
On liftings of holomorphic modular forms

Tamotsu Ikeda

Graduate school of mathematics, Kyoto University, Kitashirakawa, Kyoto, 606-8502, Japan

Introduction

In this article, we discuss a lifting of elliptic cusp forms to higher dimensional symmetric spaces. We will consider two cases. The first case is the Siegel modular case [18], and the second case is the hermitian modular case [19]. The Fourier coefficients of our liftngs are closely related to those of Eisenstein series. When the degree is 2, our lifting reduces to the classical Saito-Kurokawa lifting or hermitian Maass lifting.

The finite part of the automorphic representation generated by this lifting is isomorphic to a degenerate principal series. In particular, this is a non-tempered representation.

Part I : Siegel modular case.

1 Basic facts

We recall basic facts about Siegel modular forms. The Siegel upper half space $\mathfrak{H}_n$ of degree n is defined by

$$\mathfrak{H}_n = \{Z = {}^tZ \in \mathrm{M}_n(\mathbb{C}) \,|\, \mathrm{Im}(Z) > 0\}.$$

Here, $\mathrm{Im}(Z) > 0$ means that $\mathrm{Im}(Z)$ is positive definite. Note that $\mathfrak{H}_1 = \{\tau \in \mathbb{C} \,|\, \mathrm{Im}(\tau) > 0\}$ is equal to the upper half plane. The symplectic group

$$\begin{aligned}\mathrm{Sp}_n(\mathbb{R}) &= \left\{ g \in \mathrm{SL}_{2n}(\mathbb{R}) \,\middle|\, g \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} {}^tg = \begin{pmatrix} \mathbf{0}_n & -\mathbf{1}_n \\ \mathbf{1}_n & \mathbf{0}_n \end{pmatrix} \right\} \\ &= \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SL}_{2n}(\mathbb{R}) \,\middle|\, A\,{}^tB = B\,{}^tA,\ C\,{}^tD = D\,{}^tC,\ A\,{}^tD - B\,{}^tC = \mathbf{1}_n \right\}\end{aligned}$$

acts on $\mathfrak{H}_n$ by $\begin{pmatrix} A & B \\ C & D \end{pmatrix}(Z) = (AZ+B)(CZ+D)^{-1}$. Put

$$\begin{aligned}\mathcal{S}'_n(\mathbb{Z}) &= \text{the set of half-integral symmetric matrices} \\ &= \{B \in \tfrac{1}{2}\mathrm{M}_{2n}(\mathbb{Z}) \,|\, B = {}^tB, B_{ii} \in \mathbb{Z}\ (1 \le i \le 2n)\}, \\ \mathcal{S}'_n(\mathbb{Z})^+ &= \{B \in \mathcal{S}_{2n}(\mathbb{Z}) \,|\, B > 0\}\end{aligned}$$

A holomorphic function F on $\mathfrak{H}_n$ is called a Siegel modular form of weight l if

$$F((AZ+B)(CZ+D)^{-1}) = F(Z)\det(CZ+D)^l$$

for any $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_n(\mathbb{Z})$. When $n=1$, we need to impose that F has a Fourier expansion

$$F(Z) = \sum_{N=0}^{\infty} a_F(N)\exp(2\pi\sqrt{-1}NZ), \qquad Z \in \mathfrak{H}_1.$$

When $n \geq 2$, a Siegel modular form F automatically has a Fourier expansion

$$F(Z) = \sum_{\substack{B\in\mathcal{S}_n(\mathbb{Z}) \\ B\geq 0}} A_F(B)\mathbf{e}(BZ).$$

Here, $\mathbf{e}(X) = \exp(2\pi\sqrt{-1}\mathrm{tr}(X))$. The complex number $A_F(B)$ is called the B-th Fourier coefficient of F. A Siegel modular form F of degree n is called a cusp form if $A_F(B)=0$ unless $B \in \mathcal{S}_n(\mathbb{Z})^+$. The space of Siegel modular (resp. cusp) forms of degree n and weight l is demoted by $M_l(\mathrm{Sp}_n(\mathbb{Z}))$ (resp. $S_l(\mathrm{Sp}_n(\mathbb{Z}))$).

2 Fourier coefficients of the Eisenstein series

Now we assume $k \equiv n \bmod 2$ and $k > n+1$. Put

$$\Gamma_\infty^{(2n)} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_{2n}(\mathbb{Z}), C=0 \right\}.$$

The Siegel Eisenstein series on $\mathfrak{H}_{2n}$ of weight $k+n$ is defined by

$$E_{k+n}^{(2n)}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right)\in\Gamma_\infty^{(2n)}\backslash\mathrm{Sp}_{2n}(\mathbb{Z})} \det(CZ+D)^{-k-n}.$$

As we have assumed $k > n+1$, $E_{k+n}^{(2n)}$ is absolutely convergent. Moreover, $E_{k+n}^{(2n)}$ is a Siegel modular form of weight $k+n$. We define the normalized Eisenstein series by

$$\mathcal{E}_{k+n}^{(2n)}(Z) = 2^{-n}\zeta(1-k-n)\prod_{i=1}^{n}\zeta(1+2i-2k-2n)\cdot E_{k+n}^{(2n)}(Z).$$

For $B \in \mathcal{S}_{2n}(\mathbb{Z})^+$, we put $D_B = \det(2B)$. The absolute value of the discriminant of $\mathbb{Q}(\sqrt{(-1)^n D_B})$ is denoted by $\mathfrak{d}_B$. Put $\mathfrak{f}_B = \sqrt{D_B/\mathfrak{d}_B}$. Let χ_B be the primitive Dirichlet character modulo $\mathfrak{d}_B$ corresponding to $\mathbb{Q}(\sqrt{(-1)^n D_B})/\mathbb{Q}$.

For each prime p, let $\mathbf{e}_p : \mathbb{Q}_p \to \mathbb{C}^\times$ be the additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \mathbf{e}(-x)$ for any $x \in \mathbb{Z}[1/p]$.

