforms with the same Hecke eigenvalues as g. We define $V'\subset S_{k+r+n}(\mathrm{Sp}_{r+2n}(\mathbb{Z}))$ similarly. Then our assumption implies the map $g\mapsto\mathcal{F}_{h,g}$ is an isometry from V onto an subspace of V' up to scalar multiplication. It follows that G_i and G_j are orthogonal for $i\neq j$. $\square$

Proposition 5.2.

$$\left[\Lambda(s+k-n,\mathrm{st}(G)\boxtimes f)\prod_{i=1}^{n}\tilde{\Lambda}(s-2i+1,f,\mathrm{Ad})^{-1}\tilde{\xi}(s-2i+2)^{-1}\right]_{s=0}$$
$$=\Lambda(k+n,\mathrm{st}(g)\boxtimes f)\prod_{i=1}^{n}\tilde{\Lambda}(2i-1,f,\mathrm{Ad})\tilde{\xi}(2i).$$

Proof. By Theorem 1.1, $\Lambda(s+k-n, \mathrm{st}(G) \boxtimes f)$ is the product of

$$\prod_{i=1}^{2n} \Lambda(s+2k-i, f \times f)$$

and

$$\Lambda(s+k-n, \mathrm{st}(g) \boxtimes f) = (-1)^{k+r} \Lambda(-s+k+n, \mathrm{st}(g) \boxtimes f).$$

Since $\Lambda(s+2k-1, f \times f) = \Lambda(s, f, \mathrm{Ad})\xi(s)$, we have

$$\begin{aligned}
&\prod_{i=1}^{2n} \Lambda(s+2k-i, f \times f) \prod_{i=1}^{n} \tilde{\Lambda}(s-2i+1, f, \mathrm{Ad})^{-1} \tilde{\xi}(s-2i+2)^{-1} \\
&\quad = \prod_{i=1}^{n} \Gamma_{\mathbb{R}}(s-2i+1)^{-1} \Gamma_{\mathbb{R}}(s-2i+3)^{-1} \\
&\qquad \times \prod_{i=1}^{n} \Lambda(-s+2i-1, f, \mathrm{Ad}) \xi(-s+2i).
\end{aligned}$$

Now using $\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{R}}(-s+1) = \sin(\pi s/2)$, we have

$$\prod_{i=1}^{n} \Gamma_{\mathbb{R}}(-2i+1)^{-1} \Gamma_{\mathbb{R}}(-2i+3)^{-1} = (-1)^n \prod_{i=1}^{n} \Gamma_{\mathbb{R}}(2i-1) \Gamma_{\mathbb{R}}(2i+1).$$

Hence the proposition. □

Remark 5.3. The polynomial which shows up in the right hand side of Remark 5.1 (c) is not invariant under $(r, n) \mapsto (r+2n, -n)$.

6. SOME EVIDENCE FOR THE CASE $n = 0$

In this section, we discuss the case when $n = 0$. In this case we conjecture $\alpha(r, 0, k) = r^2 + 2rk + r - k - 1$.

By Kohnen-Zagier [23],

$$|c(|D|)|^2 \frac{\langle f, f \rangle}{\langle h, h \rangle} = 2^{k-1} |D|^{-1/2} \Lambda(k, f, \chi_D), \tag{KZ}$$

for any fundamental discriminant D such that $(-1)^k D > 0$. Here,

$$\Lambda(s, f, \chi_D) = |D|^s \Gamma_{\mathbb{C}}(s) L(s, f, \chi_D).$$

It follows that if $c(|D|) \neq 0$, our conjecture is equivalent to the following:

$$\Lambda(k, \mathrm{st}(g) \boxtimes f) = 2^{r^2+r+2rk-2} \frac{\Lambda(k, f, \chi_D)}{\sqrt{|D|}\, |c(|D|)|^2} \frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g} \rangle}{\langle g, g \rangle}. \tag{C$'$}$$

When $f = E_{2k}$ is the Eisenstein series, the equation (C) does not make sense, but (C′) makes sense. As $L(s, E_{2k}) = \zeta(2)\zeta(s-2k+1)$, we think of $L(s, \mathrm{st}(g) \boxtimes E_{2k})$ as $L(s, g, \mathrm{st})L(s-2k+1, g, \mathrm{st})$, while the gamma factor is the same as $L_\infty(s, \mathrm{st}(g)\boxtimes f)$. Let $h(\tau)$ be the Cohen Eisenstein series $\mathcal{H}_{k+(1/2)} \in M^+_{k+(1/2)}(\Gamma_0(4))$ and $F = \mathcal{E}^{(2r)}_{k+r} = 2^{-r}\mathcal{A}_{r,k} \cdot E^{(2r)}_{k+r}$ the normalized Eisenstein series, where

$$\mathcal{A}_{r,k} = \zeta(1-k-r)\prod_{i=1}^{r}\zeta(1-2k-2r+2i).$$

introduced in [18]. $F = \mathcal{E}^{(2r)}_{k+r}$ can be thought of as the Duke-Imamoglu lift of $\mathcal{H}(\tau)$.

