ON THE LIFTING OF HERMITIAN MODULAR FORMS

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To my parents

Abstract. Let K be an imaginary quadratic field with discriminant $-D$. We denote by $\mathcal{O}$ the ring of integers of K. Put $\chi(N) = \left(\frac{-D}{N}\right)$. Let $\Gamma_K^{(m)} = \mathrm{U}(m,m)(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O})$ be the hermitian modular group of degree m. We construct a lifting from $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ to $S_{2k+2n}(\Gamma_K^{(2n+1)}, \det^{-k-n})$ and a lifting from $S_{2k+1}(\Gamma_0(D), \chi)$ to $S_{2k+2n}(\Gamma_K^{(2n)}, \det^{-k-n})$. We give an explicit Fourier coefficient formula of the lifting. This is a generalization of the Maass lift considered by Kojima, Krieg and Sugano. We also discuss its extension to the adele group of $\mathrm{U}(m,m)$.

Introduction

In this paper, we are going to discuss a lifting of elliptic cusp forms to hermitian modular forms. This is a hermitian modular analogue of the lifting constructed in [9]. In [9], the author constructed a Siegel cusp form whose Fourier coefficients are closely related to the Fourier coefficients of Eisenstein series of Siegel type.

Let us describe our results. Let $K = \mathbb{Q}(\sqrt{-D_K})$ be an imaginary quadratic field. We denote the ring of integers of K by $\mathcal{O}$. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by χ. The hermitian modular group $\Gamma_K^{(m)} = \mathrm{U}(m,m)(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O})$ is the group of all elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{GL}_{2m}(\mathcal{O})$ such that

$$A\,{}^t\bar{B} = B\,{}^t\bar{A},\ C\,{}^t\bar{D} = D\,{}^t\bar{C},\ A\,{}^t\bar{D} - B\,{}^t\bar{C} = \mathbf{1}_m.$$

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We let $\Gamma_{K,\infty}^{(m)}$ be the subset of elements $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma_K^{(m)}$ such that $C = 0$. The hermitian upper half space of degree m is defined by

$$\mathcal{H}_m = \{Z \in M_m(\mathbb{C}) \mid \tfrac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0\}.$$

For $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{U}(m,m)(\mathbb{R})$ and $Z \in \mathcal{H}_m$, we put

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \qquad j(g,Z) = \det(CZ+D).$$

Let σ be a character of $\Gamma_K^{(m)}$. A holomorphic function F on $\mathcal{H}_m$ ($m \geq 2$) is called a hermitian modular form of weight l with character σ if $F(g\langle Z\rangle) = \sigma(g)F(Z)j(g,Z)^l$ for any $g \in \Gamma_K^{(m)}$.

Recall that a semi-integral hermitian matrix is a hermitian matrix $H \in \sqrt{-D_K}^{-1}\mathrm{M}_m(\mathcal{O})$ whose diagonal entries are integral. We denote the set of semi-integral hermitian matrices by $\Lambda_m(\mathcal{O})$. The set of positive definite elements of $\Lambda_m(\mathcal{O})$ is denoted by $\Lambda_m(\mathcal{O})^+$. For $H \in \Lambda_m(\mathcal{O})$, we put $\gamma(H) = (-D_K)^{[m/2]}\det H$. Note that $\gamma(H) \in \mathbb{Z}$. A hermitian modular form F is called a cusp form if it has a Fourier expansion of the form

$$F(Z) = \sum_{H \in \Lambda_m(\mathcal{O})^+} A(H)\exp(2\pi\sqrt{-1}\mathrm{tr}(HZ)).$$

We denote the space of cusp forms of weight l with character σ by $S_l(\Gamma_K^{(m)}, \sigma)$.

The Eisenstein series $E_{2l}^{(m)}(Z)$ of weight $2l$ with character $\det^{-l}$ is defined by

$$E_{2l}^{(m)}(Z) = \sum_{g \in \Gamma_{K,\infty}^{(m)} \backslash \Gamma_K^{(m)}} (\det g)^l j(g,Z)^{-2l}$$

This is absolutely convergent for $l > m$. We define the normalized Eisenstein series $\mathcal{E}_{2l}^{(m)}(Z)$ by

$$\mathcal{E}_{2l}^{(m)}(Z) = 2^{-m}\prod_{i=1}^{m} L(i-2l, \chi^{i-1}) \cdot E_{2l}^{(m)}(Z).$$

If $H \in \Lambda_m(\mathcal{O})^+$, then the H-th Fourier coefficient of $\mathcal{E}_{2l}^{(m)}(Z)$ is equal to

$$|\gamma(H)|^{l-(m/2)} \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{-l+(m/2)}).$$

(See §4.) Here, $\tilde{F}_p(H;X)$ is a certain Laurent polynomial arising from the Siegel series for H.

Then our main theorem can be stated as follows. For simplicity, we assume that $m = 2n$ is even. Let

$$f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k+1}(\Gamma_0(D_K), \chi)$$

be a primitive form, whose L-function is given by

$$L(f, s) = \prod_{p \nmid D_K} (1 - a(p)p^{-s} + \chi(p)p^{2k-2s})^{-1} \prod_{p \mid D_K} (1 - a(p)p^{-s})^{-1}.$$

For each prime $p \nmid D_K$, we define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \chi(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \chi(p)p^{2k}X^2) = (1 - p^k\alpha_p X)(1 - p^k\beta_p X).$$

For $p \mid D_K$, we put $\alpha_p = p^{-k}a(p)$. Put

$$A(H) = |\gamma(H)|^k \prod_{p \mid \gamma(H)} \tilde{F}_p(H; \alpha_p), \quad H \in \Lambda_{2n}(\mathcal{O})^+,$$

$$F(Z) = \sum_{H \in \Lambda_{2n}(\mathcal{O})^+} A(H) \exp(2\pi\sqrt{-1}\mathrm{tr}(HZ)), \quad Z \in \mathcal{H}_{2n}.$$

Then our first main theorem in the even case is as follows.