Recall that the Siegel series for $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$ is defined by

$$b_p(B,s) = \sum_{R\in\mathrm{Sym}_{2n}(\mathbb{Q}_p)/\mathrm{Sym}_{2n}(\mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(BR))p^{-\mathrm{ord}_p(\nu(R))s},$$

where

$$\begin{aligned}\mathrm{Sym}_{2n}(\mathbb{Q}_p) &= \{R = {}^tR \mid R \in \mathrm{M}_{2n}(\mathbb{Q}_p)\},\\ \mathrm{Sym}_{2n}(\mathbb{Z}_p) &= \{R = {}^tR \mid R \in \mathrm{M}_{2n}(\mathbb{Z}_p)\},\\ \nu(R) &= [R\mathbb{Z}_p^{2n} + \mathbb{Z}_p^{2n} : \mathbb{Z}_p^{2n}].\end{aligned}$$

Put

$$\gamma_p(B; X) = (1 - X)(1 - p^n\chi_B(p)X)^{-1}\prod_{i=1}^{n}(1 - p^{2i}X^2).$$

Then there exists a polynomial $F_p(B; X) \in \mathbb{Z}[X]$ such that

$$b_p(B, s) = \gamma_p(B; p^{-s})F_p(B; p^{-s}).$$

Katsurada [20] proved the following functional equation

$$F_p(B; p^{-2n-1}X^{-1}) = (p^{2n+1}X^2)^{-\mathrm{ord}_p \mathfrak{f}_B} F_p(B; X).$$

In particular, we have $\deg F_p(B; X) = 2\mathrm{ord}_p \mathfrak{f}_B$.

It is known that for $B \in \mathcal{S}'_{2n}(\mathbb{Z})^+$, the B-th Fourier coefficient of $\mathcal{E}^{(2n)}_{k+n}(Z)$ is equal to

$$L(1 - k, \chi_B)\mathfrak{f}_B^{2k-1}\prod_{p \mid D_B} F_p(B; p^{-k-n}).$$

Put $\tilde{F}_p(B; X) = X^{-\mathrm{ord}_p \mathfrak{f}_B} F_p(B; p^{-n-(1/2)}X)$. Then Katsurada's functional equation implies

$$\tilde{F}_p(B; X^{-1}) = \tilde{F}_p(B; X).$$

In terms of $\tilde{F}_p(B; X)$, the B-th Fourier coefficient of $\mathcal{E}^{(2n)}_{k+n}(Z)$ can be expressed as

$$L(1 - k, \chi_B)\mathfrak{f}_B^{k-(1/2)}\prod_{p \mid \mathfrak{f}_B} \tilde{F}_p(B; p^{-k+(1/2)}).$$

3 Kohnen plus space

Let $\mathfrak{G}$ the group which consists of all pairs $(\gamma, \phi(\tau))$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{R})$ and $\phi(\tau)$ is a holomorphic function on $\mathfrak{H}_1$ satisfying $|\phi(\tau)| = |c\tau + d|$, with group law defined by $(\gamma_1, \phi_1(\tau)) \cdot (\gamma_2, \phi_2(\tau)) = (\gamma_1\gamma_2, \phi_1(\gamma_2(\tau))\phi_2(\tau))$. If $h(\tau)$ is a function on $\mathfrak{H}_1$ and $\xi = (\gamma, \phi(\tau)) \in \mathfrak{G}$, we put

$$(h|\xi)(\tau) = \left(h|_{k+(1/2)}\xi\right)(\tau) = \phi(\tau)^{-2k-1}h(\gamma(\tau)).$$

Then $(h|\xi_1)|\xi_2 = h|(\xi_1\xi_2)$, for $\xi_1, \xi_2 \in \mathfrak{G}$. On the other hand, for $\gamma \in \mathrm{SL}_2(\mathbb{R})$, we put

$$\left(h||_{k+(1/2)}\gamma\right)(\tau) = (c\tau + d)^{-(2k+1)/2}h(\gamma(\tau)).$$

Then for $\gamma_1, \gamma_2 \in \mathrm{SL}_2(\mathbb{R})$, we have $(h||\gamma_1)||\gamma_2 = t \cdot h||(\gamma_1\gamma_2)$, for some $t \in \mathbb{C}$, $|t| = 1$.

There exists an injective homomorphism $\Gamma_0(4) \to \mathfrak{G}$ given by $\gamma \mapsto \gamma^* = (\gamma, j(\gamma, \tau))$, where

$$j(\gamma, \tau) = \left(\frac{c}{d}\right) \epsilon_d^{-1} (c\tau + d)^{1/2}, \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4).$$

Here,

$$\epsilon_d = \begin{cases} 1, & \text{if } d \equiv 1 \bmod 4, \\ \sqrt{-1}, & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

Recall that $M_{k+(1/2)}(\Gamma_0(4))$ (resp. $S_{k+(1/2)}(\Gamma_0(4))$) consists of all holomorphic functions $h(\tau)$ on $\mathfrak{H}_1$ which satisfy $h|_{k+(1/2)}\gamma^* = h$ for every $\gamma \in \Gamma_0(4)$ and which are holomorphic (resp. which vanish) at all cusps. The Kohnen plus space $M^+_{k+(1/2)}(\Gamma_0(4))$ consists of all $h(\tau) \in M_{k+(1/2)}(\Gamma_0(4))$ whose Fourier expansion is of the form

$$h(\tau) = \sum_{\substack{N \geq 0 \\ (-1)^k N \equiv 0,1 (4)}} c(N) q^N, \quad q = \exp(2\pi\sqrt{-1}\tau).$$

Similarly, the Kohnen plus space $S^+_{k+(1/2)}(\Gamma_0(4))$ is defined by

$$S^+_{k+(1/2)}(\Gamma_0(4)) = S_{k+(1/2)}(\Gamma_0(4)) \cap M^+_{k+(1/2)}(\Gamma_0(4)).$$

Kohnen [24] proved that $M^+_{k+(1/2)}(\Gamma_0(4))$ has a basis that consists of Hecke eigenforms, and that the Shimura correspondence is one to one, i.e., there is a one-to-one correspondence between Hecke eigenforms $f(\tau) \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ and Hecke eigenforms in $h(\tau) \in M^+_{k+(1/2)}(\Gamma_0(4))$, up to scalar multiplication. The form $f(\tau) \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is a cusp form if and only if the corresponding $h(\tau) \in M^+_{k+(1/2)}(\Gamma_0(4))$ is a cusp form.