Proposition 6.1. *If $f = E_{2k}$, $h = \mathcal{H}_{k+(1/2)}$, and $F = \mathcal{E}^{(2r)}_{k+r}$, then the equation* (C′) *holds.*

Proof. This is essentially a result of Böcherer [3]. When $f = E_{2k}$, $h = \mathcal{H}_{k+(1/2)}$, we have

$$c(|D|) = L(1-k, \chi_D) = (-1)^{k(k-1)/2}|D|^{k-(1/2)}2(2\pi)^{-k}\Gamma(k)L(k, \chi_D),$$

and so

$$\frac{\Lambda(k, f, \chi_D)}{\sqrt{|D|}c(|D|)^2} = (-1)^{k(k-1)/2}.$$

By the functional equation (cf. [3]) of $L(s, g, \mathrm{st})$, we have

$$\begin{aligned} &L(1-k, g, \mathrm{st}) \\ &\quad= (-1)^{k(k-1)/2}2(2\pi)^{r-2rk-k}\Gamma(k)\prod_{i=1}^{r}\frac{\Gamma(2k+i-1)}{\Gamma(i)} \cdot L(k, g, \mathrm{st}). \end{aligned}$$

Therefore, we have to prove

$$\begin{aligned} \frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle}{\langle g, g\rangle} =& 2^{-2r^2+2r-6rk-2k+4}\pi^{-r^2+r-4rk-2k} \\ &\times \Gamma(k)^2\prod_{i=1}^{r}\Gamma(2k+i-1)^2 \cdot L(k, g, \mathrm{st})^2. \end{aligned}$$

On the other hand, by the result of Böcherer [3], we have $\mathcal{F}_{h,g} = \mathcal{B}_r \cdot g$, where

$$\begin{aligned} \mathcal{B}_r =& (-1)^{r(k+r)/2}2^{(-r^2+r-2rk+2)/2}\pi^{(r^2+r)/2}\frac{\Gamma_r(k+\frac{r-1}{2})}{\Gamma_r(k+r)} \\ &\times \zeta(k+r)^{-1}\prod_{i=1}^{r}\zeta(2k+2r-2i)^{-1}L(k, g, \mathrm{st}) \cdot \mathcal{A}_{r,k}. \end{aligned}$$

Here $\Gamma_r(s) = \prod_{i=1}^{r}\Gamma(s-((i-1)/2))$.

By the functional equation of the Riemann zeta function and the definition of $\mathcal{E}^{(2r)}_{k+r}$, we have

$$\begin{aligned}\frac{\langle \mathcal{F}_{h,g}, \mathcal{F}_{h,g}\rangle}{\langle g, g\rangle} =& 2^{-3r^2+3r-6rk-2k+4}\pi^{-r^2+r-4rk-2k}\frac{\Gamma_r(k+\frac{r-1}{2})^2}{\Gamma_r(k+r)^2} \\ &\times \Gamma(k+r)^2\prod_{i=1}^{r}\Gamma(2k+2r-2i)^2 L(k,g,\mathrm{st})^2.\end{aligned}$$

Now, the next lemma proves Proposition 6.1. □

Lemma 6.1.

$$\frac{\Gamma_r(s+\frac{r-1}{2})}{\Gamma_r(s+r)} = 2^{(r^2-r)/2}\frac{\Gamma(s)}{\Gamma(s+r)}\prod_{i=1}^{r}\frac{\Gamma(2s+i-1)}{\Gamma(2s+2r-2i)}.$$

Proof. Put

$$A_r(s) = 2^{(r-r^2)/2}\frac{\Gamma_r(s+\frac{r-1}{2})\Gamma(s+r)}{\Gamma_r(s+r)\Gamma(s)}\prod_{i=1}^{r}\frac{\Gamma(2s+2r-2i)}{\Gamma(2s+i-1)}.$$

Then obviously $A_1(s) = 1$.

$$\begin{aligned}\frac{A_{r+1}(s)}{A_r(s)} =& 2^{-r}\frac{\Gamma(s+\frac{r}{2}+\frac{1}{2})\Gamma(s+\frac{r}{2})}{\Gamma(s+r+1)\Gamma(s+r+\frac{1}{2})}\frac{\Gamma(s+r+1)}{\Gamma(s+r)}\frac{\Gamma(2s+2r)}{\Gamma(2s+r)} \\ =& 2^{-r}\frac{\Gamma(s+\frac{r}{2}+\frac{1}{2})\Gamma(s+\frac{r}{2})}{\Gamma(s+r+\frac{1}{2})\Gamma(s+r)}\frac{\Gamma(2s+2r)}{\Gamma(2s+r)}\end{aligned}$$