Theorem 5.1. *Assume that $m = 2n$ is even. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n)}, \det^{-k-n})$.*

(For the case when m is odd, see Theorem 5.2.) To prove this theorem, we use the theory of compatible family of Eisenstein series as in [9]. The theory of compatible family of Eisenstein series is a method to prove that a certain Fourier series is a modular form. The Fourier series we consider can be regarded as a sum of Whittaker function, and the behavior of Whittaker function is determined by Fourier coefficients of Eisenstein series. We consider a compatible family of Eisenstein series of integral weight. This case is a little more complicated than that of [9] because the automorphic representation of $\mathrm{SL}_2(\mathbb{A})$ generated by f may be reducible. Instead of using a Whittaker function on $\mathrm{SL}_2(\mathbb{A})$, we extend it to $\mathrm{GL}_2(\mathbb{A})$. We have to show that this extension to $\mathrm{GL}_2(\mathbb{A})$ is possible in a compatible way for a family of Eisenstein series arising from the Fourier-Jacobi coefficients of $\mathcal{E}_{2k'+2n}^{(2n)}$. This is a purely local problem and is treated in §8.

In §12 and §13, we prove that the lift F can be extended to an automorphic form on the adele group of the unitary group $\mathrm{U}(m, m)$. The extension $\mathit{Lift}^{(m)}(f)$ is a common Hecke eigenform of all Hecke operators of the unitary group, if it is not identically zero (Theorem

13.6). Moreover, the standard L-function $L(s, \mathit{Lift}^{(2n)}(f), \mathrm{st})$ is given by

$$\prod_{i=1}^{m} L(s+k+n-i+(1/2), f)L(s+k+n-i+(1/2), f, \chi).$$

(See Theorem 18.1.)

Following Kohnen [14], we discuss the "linearization" of the lifting. The case when m is odd is fairly easy, and will be treated in §14. Assume now $m = 2n$ is even. Then we reformulate the main theorem in terms of a certain linear map from a subspace $S^*_{2k+1}(\Gamma_0(D_K), \chi) \subset S_{2k+1}(\Gamma_0(D_K), \chi)$ to $S_{2k+2n}(\Gamma_K^{(2n)}, \det^{-k-n})$. (In fact, we need to consider certain twisting by an ideal $\mathfrak{c}$ of K, but for simplicity we consider the case $\mathfrak{c} = \mathcal{O}$ here.) Decompose the character χ into a product $\chi = \prod_{q|D_K} \chi_q$, where χ_q is a character whose conductor is a power of a prime q. Put

$$\mathbf{a}_{D_K}(N) = \prod_{q|D_K} \left(1 + \chi_q((-1)^n N)\right).$$

Following Krieg [16], we define $S^*_{2k+1}(\Gamma_0(D_K), \chi)$ by the space of cusp forms

$$f_0(\tau) = \sum_{N>0} a_{f_0}(N) q^N \in S_{2k+1}(\Gamma_0(D_K), \chi)$$

such that $a_{f_0}(N) = 0$ whenever $\mathbf{a}_{D_K}(N) = 0$. For each primitive form $f \in S_{2k+1}(\Gamma_0(D_K), \chi)$, we define $f^* \in S^*_{2k+1}(\Gamma_0(D_K), \chi)$ as the unique element of $S_{2k+1}(\Gamma_0(D_K), \chi)$ such that $a_{f^*}(N) = \mathbf{a}_{D_K}(N) a_f(N)$ whenever $(N, D_K) = 1$. Then we can show that there exists an injective linear map

$$\iota : S^*_{2k+1}(\Gamma_0(D_K), \chi) \to S_{2k+2n}(\Gamma_K^{(2n)}, \det^{-k-n})$$

such that $F(Z)$ is equal to $\iota(f^*)$ (Theorem 15.18). It follows that $F = 0$ if and only if $f^* = 0$. It is easy to prove that $f^* = 0$ if and only if n is odd and f comes from a Hecke character of some imaginary quadratic field (Corollary 15.12). As for the lifting $\mathit{Lift}^{(2n)}(f)$ to the adele group $\mathrm{U}(2n, 2n)(\mathbb{A})$, we show that $\mathit{Lift}^{(2n)}(f) = 0$ if and only if n is odd and f comes from a Hecke character of K (Corollary 15.21).

We discuss the case $m = 2$ in §16. In this case, the theory of the lifting has been already treated by Kojima [15], Gritsenko [5], Krieg [16], Sugano [26], and Klosin [13]. In §17, we calculate the Petersson inner product of the hermitian Maass lift in the case $m = 2$, by using the results of Sugano [26].

In §18, we discuss the relation to the Arthur conjecture. The Arthur parameter associated to our lift can be described as follows. (Here, m need not be even.) We now admit the Arthur conjecture and the existence of the hypothetical Langlands group $\mathcal{L}_{\mathbb{Q}}$. Recall that the L-group of $\mathcal{G} = \mathrm{U}(m, m)$ is a semi-direct product $\mathrm{GL}_{2m}(\mathbb{C}) \rtimes W_{\mathbb{Q}}$, where $W_{\mathbb{Q}}$ is the Weil group of $\mathbb{Q}$. The canonical homomorphism $\mathcal{L}_{\mathbb{Q}} \to W_{\mathbb{Q}}$ is denoted by pr. Let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f. Note that the central character ω_τ is equal to χ^{m-1}. We denote the Langlands parameter of τ by $\rho_\tau : \mathcal{L}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. Let $\mathrm{Sym}^{m-1} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_m(\mathbb{C})$ be the m-dimensional irreducible representation of $\mathrm{SL}_2(\mathbb{C})$. We put

$$\rho_\tau^{(m)}(u) = \begin{pmatrix} \omega_\tau(u)a \cdot \mathbf{1}_m & b \cdot \mathbf{1}_m \\ \omega_\tau(u)c \cdot \mathbf{1}_m & d \cdot \mathbf{1}_m \end{pmatrix} \rtimes \mathrm{pr}(u),$$

for $u \in \mathcal{L}_{\mathbb{Q}}$, $\rho_\tau(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and put

$$\rho_\tau^{(m)}(x) = \begin{pmatrix} \mathrm{Sym}^{m-1}(x) & 0 \\ 0 & \mathrm{Sym}^{m-1}(x) \end{pmatrix} \rtimes 1.$$

for $x \in \mathrm{SL}_2(\mathbb{C})$. Then, the Arthur parameter of $\mathit{Lift}^{(m)}(f)$ should be $\rho_\tau^{(m)}$. By using this A-parameter, we will show that our result is compatible with the conjectural Arthur multiplicity formula.