Let

$$f(\tau) = \sum_{N=0}^{\infty} a(N) q^N \in M_{2k}(\mathrm{SL}_2(\mathbb{Z})), \quad q = e^{2\pi\sqrt{-1}\tau}$$

be a normalized Hecke eigenform of weight $2k$. Let $\alpha_p^{\pm 1}$ be the Satake parameter of f at p, i.e.,

$$(1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1} X) = 1 - a(p)X + p^{2k-1}X^2.$$

Let $h(\tau) = \sum_{N \geq 0} c(N) q^N \in M^+_{k+(1/2)}(\Gamma_0(4))$ be a non-zero Hecke eigenform. Then $h(\tau)$ corresponds to $f(\tau)$ by the Shimura correspondence if and only if for any fundamental discriminant D such that $(-1)^k D > 0$, we have

$$c(|D|f^2) = c(|D|) \sum_{d|f} \mu(d) \chi_{|D|}(d) d^{k-1} a(f/d).$$

Here $\mu(d)$ is the Möbius function.

For any positive integer N such that $(-1)^k N \equiv 0, 1 \bmod 4$,, we denote the absolute value of the discriminant of $\mathbb{Q}(\sqrt{(-1)^k N})/\mathbb{Q}$ by $\mathfrak{d}_N$ and the positive rational number such that $N = \mathfrak{d}_N \mathfrak{f}_N^2$ by $\mathfrak{f}_N$. Note that $\mathfrak{f}_N$ is an integer. Let χ_N be the primitive Dirichlet character corresponding to $\mathbb{Q}(\sqrt{(-1)^k N})/\mathbb{Q}$.

We define $\Psi_p(N; X_p) \in \mathbb{C}[X, X^{-1}]$ by

$$\Psi_p(N;X) = \frac{X^{e+1} - X^{-e-1}}{X - X^{-1}} - p^{-1/2}\chi_N(p)\frac{X^e - X^{-e}}{X - X^{-1}},$$

Here $e = \mathrm{ord}_p \mathfrak{f}_N$. Note that $\Psi_p(N;X) = 1$ if $\mathrm{ord}_p \mathfrak{f}_N = 0$. In terms of $\Psi_p(N;X)$, we have

$$c(N) = c(\mathfrak{d}_N)\mathfrak{f}_N^{k-(1/2)} \prod_p \Psi_p(N;\alpha_p).$$

4 Lifting of cusp forms

Now we consider cusp forms. Let k be an arbitrary positive integer such that $k \equiv n \bmod 2$.

Choose a normalized Hecke eigenform

$$f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z})), \qquad a(1) = 1$$

and a corresponding Hecke eigenform

$$h(\tau) = \sum_{\substack{N>0 \\ (-1)^k N \equiv 0,\, 1\ (4)}} c(N)q^N \in S^+_{k+(1/2)}(\varGamma_0(4)).$$

Let

$$\begin{aligned} L(s,f) &= \sum_{N=1}^{\infty} a(N)N^{-s} \\ &= \prod_p [(1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1} X)]^{-1} \end{aligned}$$

be the L-function of f. The set $\{\alpha_p, \alpha_p^{-1}\}$ is called the Satake parameter of f.

Put

$$\begin{aligned} A(B) =& c(\mathfrak{d}_B)\mathfrak{f}_B^{k-(1/2)} \prod_p \tilde{F}_p(B;\alpha_p), \quad B \in \mathcal{S}_{2n}(\mathbb{Z})^+ \\ F(Z) =& \sum_{\substack{B \in \mathcal{S}'_{2n}(\mathbb{Z})^+ \\ B = {}^t B > 0}} A(B)\mathbf{e}(BZ), \quad Z \in \mathfrak{H}_{2n} \end{aligned}$$

Note that $\tilde{F}_p(B;\alpha_p)$ does not depend on the choice of α_p by Katsurada's functional equation. Then our first main theorem as as follows.

Theorem 1. *Assume $k \equiv n$ mod 2. Then $F \in S_{k+n}(\mathrm{Sp}_{2n}(\mathbb{Z}))$ and $F \not\equiv 0$. Moreover, F is a Hecke eigenform whose standard L-function is equal to*

$$L(s,F,\mathrm{st}) = \zeta(s) \prod_{i=1}^{2n} L(s+k+n-i, f).$$

5 Outline of the proof

We consider the Foureri-Jacobi expansion

$$F\left(\begin{pmatrix}\omega & z\\ {}^t z & \tau\end{pmatrix}\right)=\sum_{S\in\mathcal{S}_{2n-1}(\mathbb{Z})^+}\sum_{\lambda\in(2S)^{-1}\mathbb{Z}^{2n-1}/\mathbb{Z}^{2n-1}}\theta_{[\lambda]}(S;\tau,z)\mathbf{e}(\mathrm{tr}(S\omega))$$
$$\times\sum_{N\in\mathbb{Z} N-{}^t\lambda S\lambda\geq 0}A_F\left(\begin{pmatrix}S & S\lambda\\ {}^t\lambda S & N\end{pmatrix}\right)\mathbf{e}((N-{}^t\lambda S\lambda)\tau).$$