By the duplication formula for the gamma function, we have

$$\begin{aligned}\Gamma(s+\frac{r}{2}+\frac{1}{2})\Gamma(s+\frac{r}{2}) =& \sqrt{\pi}2^{1-r-2s}\Gamma(2s+r), \\ \Gamma(s+r+\frac{1}{2})\Gamma(s+r) =& \sqrt{\pi}2^{1-2r-2s}\Gamma(2s+2r).\end{aligned}$$

Hence $A_{r+1}(s) = A_r(s)$. □

We restate Proposition 6.1 in the following form.

Proposition 6.2. *Assume that $k+r \equiv 2 \bmod 2$ and $g \in S_{k+r}(\mathrm{Sp}_r(\mathbb{Z}))$. Then*

$$\left|\frac{\langle E^{(2r)}_{k+r}|_{\mathfrak{h}_r\times\mathfrak{h}_r}, g^c \times g\rangle}{\langle g, g\rangle}\right| = 2^{-(r^2-r+2kr-2)/2}|\mathcal{A}_{r,k}|^{-1}\tilde{\Lambda}(k,g,\mathrm{st}).$$

Here $\tilde{\Lambda}(s,g,\mathrm{st}) = \Gamma_{\mathbb{C}}(s)\prod_{i=1}^{r}\Gamma_{\mathbb{C}}(s+k+r-i)L(s,g,\mathrm{st})$.

7. Theta functions associated with Niemeier lattices

In this section, we write $M_k^{(n)} = M_k(\mathrm{Sp}_n(\mathbb{Z}))$ and $S_k^{(n)} = S_k(\mathrm{Sp}_n(\mathbb{Z}))$, for simplicity.

We recall the results of [30]. A Niemeier lattice is a positive definite even unimodular lattice of degree 24. The number of isomorphism classes of Niemeier lattices is 24. Let L_i $(1 \le i \le 24)$ be Niemeier lattices, not isomorphic to each other.

Let V be the vector space with basis $\{[L_i] \,|\, 1 \le i \le 24\}$, where $[L_i]$ is the isomorphism class of L_i.

The theta function of degree n associated with L_i is denoted by $\Theta_{L_i}^{(n)}(Z) \in M_{12}^{(n)}$. By extending linearly, we obtain a linear map

$$\Theta^{(n)} : V \longrightarrow M_{12}^{(n)}$$
$$\sum_i c_i[L_i] \mapsto \sum_i c_i \Theta_{L_i}^{(n)}(Z).$$

Let $V_n = \mathrm{Ker}(\Theta^{(n)})$. Then $\Theta^{(12)}$ is injective (cf. [13], [5]). If $n'+n''=n$, then the restriction of $\Theta_{L_i}^{(n)}(Z)$ to $\mathfrak{h}_{n'} \times \mathfrak{h}_{n''}$ is given by

$$\Theta_{L_i}^{(n)}\left(\begin{pmatrix} Z' & 0 \\ 0 & Z'' \end{pmatrix}\right) = \Theta_{L_i}^{(n')}(Z')\Theta_{L_i}^{(n'')}(Z'').$$

As an element of V, we put $\mathtt{e}_i = [L_i]$. Following Nebe and Venkov, we define the Hermitian inner product $\mathtt{(\ ,\ )}$ on V by

$$\mathtt{(}\mathtt{e}_i, \mathtt{e}_j\mathtt{)} = \begin{cases} (\#\mathrm{Aut}(L_i)), & i = j, \\ 0, & i \ne j, \end{cases}$$

and a multiplication on V by

$$\mathtt{e}_i \circ \mathtt{e}_j = \begin{cases} (\#\mathrm{Aut}(L_i))\mathtt{e}_i, & i = j \\ 0, & i \ne j. \end{cases}$$

Nebe and Venkov defined Hecke operators $K_{p,i}$, $(1 \le i \le 12)$ and $T(p)$ acting on V and calculated Hecke eigenvectors $\mathtt{d}_1, \mathtt{d}_2, \dots, \mathtt{d}_{24}$.