Notation

Let K be an imaginary quadratic field with discriminant $-D = -D_K$. When there is no fear of confusion, we drop the subscript K. We denote by $\mathcal{O} = \mathcal{O}_K$ the ring of integers of K. The number of roots of unity contained in K is denoted by w_K. The non-trivial automorphism of K is denoted by $x \mapsto \bar{x}$. The primitive Dirichlet character corresponding to $K/\mathbb{Q}$ is denoted by χ. We denote by $\mathcal{O}^\sharp = (\sqrt{-D})^{-1}\mathcal{O}$ the inverse different ideal of $K/\mathbb{Q}$. For each prime p, we set $K_p = K \otimes \mathbb{Q}_p$ and $\mathcal{O}_p = \mathcal{O} \otimes \mathbb{Z}_p$. The set of hermitian matrices of size m with entries in K, K_p, $\mathcal{O}$, and $\mathcal{O}_p$ are denoted by $\mathcal{H}_m(K)$, $\mathcal{H}_m(K_p)$, $\mathcal{H}_m(\mathcal{O})$, and $\mathcal{H}_m(\mathcal{O}_p)$, respectively.

We set $\mathbf{e}(T) = \exp(2\pi\sqrt{-1}\mathrm{tr}(T))$ if T is a square matrix with entries in $\mathbb{C}$. When p is a prime, $\mathbf{e}_p$ is the unique additive character of $\mathbb{Q}_p$ such that $\mathbf{e}_p(x) = \exp(-2\pi\sqrt{-1}x)$ for $x \in \mathbb{Z}[p^{-1}]$. Note that $\mathbf{e}_p$ is of order 0. The adele ring of $\mathbb{Q}$ is denoted by $\mathbb{A}_{\mathbb{Q}}$ or $\mathbb{A}$. The finite part of the adele ring is denoted by $\mathbb{A}_{\mathrm{f}}$. We put $\mathbf{e}_{\mathbb{A}}(x) = \mathbf{e}(x_\infty) \prod_{p<\infty} \mathbf{e}_p(x_p)$ for an adele $x = (x_v)_v \in \mathbb{A}$.

Let $\underline{\chi} = \otimes_v \underline{\chi}_v$ be the character of the idele class group $\mathbb{A}^\times/\mathbb{Q}^\times$ determined by χ. Then $\underline{\chi}_v$ is the character of $\mathbb{Q}_v^\times$ corresponding to $\mathbb{Q}_v(\sqrt{-D})/\mathbb{Q}_v$ and is given by the Hilbert symbol $\underline{\chi}_v(t) = \left(\frac{-D,t}{\mathbb{Q}_v}\right)$ for $t \in \mathbb{Q}_v^\times$. We put $\hat{\mathbb{Z}} = \prod_p \mathbb{Z}_p$ and $\hat{\mathbb{Z}}^\times = \prod_p \mathbb{Z}_p^\times$.

In §3 and §8, F will denote a non-archimedean local field. When ψ is an additive character of F and ρ is a quasi-character of $F^\times$, the L-factor $L(s,\rho)$ and the ε-factor $\varepsilon(s,\rho,\psi)$ are defined as in Tate [28]. We set $\varepsilon'(s,\rho,\psi) = \varepsilon(s,\rho,\psi)L(1-s,\rho^{-1},\psi)L(s,\rho)^{-1}$.

1. Unitary groups and hermitian modular forms

We recall some basic facts about hermitian modular forms.

The unitary group $\mathcal{G}^{(m)} = \mathrm{U}(m,m)$ is an algebraic group defined over $\mathbb{Q}$, whose group of R-valued points is given by

$$\left\{ g \in \mathrm{GL}_{2m}(R \otimes K) \,\middle|\, g \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} {}^t\bar{g} = \begin{pmatrix} 0 & -\mathbf{1}_m \\ \mathbf{1}_m & 0 \end{pmatrix} \right\}$$

for any $\mathbb{Q}$-algebra R. When there is no fear of confusion, we drop the superscript (m). We define the hermitian modular group by $\Gamma_K^{(m)} = \mathcal{G}^{(m)}(\mathbb{Q}) \cap \mathrm{GL}_{2m}(\mathcal{O})$. Put $\Gamma_{K,\infty}^{(m)} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \tilde{\Gamma}_K^{(m)} \,\middle|\, C = 0 \right\}$. Note that $\Gamma_K^{(1)} = \mathrm{SL}_2(\mathbb{Z}) \cdot \{\alpha \cdot \mathbf{1}_2 \,|\, \alpha \in \mathcal{O}^\times\}$. A hermitian matrix $H \in \mathcal{H}_m(K)$ is called semi-integral if $\mathrm{tr}(HR) \in \mathbb{Z}$ for any $R \in \mathcal{H}_m(\mathcal{O})$. Note that $H \in \mathcal{H}_m(K)$ is semi-integral if and only if the diagonal entries of H are integral and $\sqrt{-D_K} \cdot H \in \mathrm{M}_m(\mathcal{O})$. We denote the set of semi-integral hermitian matrices of size m by $\Lambda_m(\mathcal{O})$. Similarly, we define $\Lambda_m(\mathcal{O}_p)$. Then we have $\Lambda_m(\mathcal{O}_p) = \Lambda(\mathcal{O}) \otimes_{\mathbb{Z}} \mathbb{Z}_p$. The subset of $\Lambda_m(\mathcal{O})$ consisting of all positive definite elements is denoted by $\Lambda_m(\mathcal{O})^+$.