Here $\theta_{[\lambda]}(S;\tau,z)=\sum_{x\in\mathbb{Z}^{2n-1}}\mathbf{e}({}^t(x+\lambda)S(x+\lambda)\tau+2\,{}^t(x+\lambda)Sz)$. For each $S\in\mathcal{S}_{2n-1}(\mathbb{Z})^+$, $\Delta=2\det S$, one can show that

$$\sum_{\substack{N\in\mathbb{Z}\\ N-{}^t\lambda S\lambda\geq 0}}A_{\mathcal{E}^{(2n)}_{k'+n}}\left(\begin{pmatrix}S & S\lambda\\ {}^t\lambda S & N\end{pmatrix}\right)\mathbf{e}((N-{}^t\lambda S\lambda)\tau)$$
$$=\text{(degenerate terms)}$$
$$+\sum_{N=1}^{\infty}H(k',\mathfrak{d}_N)\mathfrak{f}_N^{k'-(1/2)}\left(\prod_{p|N}\tilde{F}_p\left(\begin{pmatrix}S & S\lambda\\ {}^t\lambda S & {}^t\lambda S\lambda+N/\Delta\end{pmatrix}\right);p^{-k'+(1/2)}\right)q^{N/\Delta}$$

is in the space generated by some translates of $\mathcal{H}_{k+(1/2)}(\tau)$. Note that k' can be arbitrarily large. From this, one can show that

$$\sum_{N=1}^{\infty}c(\mathfrak{d}_N)\mathfrak{f}_N^{k-(1/2)}\left(\prod_{p|N}\tilde{F}_p\left(\begin{pmatrix}S & S\lambda\\ {}^t\lambda S & {}^t\lambda S\lambda+N/\Delta\end{pmatrix}\right);\alpha_p\right)q^{N/\Delta}$$

is in the space generated by some translates of $h(\tau)$, and has the same K-types. It follows that $F(Z)$ is modular with respect to both the Siegel parabolic and with respect to Jacobi parabolic subgroup. Since these two parabolic subgroups generates $\mathrm{Sp}_{2n}(\mathbb{Z})$, we have the desired modularity of $F(Z)$

6 Relation to the Saito-Kurokawa lifts

We shall show that when $n=1$, $F(Z)$ is equal to the Saito-Kurokawa lift of $f(\tau)$. Let k be an odd integer.

Recall that a Siegel modular form $F(Z)=\sum_{B\in\mathcal{S}'_2(\mathbb{Z})}A_F(B)\mathbf{e}(BZ)$ of weight $k+1$ satisfies a Maass relation if there is a function $\beta_F:\mathbb{Z}_{\geq 0}\to\mathbb{C}$ such that

$$A_F(B)=\sum_{\substack{d>0\\ d^{-1}B\in\mathcal{S}'_2(\mathbb{Z})}}d^k\cdot\beta_F\left(\frac{D_B}{d^2}\right).$$

The space of Siegel modular forms of weight $k+1$ which satisfies the Maass relation is called the Maass spezialschar. The Maass spezialschar is canonically isomorphic to the Kornen plus space $M^+_{k+(1/2)}(\varGamma_0(4))$ by

$$\Omega^{SK} : F(Z) = \sum_{B \in \mathcal{S}_2'(\mathbb{Z})} A_F(B)\mathbf{e}(BZ) \mapsto \sum_{\substack{n \geq 0 \\ n \equiv 0,3(4)}} \beta_F(n)\mathbf{e}(n\tau).$$

Put $h(\tau) = \Omega^{SK}(F) \in M^+_{k+(1/2)}(\Gamma_0(4))$. Then $F(Z)$ is a Hecke eigenform if and only if $h(\tau)$ is a Hecke eigenform, and $F(Z)$ is called the Saito-Kurokawa lift of $h(\tau)$. If

$$h(\tau) = \sum_{\substack{n \geq 0 \\ n \equiv 0,3(4)}} c(n)\mathbf{e}(n\tau),$$

then B-th Fourier coefficient of $F(Z)$ is equal to

$$\sum_{\substack{d>0 \\ d^{-1}B \in \mathcal{S}_2'(\mathbb{Z})}} d^k c\left(\frac{D_B}{d^2}\right).$$

Let k' be a sufficiently large odd integer. It is well-known that $\mathcal{E}^{(2)}_{k'+1}(Z)$ satisfies the Maass relation. Put $H(k',n) = \beta_{\mathcal{E}^{(2)}_{k'+1}}(n)$. The function

$$\Omega^{SK}(\mathcal{E}^{(2)}_{k'+1})(\tau) = \mathcal{H}_{k'+(1/2)}(\tau) = \sum_{\substack{n \geq 0 \\ n \equiv 0,3(4)}} H(k',n)\mathbf{e}(n\tau)$$

is called the Cohen Eisenstein series (cf. Cohen [8]. [9]).

Since $\mathcal{E}^{(2)}_{k'+1}(Z)$ satisfies the Maass relation, the B-the Fourier coefficent of $\mathcal{E}^{(2)}_{k'+1}(Z)$ is equal to

$$\sum_{\substack{d>0 \\ d^{-1}B \in \mathcal{S}_2'(\mathbb{Z})}} d^{k'} H\left(k', \frac{D_B}{d^2}\right) = H(k', \mathfrak{d}_B)\mathfrak{f}_B^{k'-(1/2)} \sum_{\substack{d>0 \\ d^{-1}B \in \mathcal{S}_2'(\mathbb{Z})}} d^{1/2} \prod_p \Psi_p\left(\frac{D_B}{d^2}; p^{k'-(1/2)}\right).$$

Since k' is arbitrary, we have

$$\prod_p \tilde{F}_p(B; X_p) = \sum_{\substack{d>0 \\ d^{-1}B \in \mathcal{S}_2'(\mathbb{Z})}} d^{1/2} \prod_p \Psi_p\left(\frac{D_B}{d^2}; X_p\right).$$

It follows that

$$\begin{aligned} A(B) =& c(\mathfrak{d}_B)\mathfrak{f}_B^{k-(1/2)} \prod_p \tilde{F}_p(B; \alpha_p) \\ =& c(\mathfrak{d}_B)\mathfrak{f}_B^{k-(1/2)} \sum_{d|(m,r,l)} d^{1/2} \prod_p \Psi_p\left(\frac{D_B}{d^2}; \alpha_p\right) \\ =& \sum_{\substack{d>0 \\ d^{-1}B \in \mathcal{S}_2'(\mathbb{Z})}} d^k c\left(\frac{D_B}{d^2}\right). \end{aligned}$$

This agrees with the well-known Fourier coefficient formula for the Saito-Kurokawa lift.