We put

$$\mathtt{d}_i = \sum_j c_{ij}\mathtt{e}_j,$$
$$\mathtt{e}_i = \sum_j b_{ij}\mathtt{d}_j.$$

A table of coefficients c_{ij} $(i, j = 1, 2, \dots, 24)$ can be found in [29]. Note that $c_{ij}, b_{ij} \in \mathbb{Q}$. As both $\{\mathtt{e}_1, \mathtt{e}_2, \dots, \mathtt{e}_{24}\}$ and $\{\mathtt{d}_1, \mathtt{d}_2, \dots, \mathtt{d}_{24}\}$

are orthogonal basis of V, we have

$$b_{ij} = (\mathtt{e}_i, \mathtt{e}_i)\,\overline{c_{ji}}\,(\mathtt{d}_j, \mathtt{d}_j)^{-1} = (\#\mathrm{Aut}(L_i))\,(\mathtt{d}_j, \mathtt{d}_j)^{-1}\,c_{ji}.$$

Nebe and Venkov showed that the degree n_i of $\mathtt{d}_i$ is as follows:

n_1	n_2	n_3	n_4	n_5	n_6	n_7	n_8	n_9	n_{10}	n_{11}	n_{12}
0	1	2	3	4	4	5	5	6	6	6	7

n_{13}	n_{14}	n_{15}	n_{16}	n_{17}	n_{18}	n_{19}	n_{20}	n_{21}	n_{22}	n_{23}	n_{24}
8	7	8	7	8	8	–	9	–	10	11	12

For the definition of the degree, see [30]. Note that they have shown that $n_i = \min\{n \mid \Theta^{(n)}(\mathtt{d}_i) \neq 0\}$ in this case (See [30], Lemma 2.5). As for n_{19} and n_{21}, they have shown that $7 \leq n_{19} \leq 9$, $8 \leq n_{21} \leq 10$, but we do not use $\mathtt{d}_{19}$ or $\mathtt{d}_{21}$.

Note that the Petersson inner product $\langle \Theta^{(n_i)}(\mathtt{d}_i), \Theta^{(n_i)}(\mathtt{d}_j)\rangle$ vanishes for $i \neq j$, since the Hecke eigenvalues are different. We put $F_i = \Theta^{(n_i)}(\mathtt{d}_i) \in S_{12}^{(n_i)}$. Note that $F_i^c = F_i$ for $i = 1, 2, \ldots, 24$.

Lemma 7.1. *Let $\mathtt{d}_i$, $\mathtt{d}_j$, and $\mathtt{d}_k$ be Hecke eigenvectors of V. Then we have*

$$\langle \Theta^{(n_i+n_j)}(\mathtt{d}_k)|_{\mathfrak{h}_{n_i}\times\mathfrak{h}_{n_j}}, F_i \times F_j\rangle = \frac{\langle F_i, F_i\rangle\,\langle F_j, F_j\rangle}{(\mathtt{d}_i, \mathtt{d}_i)\;(\mathtt{d}_j, \mathtt{d}_j)}(\mathtt{d}_k, \mathtt{d}_i \circ \mathtt{d}_j).$$

In particular, $(\mathtt{d}_k, \mathtt{d}_i \circ \mathtt{d}_j) \neq 0$ if and only if the left hand side is not zero.

Proof. The left hand side is equal to

$$\begin{aligned}
&\sum_{m=1}^{24} c_{km}\langle \Theta_{L_m}^{(n_i+n_j)}|_{\mathfrak{h}_{n_i}\times\mathfrak{h}_{n_j}}, \Theta^{(n_i)}(\mathtt{d}_i) \times \Theta^{(n_j)}(\mathtt{d}_j)\rangle \\
&= \sum_{m=1}^{24} c_{km}\langle \Theta_{L_m}^{(n_i)}, \Theta^{(n_i)}(\mathtt{d}_i)\rangle\;\langle \Theta_{L_m}^{(n_j)}, \Theta^{(n_j)}(\mathtt{d}_j)\rangle \\
&= \sum_{m=1}^{24} c_{km}\langle \sum_{l=1}^{24} b_{ml}\Theta^{(n_i)}(\mathtt{d}_l), \Theta^{(n_i)}(\mathtt{d}_i)\rangle\langle \sum_{l=1}^{24} b_{ml}\Theta^{(n_j)}(\mathtt{d}_l), \Theta^{(n_j)}(\mathtt{d}_j)\rangle \\
&= \langle F_i, F_i\rangle\,\langle F_j, F_j\rangle \sum_{m=1}^{24} c_{km} b_{mi} b_{mj} \\
&= \frac{\langle F_i, F_i\rangle\,\langle F_j, F_j\rangle}{(\mathtt{d}_i, \mathtt{d}_i)\;(\mathtt{d}_j, \mathtt{d}_j)} \sum_{m=1}^{24} (\#\mathrm{Aut}(L_m))^2 c_{km} c_{im} c_{jm}.
\end{aligned}$$