The hermitian upper half space $\mathcal{H}_m$ is defined by

$$\mathcal{H}_m = \{Z \in M_m(\mathbb{C}) \,|\, \tfrac{1}{2\sqrt{-1}}(Z - {}^t\bar{Z}) > 0\}.$$

Note that $\mathcal{H}_1 = \mathfrak{H}_1$ is the usual upper half plane. Then the unitary group $\mathcal{G}^{(m)}(\mathbb{R})$ acts on $\mathcal{H}_m$ by

$$g\langle Z\rangle = (AZ+B)(CZ+D)^{-1}, \quad Z \in \mathcal{H}_m,\ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{G}^{(m)}(\mathbb{R}).$$

We put $j(g,Z) = \det(CZ+D)$ for $Z \in \mathcal{H}_m$ and $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{G}^{(m)}(\mathbb{R})$. If $F(Z)$ is a function on $\mathcal{H}_m$, we put

$$(F|_l\, g)(Z) = F(g\langle Z\rangle) j(g,Z)^{-l}$$

for $g \in \mathcal{G}^{(m)}(\mathbb{R})$. When l is clear from the context, we sometimes drop it. Let σ be a character of $\Gamma_K^{(m)}$, which is trivial on $\left\{ \begin{pmatrix} \mathbf{1}_m & B \\ 0 & \mathbf{1}_m \end{pmatrix} \in \tilde{\Gamma}_K^{(m)} \right\}$. A holomorphic function F on $\mathcal{H}_m$ is called a hermitian modular form of weight l with character σ if $F|_l\, g = \sigma(g)F$ for any $g \in \Gamma_K^{(m)}$. When $m = 1$, the usual holomorphy condition at the cusp is required. A hermitian modular form $F(Z)$ has a Fourier expansion of the form

$$F(Z) = \sum_{\substack{H \in \Lambda_m(\mathcal{O}) \\ H \geq 0}} A(H)\mathbf{e}(HZ).$$

Here, $H \geq 0$ means that the hermitian matrix H is positive semi-definite. A hermitian modular form F is called a cusp form if Fourier coefficients $A(H)$ vanish unless $H \in \Lambda(\mathcal{O})^+$. The space of hermitian modular forms (resp. hermitian cusp forms) of weight l with character σ is denoted by $M_l(\Gamma_K^{(m)}, \sigma)$ (resp. $S_l(\Gamma_K^{(m)}, \sigma)$).

2. **Siegel series for unitary groups**

In this section, we consider Siegel series associated to non-degenerate semi-integral hermitian matrices. Fix a prime p. Put $\xi_p = \chi(p)$, i.e.,

$$\xi_p = \begin{cases} 1 & \text{if } K_p \simeq \mathbb{Q}_p \oplus \mathbb{Q}_p \\ -1 & \text{if } K_p/\mathbb{Q}_p \text{ is an unramified quadratic extension} \\ 0 & \text{if } K_p/\mathbb{Q}_p \text{ is a ramified quadratic extension.} \end{cases}$$

For $H \in \Lambda_m(\mathcal{O}_p)$, $\det H \neq 0$, we put

$$\gamma(H) = (-D_K)^{[m/2]} \det H.$$

It is easily seen that $\gamma(H) \in \mathbb{Z}_p$. Note that $\gamma(H) \in \mathbb{Z}$ for $H \in \Lambda_m(\mathcal{O})$. The Siegel series for H is defined by

$$b_p(H, s) = \sum_{R \in \mathcal{H}_m(K_p)/\mathcal{H}_m(\mathcal{O}_p)} \mathbf{e}_p(\mathrm{tr}(HR)) p^{-\mathrm{ord}_p(\nu(R))s}, \quad \mathrm{Re}(s) \gg 0.$$

The ideal $\nu(R) \subset \mathbb{Z}_p$ is defined as follows: Choose an element $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathcal{G}^{(m)}(\mathbb{Q}_p) \cap \mathrm{SL}_{2m}(\mathcal{O}_p)$ such that $\det D \neq 0$, $D^{-1}C = R$. Then $\nu(R) = (\det D)\mathbb{Z}_p$. Note that $\det D \in \mathbb{Q}_p$.

We define a polynomial $t_p(K/\mathbb{Q}; X) \in \mathbb{Z}[X]$ by

$$t_p(K/\mathbb{Q}; X) = \prod_{i=1}^{[(m+1)/2]} (1 - p^{2i}X) \prod_{i=1}^{[m/2]} (1 - p^{2i-1}\xi_p X).$$

There exists a polynomial $F_p(H;X) \in \mathbb{Z}[X]$ with constant term 1 such that

$$F_p(H;p^{-s}) = b_p(H,s)t_p(K/\mathbb{Q};p^{-s})^{-1}.$$

For a proof of this fact, see [24]. Clearly, $F_p({}^t\bar{A}HA;X) = F_p(H;X)$ for any $A \in \mathrm{GL}_m(\mathcal{O}_p)$. Moreover, $F_p(H;X)$ satisfies the following functional equation:

$$F_p(H;p^{-2m}X^{-1}) = \underline{\chi}_p(\gamma(H))^{m-1}(p^mX)^{-\mathrm{ord}_p\,\gamma(H)}F_p(H;X).$$

This functional equation follows from the results of Kudla and Sweet [17]. We will discuss it in the next section.

The functional equation implies that $\deg F_p(H;X) = \mathrm{ord}_p\,\gamma(H)$. In particular, if $p \nmid \gamma(H)$, then $F_p(H;X) = 1$.

Definition 2.1. For $H \in \Lambda_m(\mathcal{O}_p)$, $\det H \neq 0$, we put

$$\tilde{F}_p(H;X) = X^{\mathrm{ord}_p\,\gamma(H)}F_p(H;p^{-m}X^{-2}).$$

Note that the highest term and the lowest term of $\tilde{F}_p(H;X)$ are $X^{\mathrm{ord}_p\,\gamma(H)}$ and $\underline{\chi}_p(\gamma(H))^{m-1}X^{-\mathrm{ord}_p\,\gamma(H)}$, respectively. The following lemma follows immediately from the functional equation of $F_p(H;X)$.

Lemma 2.2. *For $H \in \Lambda_m(\mathcal{O}_p)$, $\det H \neq 0$, we have*

$$\begin{aligned} \tilde{F}_p(H;X^{-1}) =& \tilde{F}_p(H;X), && \text{if } m \text{ is odd} \\ \tilde{F}_p(H;X^{-1}) =& \underline{\chi}_p(\gamma(H))\tilde{F}_p(H;X), && \text{if } m \text{ is even} \\ \tilde{F}_p(H;\xi_pX^{-1}) =& \tilde{F}_p(H;X), && \text{if } m \text{ is even and } p \nmid D_K. \end{aligned}$$

We will need the following lemma later.

Lemma 2.3. *There exists a constant $M > 0$ such that*

$$|\tilde{F}_p(H;\omega)| \leq p^{M\cdot\mathrm{ord}_p(\gamma(H))}$$

for any $H \in \Lambda_m(\mathcal{O}_p)$, $\det H \neq 0$ and any $\omega \in \mathbb{C}$, $|\omega| = 1$. The constant M does not depend on p.