Part II : Hermitian modular case

7 Hermitian modular forms and hermitian Eisensetein series

Now we consider the hermitian modular case. Let $K=\mathbb{Q}(\sqrt{-D_K})$ be an imaginary quadratic field. We denote the ring of integers of K by $\mathcal{O}$. The non-trivial automorphism of K is denoted by $x\mapsto \bar{x}$. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by χ.We denote by $\mathcal{O}^\sharp=(\sqrt{-D})^{-1}\mathcal{O}$ the inverse different ideal of $K/\mathbb{Q}$. For each prime p, we set $K_p=K\otimes\mathbb{Q}_p$ and $\mathcal{O}_p=\mathcal{O}\otimes\mathbb{Z}_p$.

The special unitary group $\mathrm{SU}(m,m)$ is an algebraic group defined over $\mathbb{Q}$, whose group of R-valued points is given by

$$\left\{g\in \mathrm{GL}_{2m}(R\otimes K)\,\middle|\, g\begin{pmatrix}\mathbf{0}_m & -\mathbf{1}_m\\ \mathbf{1}_m & \mathbf{0}_m\end{pmatrix}{}^t\bar{g}=\begin{pmatrix}\mathbf{0}_m & -\mathbf{1}_m\\ \mathbf{1}_m & \mathbf{0}_m\end{pmatrix},\ \det g=1\right\}$$

for any $\mathbb{Q}$-algebra R.

The special hermitian modular group $\Gamma_K^{(m)}$ is defined by $\mathrm{SU}(m,m)(\mathbb{Q})\cap\mathrm{SL}_{2m}(\mathcal{O})$. Note that $\Gamma_K^{(1)}=\mathrm{SL}_2(\mathbb{Z})$.

Put

$$\Gamma_{K,\infty}^{(m)}=\{\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_K^{(m)}\,\Big|\, C=0\}.$$

We define the hermitian upper half space $\mathcal{H}_m$ by

$$\mathcal{H}_m=\{Z\in M_m(\mathbb{C})\,|\,\frac{1}{2\sqrt{-1}}(Z-{}^t\bar{Z})>0\}.$$

The action of $\mathrm{SU}(2,2)(\mathbb{R})$ on $\mathcal{H}_m$ is given by

$$g\langle Z\rangle=(AZ+B)(CZ+D)^{-1},\quad Z\in\mathcal{H}_m,\ g=\begin{pmatrix}A & B\\ C & D\end{pmatrix}.$$

We put

$$\begin{aligned}\Lambda_m(\mathcal{O})&=\{h=(h_{ij})\in\mathrm{M}_m(K)\,|\,h_{ii}\in\mathbb{Z},h_{ij}=\bar{h}_{ji}\in\mathcal{O}^\sharp,\ (i\neq j)\},\\ \Lambda_m(\mathcal{O})^+&=\{h\in\Lambda_m(\mathcal{O})\,|\,h>0\}.\end{aligned}$$

For $H\in\Lambda_m(\mathcal{O})$, $\det H\neq 0$, we put

$$\gamma(H)=(-D_K)^{[m/2]}\det(H).$$

A holomorphic function F on $\mathcal{H}_m$ is called a hermitian modular form of weight l if

$$F((AZ+B)(CZ+D)^{-1})=F(Z)\det(CZ+D)^l$$

for any $\begin{pmatrix}A & B\\ C & D\end{pmatrix}\in\Gamma_K^{(m)}$. Again, we need a condition on Fourier expantion when $n=1$. When $m\geq 2$, a hermitian modular form F automatically has a Fourier expansion

$$F(Z) = \sum_{\substack{H \in \Lambda_n(\mathcal{O}) \\ H \geq 0}} A_F(H)\mathbf{e}(HZ).$$

The complex number $A_F(H)$ is called the H-th Fourier coefficient of F. A hermitian modular form F of degree m is called a cusp form if $A_F(H) = 0$ unless $H \in \Lambda_m(\mathcal{O})^+$. The space of hermitian modular (resp. cusp) form of degree m and weight l is demoted by $S_l(\Gamma_K^{(m)})$ (resp.$S_l(\Gamma_K^{(m)})$).

The Siegel series for $H \in \Lambda_m(\mathcal{O})^+$ is defined by

$$b_p(H, s) = \sum_{R \in \mathcal{H}_m(K_p)/\mathcal{H}_m(\mathcal{O} \otimes \mathbb{Z}_p)} \mathbf{e}_p(\mathrm{tr}(HR)) p^{-\mathrm{ord}_p(\nu(R))s}$$

for $\mathrm{Re}(s) \gg 0$. Here, $\mathcal{H}_m(K_p)$ (resp. $\mathcal{H}_m(\mathcal{O} \otimes \mathbb{Z}_p)$) is the additive group of all hermitian matrices with entries in K_p (resp. $\mathcal{O} \otimes \mathbb{Z}_p$).

The ideal $\nu(R) \subset \mathbb{Z}_p$ is defined as follows: Choose an element $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{SU}(2,2)(\mathbb{Q}_p) \cap \mathrm{SL}_{2m}(\mathcal{O} \otimes \mathbb{Z}_p)$ such that $\det D \neq 0$, $D^{-1}C = R$. Then $\nu(R) = \det(D) \in \mathbb{Z}_p$.