On the other hand,

$$
\begin{aligned}
(\mathtt{d}_k, \mathtt{d}_i \circ \mathtt{d}_j) &= (\mathtt{d}_k, \left(\sum_{m=1}^{24} c_{im} \mathtt{e}_m\right) \circ \left(\sum_{l=1}^{24} c_{jl} \mathtt{e}_m\right)) \\
&= (\mathtt{d}_k, \sum_{m=1}^{24} (\#\mathrm{Aut}(L_m))\, c_{im} c_{jm} \mathtt{e}_m) \\
&= \sum_{m=1}^{24} (\#\mathrm{Aut}(L_m))\, c_{im} c_{jm} (\mathtt{d}_k, \mathtt{e}_m) \\
&= \sum_{m=1}^{24} (\#\mathrm{Aut}(L_m))^2\, c_{im} c_{jm} c_{km}.
\end{aligned}
$$

Hence the lemma. □

Nebe and Venkov [30] claimed that $F_{11} \in S_{12}^{(6)}$, $F_{13} \in S_{12}^{(8)}$, and $F_{24} \in S_{12}^{(12)}$ are the Duke-Imamoglu lift of $\phi_{18} \in S_{18}^{(1)}$, $\phi_{16} \in S_{16}^{(1)}$, and $\Delta \in S_{12}^{(1)}$, respectively. In fact this is easily verified by comparing the eigenvalue of $T(2)$ (See [29]). Nebe and Venkov [30] have shown that $(\mathtt{d}_{24}, \mathtt{d}_i \circ \mathtt{d}_j) \neq 0$ for

$$(i, j) = (2, 23), (3, 22), (4, 20), (5, 17), (6, 18), (7, 14), (8, 16).$$

Proposition 3.1 implies that F_j is the Miyawaki lift of F_i with respect to $F_{24} \in S_{12}^{(12)}$. Similarly, using the structure constants found in [29], one can prove that $F_8 \in S_{12}^{(5)}$ and $F_6 \in S_{12}^{(4)}$ are Miyawaki lift of $F_2 \in S_{12}^{(1)}$ and $F_3 \in S_{12}^{(2)}$, respectively. One can also prove that $F_{12} \in S_{12}^{(7)}$, $F_9 \in S_{12}^{(6)}$, and $F_7 \in S_{12}^{(5)}$ are the Miyawaki lift of $F_2 \in S_{12}^{(1)}$, $F_3 \in S_{12}^{(2)}$, and $F_4 \in S_{12}^{(3)}$ with respect to $F_{13} \in S_{12}^{(8)}$, respectively. We summarize these as Table A and Table B.

8. Appendix

We briefly explain how to calculate both sides of (C) by computers. For the calculation of various L-values, we have used a very useful program due to Dokchitser [10]. The Petersson norm $\langle f, f\rangle$ can be easily computed by $\tilde{\Lambda}(1, f, \mathrm{Ad}) = 2^{2k}\langle f, f\rangle$. Similarly, $\langle h, h\rangle$ can be computed by Kohnen-Zagier formula (KZ). The Petersson norm of g or G can be computed by Proposition 6.2 and Lemma 7.1. Finally, $\langle F|_{\mathfrak{h}_r \times \mathfrak{h}_{r+2n}}, g \times G\rangle$ is computed by Lemma 7.1. Note that the structure constants $(\mathtt{d}_k, \mathtt{d}_i \circ \mathtt{d}_j)$ are already computed by Nebe [29].

We discuss the case $f = \phi_{20} \in S_{20}^{(1)}$, $g = \Delta \in S_{12}^{(1)}$, and $G \in S_{12}^{(3)}$. We put

$$\begin{aligned}
\mathtt{d}_1' =& \mathtt{d}_1/1027637932586061520960267,\\
\mathtt{d}_2' =& -\mathtt{d}_2/8104867379578640543040,\\
\mathtt{d}_4' =& \mathtt{d}_4/846305351287603200,\\
\mathtt{d}_5' =& -\mathtt{d}_5/212694241858560.
\end{aligned}$$

We give a table of coefficients of $\mathtt{d}_2$, $\mathtt{d}_4$, and $\mathtt{d}_5$ below (See Nebe [29]). The coefficients of $\mathtt{d}_1$ can be found in [29] or [8], p. 413. Then $E_{12}^{(2r)} = \Theta^{(2r)}(\mathtt{d}_1')$, $F_2' = \Theta^{(1)}(\mathtt{d}_2') = \Delta \in S_{12}^{(1)}$, and $F_4' = \Theta^{(3)}(\mathtt{d}_4') \in S_{12}^{(3)}$ is the Miyawaki's cusp form [27]. Put $h = q - 56q^4 + 360q^5 - 13680q^8 + \cdots \in S_{21/2}^{+}(\Gamma(4))$. Then $F_5' = \Theta^{(4)}(\mathtt{d}_5') \in S_{12}^{(4)}$ is the Duke-Imamoglu lift of $h(\tau)$ to degree 4.