Proof. We may assume $p|\gamma(H)$. Put $d = \mathrm{ord}_p(\gamma(H)) = \deg F_p(H;X)$. For each

$$\varphi(X) = \sum_{N=0}^{\infty} a_NX^N \in \mathbb{C}[[X]],$$

we put $H_d(\varphi) = \max(|a_0|,\ldots,|a_d|)$. Then

$$H_d(\varphi_1\varphi_2) \leq (d+1)H_d(\varphi_1)H_d(\varphi_2)$$

for $\varphi_1,\ \varphi_2 \in \mathbb{C}[[X]]$. For each positive integer l, we define a formal power series $\alpha_H^l(t)$ such that

$$\alpha_H^l(p^{-s}) = \sum_{R\in p^{-l}\Lambda_m(\mathcal{O}_p)/\Lambda_m(\mathcal{O}_p)} \mathbf{e}_p(\mathrm{tr}(HR))p^{-\mathrm{ord}_p(\nu(R))s}.$$

Then it is proved by Shimura [24] that if $l \geq 2d+1$, then $\alpha_H^l(p^{-s}) = b_p(H,s)$. Since $b_p(H,s) = t_p(K/\mathbb{Q};p^{-s})F_p(H;p^{-s})$, we have

$$H_d(t_p(K/\mathbb{Q};X)F_p(H;X)) \leq p^{(2d+1)m^2} \leq p^{3dm^2}.$$

On the other hand,

$$\begin{aligned} H_d(t_p(K/\mathbb{Q};X)^{-1}) \leq & H_d(\prod_{i=1}^m (1-p^iX)^{-1}) \\ \leq & (d+1)^{m-1}\prod_{i=1}^m H_d((1-p^iX)^{-1}) \\ \leq & (d+1)^{m-1}p^{dm(m+1)/2}. \end{aligned}$$

By the obvious estimate $(d+1) \leq p^d$, we have

$$H_d(F_p(H;X)) \leq (d+1)^m p^{dm(m+1)/2}p^{3dm^2} \leq p^{d(7m^2+3m)/2}.$$

Since $|\omega| = 1$, we have

$$|\tilde{F}_p(H;\omega)| = |F_p(H;p^{-m}\omega^{-2})| \leq (d+1)p^{d(7m^2+3m)/2} \leq p^{d(7m^2+3m+2)/2}.$$

It follows that $|\tilde{F}_p(H;\omega)| \leq p^{d(7m^2+3m+2)/2}$. □

3. **A proof of the functional equation**

In this section, $F = F_v$ is a non-archimedean local field. We denote the ring of integers, the absolute value, and the order of the residue field by $\mathfrak{o}$, $|\ |$ and q, respectively. Let E be either a quadratic extension of F or $F \oplus F$. We denote the character corresponding to E/F by $\underline{\chi}$. We denote the discriminant ideal of E/F by $\mathfrak{D}$. Put $G = \mathrm{SU}(m,m)_{E/F}$. For a quasi-character $\rho : F^\times \to \mathbb{C}^\times$, we denote the degenerate principal series $\mathrm{Ind}_P^G(\rho\cdot|\ |^s)$ by $I(\rho,s)$. Here P is the Siegel parabolic subgroup of G. The space of $I(\rho,s)$ consists of all locally constant functions $\Phi(g)$ on G such that

$$\Phi\left(\begin{pmatrix} A & B \\ 0 & {}^t\bar{A}^{-1}\end{pmatrix} g\right) = |\det A|^{s+m}\Phi(g)$$

for any $\begin{pmatrix} A & B \\ 0 & {}^t\bar{A}^{-1}\end{pmatrix} \in P$ and any $g \in G$.

Fix an additive character ψ of F. For $\Phi(g) \in I(\rho, s)$ and a non-degenerate hermitian matrix $H \in \mathcal{H}_m(E)$, put

$$M(s,\rho)\Phi(g) = \int_{\mathcal{H}_m(E)} \Phi(w \begin{pmatrix} \mathbf{1}_m & x \\ 0 & \mathbf{1}_m \end{pmatrix} g, s)\, dx.$$

$$\mathrm{Wh}_H(s)\Phi(g) = \int_{\mathcal{H}_m(E)} \Phi(w \begin{pmatrix} \mathbf{1}_m & x \\ 0 & \mathbf{1}_m \end{pmatrix} g, s)\overline{\psi(\mathrm{tr} Hx)}\, dx.$$

These integrals are absolutely convergent for $\mathrm{Re}(s) \gg 0$ and can be meromorphically continued to the whole complex plane. If s is not a pole of $M(s,\rho)$, then $M(s,\rho)\Phi(g) \in I(\rho^{-1}, -s)$. Moreover, it is known that $\mathrm{Wh}_H(s)$ is entire.

Proposition 3.1. (Kudla and Sweet) *The following functional equation holds:*

$$\mathrm{Wh}_H(-s) \circ M(s,\rho) = \kappa_H(s,\rho,\psi)\mathrm{Wh}_H(s),$$

where

$$\begin{aligned} \kappa_H(s,\rho,\psi) =& \rho(\det H)^{-1} |\det H|^{-s} \gamma(E/F,\psi)^{m(m-1)/2} \\ & \times \underline{\chi}(\det H)^{m-1} \prod_{r=1}^{m} \varepsilon'(s-m+r, \rho\underline{\chi}^{r-1}, \psi)^{-1}. \end{aligned}$$

Here $\gamma(E/F,\psi)$ is the Weil factor for E/F with respect to ψ, and $\varepsilon'(s,\rho,\psi) = \varepsilon(s,\rho,\psi)L(1-s,\rho^{-1})L(s,\rho)^{-1}$. This proposition is Proposition 3.1 of Kudla and Sweet [17] when E is a quadratic extension of F. When $E = F \oplus F$, see [17], p. 303.