We define a polynomial $t_p(K/\mathbb{Q}; X) \in \mathbb{Z}[X]$ by

$$t_p(K/\mathbb{Q}; X) = \prod_{i=1}^{[(m+1)/2]} (1 - p^{2i}X) \prod_{i=1}^{[m/2]} (1 - p^{2i-1}\chi(p)X).$$

Then there exists a polynomial $F_p(H; X) \in \mathbb{Z}[X]$ such that

$$b_p(H, s) = t_p(K/\mathbb{Q}; p^{-s}) F_p(H; p^{-s}).$$

$$\mathrm{deg} F_p(H; X) = \mathrm{ord}_p \gamma(H).$$

The functional equation of $F_p(H; X)$ is as follows:

$$F_p(H; p^{-2m}X^{-1}) = \underline{\chi}_p(\gamma(H))^{m-1}(p^m X)^{-\mathrm{ord}_p \gamma(H)} F_p(H; X).$$

Here, $\underline{\chi}_p$ is the p-component of the idele character $\mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times} \to \mathbb{C}^{\times}$ corresponding to χ.

Put

$$\tilde{F}_p(H; X) = X^{\mathrm{ord}_p \gamma(H)} F_p(H; p^{-m}X^{-2}).$$

Then

$$\tilde{F}_p(H; X^{-1}) = \tilde{F}_p(H; X), \quad 2 \nmid m$$
$$\tilde{F}_p(H; \chi(p)X^{-1}) = \tilde{F}_p(H; X), \quad 2|m, \text{ and } \chi(p) \neq 0.$$

Assume $k \gg 0$. Put $n = [m/2]$. We define the Eisenstein series

$$E_{2k+2n}^{(m)}(Z) = \sum_{\left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma_{K.\infty}^{(m)} \backslash \Gamma_K^{(m)}} \det(CZ + D)^{-2k-2n}$$

and its normalization

$$\mathcal{E}^{(m)}_{2k+2n}(Z) = 2^{-m} \prod_{i=1}^{m} L(1+i-2k-2n, \chi^{i-1}) \cdot E^{(m)}_{2k+2n}(Z).$$

Then for each $H \in \Lambda_m(\mathcal{O})^+$, the H-th Fourier coefficient of $\mathcal{E}^{(m)}_{2k+2n}$ is equal to

$$|\gamma(H)|^{k+n-(m/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k-n+(m/2)}).$$

8 The case $m = 2n + 1$.

When $m = 2n+1$, the H-th Fourier coefficient of $\mathcal{E}^{(2n+1)}_{2k+2n}(Z)$ is equal to

$$|\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k+(1/2)})$$

for any $H \in \Lambda_{2n+1}(\mathcal{O})^+$.

Now let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, whose L-function is given by

$$L(f,s) = \sum_{N=1}^{\infty} a(N)N^{-s} = \prod_{p} \left[(1-p^{k-(1/2)}\alpha_p X)(1-p^{k-(1/2)}\alpha_p^{-1} X)\right]^{-1}.$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H, \alpha_p)$$

for $H \in \Lambda_{2n+1}(\mathcal{O})^+$ and

$$F(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})^+} A(H)\mathbf{e}(HZ)$$

for $Z \in \mathcal{H}_{2n+1}$.

Then we have

Theorem 2. *Assume that $m = 2n+1$ is odd. Then $F \in S_{2k+2n}(\Gamma_K^{(2n+1)})$ and $F \not\equiv 0$. Moreover, F is a Hecke eigenform.*

9 The case $m = 2n$.

When $m = 2n$, the H-th Fourier coefficient of $\mathcal{E}^{(2n)}_{2k+2n}(Z)$ is equal to

$$|\gamma(H)|^{k} \prod_{p|\gamma(H)} \tilde{F}_p(H; p^{-k})$$

for any $H \in \Lambda_{2n}(\mathcal{O})^+$.

Now let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k+1}(\Gamma_0(D_K), \chi)$ be a primitive form, whose L-function is given by

$$\begin{aligned} L(f,s) &= \sum_{N=1}^{\infty} a(N)N^{-s} \\ &= \prod_{p \nmid D_K} (1 - a(p)p^{-s} + \chi(p)p^{2k-2s})^{-1} \times \prod_{p \mid D_K} (1 - a(p)p^{-s})^{-1}. \end{aligned}$$

For each prime $p \nmid D_K$, we define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \chi(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \chi(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X).$$

For $p \mid D_K$, we put $\alpha_p = p^{-k}a(p)$.

Put

$$A(H) = |\gamma(H)|^k \prod_{p \mid \gamma(H)} \tilde{F}_p(H, \alpha_p)$$

for $H \in \Lambda_{2n}(\mathcal{O})^+$ and

$$F(Z) = \sum_{H \in \Lambda_{2n}(\mathcal{O})^+} A(H)\mathbf{e}(HZ)$$

for $Z \in \mathcal{H}_{2n}$. Then we have

Theorem 3. *Assume that $m = 2n$ is even. Then $F \in S_{2k+2n}(\Gamma_K^{(2n)})$. Moreover, F is a Hecke eigenform. $F \equiv 0$ if and only if n is odd and $f(\tau)$ comes from a Hecke character of some imaginary quadratic field.*

10 *L*-functions.

For simplicity, we assume the class number of K is one. Then, the L-function of F is as follows.

$$\begin{aligned} L(s, F, \rho) = &\prod_{i=1}^{2n+1} L(s + k + n - i + (1/2), f) \\ &\times \prod_{i=1}^{2n+1} L(s + k + n - i + (1/2), f, \chi) \end{aligned}$$

for $m = 2n + 1$, and

$$\begin{aligned} L(s, F, \rho) = &\prod_{i=1}^{2n} L(s + k + n - i + (1/2), f) \\ &\times \prod_{i=1}^{2n} L(s + k + n - i + (1/2), f, \chi) \end{aligned}$$

for $m = 2n$ and $F \not\equiv 0$.