	$\mathtt{d}_2$	$\mathtt{d}_4$	$\mathtt{d}_5$
Leech	21625795628236800	-1992646656000	214592716800
A_1^{24}	21618140012108640000	-462916726272000	22783711104000
A_2^{12}	104595874904801280000	385220419584000	-56204746752000
A_3^{8}	-7569380452233600000	865252948560000	22644338640000
A_4^{6}	-66640754260236828672	-625041225768960	21173267275776
$A_5^4D_4$	-37660962656647249920	-318497556529152	2319747268608
D_4^{6}	-861991027602705000	-7289830548000	4817683332000
A_6^{4}	-8962553548174786560	25632591249408	-23357975494656
$A_7^2D_5^2$	-3844278424500433920	89124325640064	6074130446208
A_8^{3}	-400803255218995200	20932199608320	-1962418360320
$A_9^2D_6$	-226886348300451840	20394416373760	168373460992
D_6^{4}	-40713248535359400	3659642586600	716314247880
$A_{11}D_7E_6$	-22871209751470080	4366739579904	500824507392
E_6^{4}	-1056891465710080	201789491904	52888473792
A_{12}^{2}	-2655635220725760	675250266112	11615002624
D_8^{3}	-554584334604300	180878892480	32784927120
$A_{15}D_9$	-141086166819840	69909993856	8326316416
$D_{10}E_7^2$	-20420264058480	14273509536	4257598752
$A_{17}E_7$	-17203085475840	12024741888	2130518016
D_{12}^{2}	-426847644405	515734934	139737422
A_{24}	-30884364288	51875840	11128832
$D_{16}E_8$	-2482214625	6542775	2974851
E_8^{3}	-584290850	1540110	927894
D_{24}	-367740	2621	1601

We need the following computer calculations.

$$\begin{aligned}
(\mathtt{d}_2', \mathtt{d}_2') =& 2^{31} \cdot 3^{10} \cdot 5^4 \cdot 7 \cdot 11^2 \cdot 13 \cdot 17 \cdot 19 \cdot 23 \cdot 283^{-1} \cdot 617^{-1} \cdot 3617^{-1} \cdot 43867^{-1},\\
(\mathtt{d}_4', \mathtt{d}_4') =& 2^{16} \cdot 3^{-1} \cdot 5^5 \cdot 7 \cdot 11 \cdot 13 \cdot 283 \cdot 617 \cdot 691^{-1} \cdot 3617^{-1},\\
(\mathtt{d}_1', \mathtt{d}_4' \circ \mathtt{d}_4') =& \frac{2^{61} \cdot 3^{16} \cdot 5^{12} \cdot 7^5 \cdot 11^3 \cdot 13^3 \cdot 17 \cdot 19 \cdot 23}{131 \cdot 593 \cdot 691^3 \cdot 3617^2 \cdot 43867},\\
(\mathtt{d}_5', \mathtt{d}_2' \circ \mathtt{d}_4') =& -2^{54} \cdot 3^{12} \cdot 5^{10} \cdot 7^2 \cdot 11^3 \cdot 13^2 \cdot 17 \cdot 19 \cdot 23 \cdot 691^{-1} \cdot 3617^{-2} \cdot 43867^{-1}.
\end{aligned}$$

$$
\begin{aligned}
\langle\Delta,\Delta\rangle =&0.000001035362056804320922347816812225164593224907 \ \cdots \\
\langle\phi_{20},\phi_{20}\rangle =&0.000008265541531659703164230062760258225715343908 \ \cdots \\
\frac{\langle\Delta,\Delta\rangle}{\langle h,h\rangle} =&0.098872279065281741186752369945336382997115288715 \ \cdots \\
\tilde{\Lambda}(9,\Delta,\mathrm{Ad}) =&0.139584317666868979132086560789461824236408711579 \ \cdots \\
&\doteqdot 2^{19}\cdot 3^2\cdot 5^{-1}\cdot 7^{-1}\langle\Delta,\Delta\rangle, \\
\Lambda(18,\phi_{20})\Lambda(19,\phi_{20}) =&2^{23}\cdot 3^4\cdot 7^2\cdot 17\cdot 283^{-1}\cdot 617^{-1}\langle\phi_{20},\phi_{20}\rangle, \\
\Lambda(11,\mathrm{Ad}(\Delta)\boxtimes\phi_{20}) =&0.000000033447080614408498864020192110373963031495 \ \cdots \\
&\doteqdot 2^{24}\cdot 3^2\cdot 5^2\langle\Delta,\Delta\rangle^2\langle\phi_{20},\phi_{20}\rangle\langle h,h\rangle^{-1}.
\end{aligned}
$$