We assume ψ has order 0 and $\rho = \mathbf{1}$ is the trivial character. Let $\Phi_0^{(s)} \in I(s,\mathbf{1})$ be the unique element such that $\Phi_0(g) = 1$ for $g \in G \cap \mathrm{SL}_{2m}(\mathfrak{o})$. Then $\mathrm{Wh}_H(s)\Phi_0^{(s)}(\mathbf{1}_{2m})$ is an analogue of the Siegel series considered in the last section. It is known (cf. Shimura [24]) that there exists a polynomial $F_v(H;X) \in \mathbb{Z}[X]$ such that $\mathrm{Wh}_H(s)\Phi_0^{(s)}(\mathbf{1}_{2m})$ is equal to

$$|\mathfrak{D}|^{m(m-1)/4} \prod_{r=1}^{m} \frac{1}{L(s+m+1-r, \underline{\chi}^{r-1})} \cdot F_v(H; q^{-s-m}).$$

By standard Gindikin-Karpelevich argument, we have

$$M(s,1)\Phi_0^{(s)} = |\mathfrak{D}|^{m(m-1)/2} \prod_{r=1}^{m} \frac{L(s-m+r, \underline{\chi}^{r-1})}{L(s+m+1-r, \underline{\chi}^{r-1})} \Phi_0^{(-s)}.$$

It follows that

$$\begin{aligned}
&\mathrm{Wh}_H(-s)\circ M(s,1)\Phi_0^{(s)}(\mathbf{1}_{2m})\\
&\qquad=|\mathfrak{D}|^{m(m-1)/2}\prod_{r=1}^{m}\frac{L(s-m+r,\underline{\chi}^{r-1})}{L(s+m+1-r,\underline{\chi}^{r-1})}\\
&\qquad\quad\times\prod_{r=1}^{m}\frac{1}{L(-s+m-r+1,\underline{\chi}^{r-1})}\cdot F_v(H;q^{s-m}).
\end{aligned}$$

By Proposition 3.1, we have

$$\begin{aligned}
&|\det H|^{-s}\gamma(E/F,\psi)^{m(m-1)/2}\underline{\chi}(\det H)^{m-1}\\
&\quad\times\prod_{r=1}^{m}\varepsilon(s-m+r,\underline{\chi}^{r-1},\psi)^{-1}\cdot F_v(H;q^{-s-m})\\
&=|\mathfrak{D}|^{m(m-1)/4}F_v(H;q^{s-m}).
\end{aligned}$$

Since $\varepsilon(s,\underline{\chi},\psi)=\gamma(E/F,\psi)^{-1}|\mathfrak{D}|^{s-(1/2)}$ and $\gamma(E/F,\psi)^2=\underline{\chi}(-1)$, we obtain the following functional equation for $F_v(H;X)$.

Corollary 3.2. *The polynomial $F_v(H;X)$ satisfies the following functional equation:*
If $m=2n$,

$$F_v(H;q^{-2m}X^{-1})=\underline{\chi}((-1)^n\det H)(q^mX)^{-\operatorname{ord}(\mathfrak{D}^n\det H)}F_v(H;X).$$

If $m=2n+1$,

$$F_v(H;q^{-2m}X^{-1})=(q^mX)^{-\operatorname{ord}(\mathfrak{D}^n\det H)}F_v(H;X).$$

Remark 3.3. When $E/F=K_p/\mathbb{Q}_p$, we obtain the functional equation of $F_p(H;X)$. Note that $\underline{\chi}_p(-1)=\underline{\chi}_p(-D)$, since $\underline{\chi}_p(D)=\left(\frac{-D,D}{\mathbb{Q}_p}\right)=1$.

4. Fourier coefficients of hermitian Eisenstein series

For simplicity, we assume that l is a sufficiently large integer. Let $E_{2l}^{(m)}(Z,s)$ be the Eisenstein series of weight $2l$ on the hermitian upper-half space $\mathcal{H}_m$. For $Z\in\mathcal{H}_m$, we put $X=(Z+{}^t\bar{Z})/2$ and $Y=(Z-{}^t\bar{Z})/(2\sqrt{-1})$. Then the hermitian Eisenstein series $E_{2l}^{(m)}(Z,s)$ is defined by

$$E_{2l}^{(m)}(Z,s)=(\det Y)^{s-l}\sum_{g\in\Gamma_{K,\infty}^{(m)}\backslash\Gamma_K^{(m)}}(\det g)^l j(g,Z)^{-2l}\,|j(g,Z)|^{-2s+2l}$$

$E_{2l}^{(m)}(Z,s)$ is absolutely convergent for $\mathrm{Re}(s)>m$ and

$$E_{2l}^{(m)}(*,s)|_{2l}\,g=(\det g)^{-l}E_{2l}^{(m)}(*,s)$$

for any $g \in \Gamma_K^{(m)}$. Then the Fourier expansion of $E_{2l}^{(m)}(Z,s)$ is as follows.

$$E_{2l}^{(m)}(Z,s) = \sum_{H \in \Lambda_m(\mathcal{O})} c_{2l}^{(m)}(H;Y,s)\mathbf{e}(HX).$$

If H is non-degenerate, we have

$$\begin{aligned} c_{2l}^{(m)}(H;Y,s) &= \left(\frac{D}{4}\right)^{-m(m-1)/4} (\det Y)^{s-l}\, \Xi(Y,H;s+l,s-l) \\ &\qquad \times \prod_p b_p(H,p^{-2s}) \\ &= \left(\frac{D}{4}\right)^{-m(m-1)/4} (\det Y)^{s-l} \frac{\Xi(Y,H;s+l,s-l)}{\prod_{i=1}^m L(2s+1-i,\chi^{i-1})} \\ &\qquad \times \prod_{p \mid \gamma(H)} F_p(H;p^{-2s}). \end{aligned}$$

Here,

$$\Xi(g,h;s,s') = \int_{\mathcal{H}_m(\mathbb{C})} \mathbf{e}(-hx)\det(x+\sqrt{-1}g)^{-s}\det(x-\sqrt{-1}g)^{-s'}\,dx.$$