Here, ρ is a $2m$-dimensional representation of the L-group of $U(m, m)$.

11 The case $m = 2$

The case $m = 2$ was first considered by Kojima [27] for $K = \mathbb{Q}(\sqrt{-1})$ and later by Krieg [28] and Sugano [43] for arbitrary imaginary quadratic field. For simplicity, we assume $D_K \neq 3, 4$.

Recall that

$$G(Z) = \sum_{H \in \Lambda_2(\mathcal{O})} A_G(H)\mathbf{e}(HZ) \in M_{2k+2}(\Gamma_2^{(2)})$$

satisfies the Maass relation if and only if there is a function

$$\alpha_G^* : \mathbb{Z}_{\geq 0} \to \mathbb{C}$$

such that

$$A_G(H) = \sum_{d \mid \varepsilon(H)} d^{2k+1} \alpha_G^*(\tfrac{|\gamma(H)|}{d^2}).$$

Here

$$\varepsilon(H) = \max\{q \in \mathbb{Z}_{>0} \mid q^{-1}H \in \Lambda_2(\mathcal{O})\}.$$

We denote the space of elements of $M_{2k+2}(\Gamma_2^{(2)})$ satisfying the Maass relation by $M_{2k+2}^{\text{Maass}}(\Gamma_2^{(2)})$. We put

$$\mathbf{a}_{D_K}(N) = \prod_{p \mid D_K} (1 + \chi_p(-N)).$$

Here, $\chi = \prod_{p \mid D_K} \chi_p$ is the decomposition to a product of Dirichlet characters with prime power conductors. The linear map

$$\Omega : M_{2k+2}^{\text{Maass}}(\Gamma_K^{(2)}) \to M_{2k+1}(\Gamma_0(D_K), \chi)$$

is defined by

$$\Omega(G)(\tau) = \sum_{N=0}^{\infty} \mathbf{a}_{D_K}(N)\alpha_G^*(H)q^N.$$

(This definition is slightly modified by a scalar from Krieg's definition.) Then Krieg proved that the image of Ω is equal to the space

$$M_{2k+1}^*(\Gamma_0(D_K), \chi) = \left\{ f = \sum_{N \geq 0} a(N)q^N \in M_{2k+1}(\Gamma_0(D_K), \chi) \,\middle|\, a(N) = 0 \text{ for } \mathbf{a}_{D_K}(N) = 0 \right\}.$$

Moreover, Ω induces an isomorphism between $M_{2k+2}^{\text{Maass}}(\Gamma_2^{(2)})$ and $M_{2k+1}^*(\Gamma_0(D_K), \chi)$. For each primitive form $f \in S_{2k+1}(\Gamma_0(D_K), \chi)$, there exists an element $f^* \in S_{2k+1}^*(\Gamma_0(D_K), \chi)$ such that

$$a_{f^*}(n) = \mathbf{a}_{D_K}(n)a_f(n), \quad (n, D_K) = 1.$$

When G is equal to the normalized hermitian Eisenstein series

$$\mathcal{E}_{2k+2}^{(2)}(Z) = \frac{B_{2k+2}B_{2k+1,\chi}}{8(k+1)(2k+1)} E_{2k+2}^{(2)}(Z),$$

then we have (cf. Krieg [28], p. 679)

$$\alpha_G^*(N) = \begin{cases} 0 & \mathbf{a}_{D_K}(N) = 0 \\ -\frac{B_{2k+1,\chi}}{4k+2} & N = 0 \\ \frac{1}{\mathbf{a}_{D_K}(N)} \sum_{d|N} \sum_{Q \subset Q_{D_K}} \chi_Q(\frac{-N}{d}) \chi'_Q(d) d^{2k} & N > 0, \mathbf{a}_{D_K}(N) \neq 0. \end{cases}$$

Using these results, one can calculate the polynomial $F_p(H; X)$ as follows: If $p \nmid D_K$, then

$$F_p(H; X) = \sum_{i=0}^{b} p^{3i} X^i \sum_{j=0}^{a-2i} \underline{\chi}_p(p)^j p^{2j} X^j.$$

If $p|D_K$, then

$$F_p(H; X) = \begin{cases} \sum_{i=0}^{b} p^{3i} X^i (1 + \underline{\chi}_p(\gamma(H)) p^{2(a-2i)} X^{a-2i}) & 2b < a \\ p^{3b} X^b + \sum_{i=0}^{b-1} (p^{3i} X^i + p^{4b-i} X^{2b-i}) & 2b = a \end{cases}$$

Here $a = \operatorname{ord}_p \gamma(H)$, $b = \operatorname{ord}_p \varepsilon(H)$. When the class number of K is one, this has been already calculated by Nagaoka [35]. Using this result, one can show that $\Omega^{-1}(f^*)$ is the lift of the primitive form $f \in S_{2k+1}(\Gamma_0(D_K), \chi)$.