We can now calculate the Petersson norm $\langle F_4', F_4'\rangle$. By Proposition 6.2 and Lemma 7.1, we have

$$
\begin{aligned}
\langle F_4', F_4'\rangle =&2^{-29}\frac{(\mathtt{d}_4',\mathtt{d}_4')^2}{(\mathtt{d}_1',\mathtt{d}_4'\circ\mathtt{d}_4')}|\mathcal{A}_{3,9}|^{-1}\tilde{\Lambda}(9,\Delta,\mathrm{Ad})\Lambda(18,\phi_{20})\Lambda(19,\phi_{20}) \\
&\doteqdot 2^{-6}\cdot 3^{-5}\langle\phi_{20},\phi_{20}\rangle\langle\Delta,\Delta\rangle.
\end{aligned}
$$

Here, $\mathcal{A}_{3,9}=\zeta(-11)\zeta(-21)\zeta(-19)\zeta(-17)$. By Lemma 7.1, we have

$$
\begin{aligned}
\frac{\langle F_5'|_{\mathfrak{h}_1\times\mathfrak{h}_3}, F_2'\times F_4'\rangle^2}{\langle F_2',F_2'\rangle\langle F_4',F_4'\rangle} &= \langle F_2',F_2'\rangle\langle F_4',F_4'\rangle\left(\frac{(\mathtt{d}_5',\mathtt{d}_2'\circ\mathtt{d}_4')}{(\mathtt{d}_2',\mathtt{d}_2')\ (\mathtt{d}_4',\mathtt{d}_4')}\right)^2 \\
&\doteqdot 2^8\cdot 3\cdot 5^2\langle\Delta,\Delta\rangle^2\langle\phi_{20},\phi_{20}\rangle.
\end{aligned}
$$

On the other hand, we have

$$
\Lambda(11,\mathrm{st}(g)\boxtimes f)\tilde{\Lambda}(1,f,\mathrm{Ad})\tilde{\xi}(2)\doteqdot 2^{42}\cdot 3\cdot 5^2\langle\Delta,\Delta\rangle^3\langle\phi_{20},\phi_{20}\rangle^2\langle h,h\rangle^{-1}
$$

Hence the equation (C) holds approximately in this case with $\alpha=34$. Other examples are shown in Table C.

We give another example $n=k=6$, $r=0$, $g=1$, $f=\Delta$, and $F=G=F_{24}$. Then by computer calculation,

$$
\Lambda(12,\mathrm{st}(g)\boxtimes f)\prod_{i=1}^{6}\tilde{\Lambda}(2i-1,f,\mathrm{Ad})\tilde{\xi}(2i)\doteqdot\frac{2^{73}\langle\Delta,\Delta\rangle^6\Lambda(12,\Delta)}{3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 23}.
$$

On the other hand, using Böcherer's result [3], one can show

$$
\frac{\langle f,f\rangle}{\langle h,h\rangle}\langle F,F\rangle=\frac{\langle\Delta,\Delta\rangle^6\Lambda(12,\Delta)}{2^5\cdot 3^3\cdot 5^2\cdot 7^2\cdot 11\cdot 13\cdot 23}.
$$

Therefore it seems (C) holds in this case as well. Notice that the assumption $k>n$ is not satisfied in this case and that $\Lambda(12,\Delta)$ is not a critical value in the sense of Deligne [9].

• Tabe A: Standard L-functions

$$
\begin{aligned}
L(s,F_3,\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11} L(s+i,\phi_{22}),\\
L(s,F_4,\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10} L(s+i,\phi_{20}),\\
L(s,F_5,\mathrm{st}) =&\zeta(s)\prod_{8\le i\le 11} L(s+i,\phi_{20}),\\
L(s,F_6,\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11} L(s+i,\phi_{22})\prod_{8\le i\le 9} L(s+i,\phi_{18}),\\
L(s,F_7,\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10} L(s+i,\phi_{20})\prod_{7\le i\le 8} L(s+i,\phi_{16}),\\
L(s,F_8,\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{7\le i\le 10} L(s+i,\phi_{18}),\\
L(s,F_9,\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11} L(s+i,\phi_{22})\prod_{6\le i\le 9} L(s+i,\phi_{16}),\\
L(s,F_{11},\mathrm{st}) =&\zeta(s)\prod_{6\le i\le 11} L(s+i,\phi_{18}),\\
L(s,F_{12},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{5\le i\le 10} L(s+i,\phi_{16}),\\
L(s,F_{14},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10} L(s+i,\phi_{20})\prod_{7\le i\le 8} L(s+i,\phi_{16})\prod_{5\le i\le 6} L(s+i,\Delta),\\
L(s,F_{16},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{7\le i\le 10} L(s+i,\phi_{18})\prod_{5\le i\le 6} L(s+i,\Delta),\\
L(s,F_{13},\mathrm{st}) =&\zeta(s)\prod_{4\le i\le 11} L(s+i,\phi_{16}),\\
L(s,F_{17},\mathrm{st}) =&\zeta(s)\prod_{8\le i\le 11} L(s+i,\phi_{20})\prod_{4\le i\le 7} L(s+i,\Delta),\\
L(s,F_{18},\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11} L(s+i,\phi_{22})\prod_{8\le i\le 9} L(s+i,\phi_{18})\prod_{4\le i\le 7} L(s+i,\Delta),\\
L(s,F_{20},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{9\le i\le 10} L(s+i,\phi_{20})\prod_{3\le i\le 8} L(s+i,\Delta),\\
L(s,F_{22},\mathrm{st}) =&\zeta(s)\prod_{10\le i\le 11} L(s+i,\phi_{22})\prod_{2\le i\le 9} L(s+i,\Delta),\\
L(s,F_{23},\mathrm{st}) =&L(s,\Delta,\mathrm{Ad})\prod_{i=1}^{10} L(s+i,\Delta),\\
L(s,F_{24},\mathrm{st}) =&\zeta(s)\prod_{i=0}^{11} L(s+i,\Delta).
\end{aligned}
$$