(See Shimura [24].) When $s=l$, we get the Fourier expansion of the holomorphic hermitian Eisenstein series $E_{2l}^{(m)}(Z) \in M_{2l}(\tilde{\Gamma}_K^{(m)};\det^{-l})$. If $H \in \Lambda_m(\mathcal{O})^+$,

$$\Xi(Y,H;2l,0) = \frac{(-1)^{ml}2^{m(2l-m+1)}\pi^{2ml}}{\Gamma_m(2l)}(\det H)^{2l-m}\mathbf{e}(\sqrt{-1}HY),$$

where

$$\Gamma_m(s) = \pi^{m(m-1)/2}\prod_{i=1}^m \Gamma(s+1-i).$$

The H-th Fourier coefficient of $E_{2l}^{(m)}(Z)$ for $H \in \Lambda_m(\mathcal{O})^+$ is equal to

$$\frac{(-1)^{ml}2^{(4ml-m^2+m)/2}D^{-m(m-1)/4}\pi^{2ml}}{\Gamma_m(2l)\prod_{i=1}^m L(2l+1-i,\chi^{i-1})}(\det H)^{(2l-m)}\prod_{p \mid \gamma(H)} F_p(H;p^{-2l}).$$

It follows that the H-th Fourier coefficient of $E_{2l}^{(m)}(Z)$ for $H \in \Lambda_m(\mathcal{O})^+$ is equal to the product of

$$A_m 2^m \prod_{i=1}^m L(i-2l,\chi^{i-1})^{-1}$$

and

$$|\gamma(H)|^{2l-m}\prod_{p \mid \gamma(H)} F_p(H;p^{-2l}) = |\gamma(H)|^{l-(m/2)}\prod_{p \mid \gamma(H)} \tilde{F}_p(H;p^{l-(m/2)}).$$

Here

$$A_m = \begin{cases} 1 & \text{if } m = 2n+1, \\ (-1)^n & \text{if } m = 2n. \end{cases}$$

Observe that

$$\begin{aligned} \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{l-(m/2)}) &= \prod_{p \mid \gamma(H)} \underline{\chi}_p(\gamma(H))^{m-1} \tilde{F}_p(H; p^{-l+(m/2)}) \\ &= A_m \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{-l+(m/2)}) \end{aligned}$$

by Lemma 2.2. We define the normalized Eisenstein series by

$$\mathcal{E}_{2l}^{(m)}(Z) = 2^{-m} \prod_{i=1}^{m} L(i-2l, \chi^{i-1}) \cdot E_{2l}^{(m)}(Z) \in M_{2l}(\Gamma_K^{(m)}, \det{}^{-l}).$$

When $m = 2n+1$, the H-th Fourier coefficient of $\mathcal{E}_{2k'+2n}^{(2n+1)}(Z)$ is equal to

$$|\gamma(H)|^{k'-(1/2)} \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{-k'+(1/2)})$$

for any $H \in \Lambda_{2n+1}(\mathcal{O})^+$ and any sufficiently large integer k'. When $m = 2n$, the H-th Fourier coefficient of $\mathcal{E}_{2k'+2n}^{(2n)}(Z)$ is equal to

$$|\gamma(H)|^{k'} \prod_{p \mid \gamma(H)} \tilde{F}_p(H; p^{-k'})$$

for any $H \in \Lambda_{2n}(\mathcal{O})^+$ and any sufficiently large integer k'.

5. Main theorems

We first consider the case when $m = 2n$ is even. We refer this case as Case E. In this case, let $f(\tau) = \sum_{N=1}^{\infty} a(N) q^N \in S_{2k+1}(\Gamma_0(D), \chi)$ be a primitive form, whose L-function is given by

$$L(f, s) = \prod_{p \nmid D} (1 - a(p)p^{-s} + \chi(p)p^{2k-2s})^{-1} \prod_{q \mid D} (1 - a(q)q^{-s})^{-1}.$$

For each prime $p \nmid D$, we define the Satake parameter $\{\alpha_p, \beta_p\} = \{\alpha_p, \chi(p)\alpha_p^{-1}\}$ by

$$(1 - a(p)X + \chi(p)p^{2k}X^2) = (1 - p^k \alpha_p X)(1 - p^k \beta_p X).$$

For $q \mid D$, we put $\alpha_q = q^{-k} a(q)$.

For each $H \in \Lambda_{2n}(\mathcal{O})^+$, we put

$$A(H) = |\gamma(H)|^k \prod_{p \mid \gamma(H)} \tilde{F}_p(H; \alpha_p).$$

Here p extends over all primes which divide $\gamma(H)$. We put

$$F(Z) = \sum_{H \in \Lambda_{2n}(\mathcal{O})^+} A(H)\mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n}.$$

Then our main theorem for Case E is as follows:

Theorem 5.1. (Case E) *Assume that $m = 2n$. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n)}, \det^{-k-n})$.*

Now we consider the case when $m = 2n+1$ is odd. We refer this case as Case O. In this case, let $f(\tau) = \sum_{N=1}^{\infty} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform, whose L-function is given by

$$L(f,s) = \prod_p (1 - a(p)p^{-s} + p^{2k-1-2s})^{-1}.$$

For each prime p, we define the Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$ by

$$(1 - a(p)X + p^{2k-1}X^2) = (1 - p^{k-(1/2)}\alpha_p X)(1 - p^{k-(1/2)}\alpha_p^{-1} X).$$

Put

$$A(H) = |\gamma(H)|^{k-(1/2)} \prod_{p|\gamma(H)} \tilde{F}_p(H; \alpha_p), \quad H \in \Lambda_{2n+1}(\mathcal{O})^+,$$

$$F(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})^+} A(H)\mathbf{e}(HZ), \quad Z \in \mathcal{H}_{2n+1}.$$

Then our main theorem for Case O is as follows:

Theorem 5.2. (Case O) *Assume that $m = 2n+1$. Let $f(\tau)$, $A(H)$ and $F(Z)$ be as above. Then we have $F \in S_{2k+2n}(\Gamma_K^{(2n+1)}, \det^{-k-n})$.*

In both Case E and Case O, the definition of $F(Z)$ is independent of the choice of α_p by Lemma 2.2. Observe that $F(Z)$ is absolutely convergent on $\mathcal{H}_m$ by Lemma 2.3 for any m. Since $F_p({}^t\bar{A}HA; X) = F_p(H; X)$ for any $H \in \Lambda_m(\mathcal{O})$ and any $A \in \mathrm{GL}_m(\mathcal{O})$, we have $F|_{2k+2n}\, g = (\det g)^{-k-n} F$ for any $g \in \Gamma_{K,\infty}^{(m)}$. We call $F(Z)$ the lift of $f(\tau)$ to $S_{2k+2n}(\Gamma_K^{(m)}, \det^{-k-n})$ and denote it by $\mathrm{Lift}^{(m)}(f)$.