References

1. J. Arthur, *Unipotent automorphic representations: conjectures*, Astérisque **171-172** (1989), 13–71.
2. S. Böcherer *Über die Fourier-Jacobi-Entwicklung Siegelscher Eisensteinreihen. I* Math. Zeitschr. **183** (1983) 21–46
3. S. Böcherer *Über die Fourierkoeffizienten der Siegelschen Eisensteinreihen* Manuscripta Math. **45** (1984) 273–288
4. S. Böcherer *Siegel modular forms and theta series* Proc. Symp. Pure Math., **49-2** (1989) 3–17
5. R. E. Borcherds, E. Freitag, and R. Weissauer *A Siegel cusp form of degree 12 and weight 12* J. Reine Angew. Math. **494** (1998) 141–153
6. H. Braun, *Hermitian modular functions. III*, Ann. of Math. **53**, (1951), 143–160.
7. S. Breulmann and M. Kuss *On a conjecture of Duke-Imamoglu* Proc. Amer. Math. Soc. **128** (2000), 1595–1604
8. H. Cohen *Sommes de carres, fonctions L et formes modulaires* C. R. Acad. Sci. Paris Sér. A-B 277 (1973) A827–A830
9. H. Cohen *Sums involving the values at negative integers of L-functions of quadratic characters* Math. Ann. **217** (1975) 271–285
10. M. Eichler and D. Zagier, *The theory of Jacobi forms*, Progress in Mathematics **55** Birkhäuser Boston, Inc., Boston, Mass. 1985.
11. P. Feit *Poles and residues of Eisenstein series for symplectic and unitary groups* Momoirs of the AMS. **346** (1986)
12. P. Feit *Explicit formulas for local factors in the Euler products for Eisenstein series* Nagoya Math. J. **113** (1989) 37–87
13. E. Freitag *Siegelsche Modulfunktionen* Springer-Verlag, Berlin-New York (1983)
14. V. A. Gritsenko, *The Maass space for* SU(2, 2)*. The Hecke ring, and zeta functions* (Russian), Translated in Proc. Steklov Inst. Math. 1991, no. 4, 75–86. Galois theory, rings, algebraic groups and their applications (Russian). Trudy Mat. Inst. Steklov. **183** (1990), 68–78, 223–225.

15. T. Ibukiyama *Conjecture on the lifting of modular forms to Siegel modular forms* informal note.
16. A. Ichino and T. Ikeda, *On Maass lifts and the central critical values of triple product L-functions*, preprint.
17. T. Ikeda, *On the theory of Jacobi forms and the Fourier-Jacobi coefficients of Eisenstein series*, J. Math. Kyoto Univ. **34** (1994), 615–636.
18. T. Ikeda, *On the lifting of elliptic cusp forms to Siegel cusp forms of degree* $2n$, Ann. of Math. **154** (2001), 641–681.
19. T. Ikeda, *On the lifting of hermitian modular forms*, preprint
20. H. Katsurada *An explicit formula for Siegel series* Amer. J. Math. **121** (1999) 415–452
21. Y. Kitaoka *Dirichlet series in the theory of Siegel modular forms* Nagoya Math. J. **95**] (1984) 73–84
22. Y. Kitaoka *A space of Siegel modular forms closed under the action of Hecke operators* Proc. Japan Acad. Ser. A Math. Sci. **70** (1994) 194–197
23. H. Klingen, *Über die Erzeugenden gewisser Modulgruppen*, Nachr. Akad. Wiss. Goettingen. Math.-Phys. Kl. IIa. 1956 (1956), 173–185.
24. W. Kohnen *Modular forms of half-integral weight on $\Gamma_0(4)$* Math. Ann. **248** (1980) 249–266
25. W. Kohnen, *Lifting modular forms of half-integral weight to Siegel modular forms of even genus*, Math. Ann. **322** (2002), 787–809.
26. W. Kohnen and D. Zagier *Values of L-series of modular forms at the center of the critical strip* Invent. Math. **64** (1981) 175–198
27. H. Kojima, *An arithmetic of Hermitian modular forms of degree two*, Invent. Math. **69** (1982), 217–227.
28. A. Krieg, *The Maass spaces on the Hermitian half-space of degree* 2, Math. Ann. **289** (1991), 663–681.
29. S. Kudla and W. J. Sweet Jr., *Degenerate principal series representations for* $\mathrm{U}(n,n)$, Israel J. Math. **98** (1997), 253–306.
30. N. Kurokawa *Examples of eigenvalues of Hecke operators on Siegel cusp forms of degree two* Invent. Math. **49** (1978) 149–165
31. H. Maass *Über eine Spezialschar von Modulformen zweiten Grades.* Invent. Math **52** (1979) 95–104
32. H. Maass *Über eine Spezialschar von Modulformen zweiten Grades. II* Invent. Math. **53** (1979) 249–253
33. H. Maass *Über eine Spezialschar von Modulformen zweiten Grades. III* Invent. Math. **53** (1979) 255–265
34. T. Miyake, *Modular Forms*, Springer, (1989).
35. S. Nagaoka, *On Eisenstein series for the Hermitian modular groups and the Jacobi groups*, Abh. Math. Sem. Univ. Hamburg **62** (1992), 117–146.
36. S. Raghavan and J. Sengupta, *A Dirichlet series for Hermitian modular forms of degree* 2, Acta Arith. **58** (1991), 181–201.
37. G. Shimura, *Introduction to the arithmetic theory of automorphic functions*, Publ. Math. Soc. Japan **11** Iwanami Shoten and Princeton University Press, 1971.
38. G. Shimura *On modular forms of half integral weight* Ann. of Math. **97** (1973) 440–481
39. G. Shimura *On certain reciprocity-laws for theta functions and modular forms* Acta Math. **141** (1978) 35–71
40. G. Shimura *On Eisenstein series* Duke Math. J. **50** (1983) 417–476
41. G. Shimura *Euler products and Fourier coefficients of automorphic forms on symplectic groups* Invent. Math. **116** (1994) 531–576
42. G. Shimura, *Euler products and Eisenstein series*, CBMS Regional Conference Series in Mathematics **93** the American Mathematical Society, Providence, RI, 1997.
43. T. Sugano, *On Maass space for* $\mathrm{SU}(2,2)$ (Japanese), Surikaisekikenkyusho Kokyuroku (RIMS Kokyuroku) **546** (1985) 1–16.
44. J. Tate, *Number theoretic background*, Proc. Sympos. Pure Math., XXXIII, patr 2 (1979) 3–26.
45. J.-L. Waldspurger *Correspondances de Shimura et quaternions* Forum Math. **3** (1991) 219–307
46. D. Zagier *Sur la conjecture de Saito-Kurokawa* (d'aprés H. Maass) Seminar on Number Theory 1979–80, Paris, Progr. Math., 12, Birkhäuser, Boston, Mass. 371–394 (1981)