• Table B: Liftings

type	form	degree	g	f	F	r	n	k
Duke-Imamoglu	F_3	2		ϕ_{22}				
Miyawaki	F_4	3	Δ	ϕ_{20}	F_5	1	1	10
Duke-Imamoglu	F_5	4		ϕ_{20}				
Miyawaki	F_6	4	F_3	ϕ_{18}	F_{11}	2	1	9
Miyawaki	F_7	5	F_4	ϕ_{16}	F_{13}	3	1	8
Miyawaki	F_8	5	Δ	ϕ_{18}	F_{11}	1	2	9
Miyawaki	F_9	6	F_3	ϕ_{16}	F_{13}	2	2	8
Duke-Imamoglu	F_{11}	6		ϕ_{18}				
Miyawaki	F_{12}	7	Δ	ϕ_{16}	F_{13}	1	3	8
Miyawaki	F_{14}	7	F_7	Δ	F_{24}	5	1	6
Miyawaki	F_{16}	7	F_8	Δ	F_{24}	5	1	6
Duke-Imamoglu	F_{13}	8		ϕ_{16}				
Miyawaki	F_{17}	8	F_5	Δ	F_{24}	4	2	6
Miyawaki	F_{18}	8	F_6	Δ	F_{24}	4	2	6
Miyawaki	F_{20}	9	F_4	Δ	F_{24}	3	3	6
Miyawaki	F_{22}	10	F_3	Δ	F_{24}	2	4	6
Miyawaki	F_{23}	11	Δ	Δ	F_{24}	1	5	6
Duke-Imamoglu	F_{24}	12		Δ				

• Table C: The autor has checked that the equation (C) holds up to at least 30 decimals in the following cases:

G	g	f	F	r	n	k	α
Δ	Δ	ϕ_{22}	F_3	1	0	11	12
F_3	F_3	ϕ_{20}	F_5	2	0	10	35
F_4	F_4	ϕ_{18}	F_{11}	3	0	9	56
F_5	F_5	ϕ_{16}	F_{13}	4	0	8	75
F_6	F_6	ϕ_{16}	F_{13}	4	0	8	75
F_9	F_9	Δ	F_{24}	6	0	6	107
F_{11}	F_{11}	Δ	F_{24}	6	0	6	107
F_3	1	ϕ_{22}	F_3	0	1	11	12
F_4	Δ	ϕ_{20}	F_5	1	1	10	34
F_6	F_3	ϕ_{18}	F_{11}	2	1	9	55
F_7	F_4	ϕ_{16}	F_{13}	3	1	8	74
F_{14}	F_7	Δ	F_{24}	5	1	6	106
F_{16}	F_8	Δ	F_{24}	5	1	6	106
F_5	1	ϕ_{20}	F_5	0	2	10	33
F_8	Δ	ϕ_{18}	F_{11}	1	2	9	53
F_9	F_3	ϕ_{16}	F_{13}	2	2	8	72
F_{17}	F_5	Δ	F_{24}	4	2	6	104
F_{18}	F_6	Δ	F_{24}	4	2	6	104
F_{11}	1	ϕ_{18}	F_{11}	0	3	9	50
F_{12}	Δ	ϕ_{16}	F_{13}	1	3	8	68
F_{20}	F_4	Δ	F_{24}	3	3	6	100
F_{13}	1	ϕ_{16}	F_{13}	0	4	8	63
F_{22}	F_3	Δ	F_{24}	2	4	6	94
F_{23}	Δ	Δ	F_{24}	1	5	6	86

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