6. Fourier-Jacobi expansions

We define the Jacobi group $J_{m,r}(\mathcal{O})$ by

$$\left\{ M = \left(\begin{array}{cc|cc} \mathbf{1}_r & * & * & * \\ 0 & * & * & * \\ \hline 0 & 0 & \mathbf{1}_r & 0 \\ 0 & * & * & * \end{array} \right) \in \Gamma_K^{(m)} \,\middle|\, \det M = 1 \right\}.$$

We consider only the case $r = m-1$. Fix $S \in \Lambda_{m-1}(\mathcal{O})^+$.

For a holomorphic function $\phi(\tau, z_1, z_2)$ on $\mathfrak{H}_1 \times \mathbb{C}^{m-1} \times \mathbb{C}^{m-1}$, we define a function on $\mathcal{H}_{m-1} \times \mathfrak{H}_1 \times \mathbb{C}^{m-1} \times \mathbb{C}^{m-1}$ by $\widetilde{\phi}(Z) = \mathbf{e}(S\omega)\phi(\tau, z_1, z_2)$. Here

$$Z = \begin{pmatrix} \omega & z_1 \\ {}^t z_2 & \tau \end{pmatrix}, \qquad \omega \in \mathcal{H}_{m-1}, \tau \in \mathfrak{H}_1, z_1, z_2 \in \mathbb{C}^{m-1}.$$

We shall say that the function ϕ is a weak Jacobi form of index S and weight l, if and only if

$$\widetilde{\phi}|_l \, M = \widetilde{\phi}$$

for any $M \in J_{m,m-1}(\mathcal{O})$. A weak Jacobi form is called a Jacobi form if ϕ has a Fourier expansion

$$\phi(\tau, z_1, z_2) = \sum_{x \in (\mathcal{O}^\sharp)^{m-1}} \sum_{N \in \mathbb{Z}} c(x, N) \mathbf{e}({}^t\overline{x} z_1 + {}^t x z_2) \mathbf{e}(N\tau)$$

such that $c(x, N) = 0$ unless $N - {}^t\overline{x} S^{-1} x \geq 0$.

For each $\xi \in K^{m-1}$, we define the theta function $\theta_{[\xi]}(S; \tau, z_1, z_2)$ by

$$\theta_{[\xi]}(S; \tau, z_1, z_2) = \sum_{x \in \mathcal{O}^{m-1}} \mathbf{e}({}^t\overline{(x+\xi)} S (x+\xi)\tau + {}^t\overline{(x+\xi) S} z_1 + {}^t(x+\xi) S z_2).$$

Choose a complete representative $\Xi = \Xi(S)$ for $S^{-1}(\mathcal{O}^\sharp)^{m-1}/\mathcal{O}^{m-1}$.

Then a Jacobi form $\phi(\tau, z_1, z_2)$ of index S can be expressed as a sum

$$\begin{aligned} \phi(\tau, z_1, z_2) &= \sum_{\xi \in \Xi} \theta_{[\xi]}(S; \tau, z_1, z_2) \phi_\xi(\tau), \\ \phi_\xi(\tau) &= \sum_{\substack{N \in \mathbb{Z} \\ N - {}^t\overline{\xi} S \xi \geq 0}} c(S\xi, N) \mathbf{e}((N - {}^t\overline{\xi} S \xi)\tau). \end{aligned}$$

It is well-known that for each $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there exists a unitary representation $u_S : \mathrm{SL}_2(\mathbb{Z}) \to \mathrm{GL}(\mathbb{C}^\Xi)$ with kernel containing some congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that

$$\begin{aligned} &\theta_{[\xi]}(S; \tau', z_1', z_2') \\ =&(c\tau + d)^{m-1} \sum_{\eta \in \Xi} \overline{u_S(\gamma)_{\eta\xi}} \, \mathbf{e}(S z_1 (c\tau + d)^{-1} c \, {}^t z_2) \theta_{[\eta]}(S; \tau, z_1, z_2), \end{aligned}$$

where $\tau' = (a\tau + b)(c\tau + d)^{-1}$, $z_i' = z_i(c\tau + d)^{-1}$, $(i = 1, 2)$. It follows that

$$\phi_\xi(\gamma\langle\tau\rangle) = (c\tau + d)^{l-m+1} \sum_{\eta \in \Xi} u_S(\gamma)_{\eta\xi} \, \phi_\eta(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and any $\xi \in \Xi$.

Let $\mathfrak{F}(Z)$ be a holomorphic function on $\mathcal{H}_m$ which has a Fourier expansion

$$\mathfrak{F}(Z) = \sum_{H \geq 0} \mathfrak{A}(H)\mathbf{e}(HZ).$$

Here H extends over positive semi-definite elements of $\Lambda_m(\mathcal{O})$. We assume that $\mathfrak{A}({}^t\bar{X}HX) = \mathfrak{A}(H)$ for any $X \in \mathrm{GL}_m(\mathcal{O})$ and any $H \in \Lambda_m(\mathcal{O})$. For each $S \in \Lambda_{m-1}(\mathcal{O})$, we put

$$\mathfrak{F}_S(\tau, z_1, z_2) = \sum_{H = \begin{pmatrix} S & x \\ {}^t\bar{x} & N \end{pmatrix}} \mathfrak{A}(H)\mathbf{e}(N\tau)\mathbf{e}({}^t\bar{x}z_1 + {}^txz_2).$$

Here H extends over all positive semi-definite elements of $\Lambda_m(\mathcal{O})$ which is of the form $H = \begin{pmatrix} S & x \\ {}^t\bar{x} & N \end{pmatrix}$. As in [9], we have an expansion

$$\mathfrak{F}_S(\tau, z_1, z_2) = \sum_{\xi \in \Xi} \theta_{[\xi]}(S; \tau, z_1, z_2)\mathfrak{F}_{S,\xi}(\tau),$$

where

$$\mathfrak{F}_{S,\xi}(\tau) = \sum_{\substack{N \in \mathbb{Z} \\ N - {}^t\bar{\xi}S\xi \geq 0}} \mathfrak{A}\left(\begin{pmatrix} S & S\xi \\ {}^t\bar{\xi}S & N \end{pmatrix}\right)\mathbf{e}((N - {}^t\bar{\xi}S\xi)\tau).$$

We call $\mathfrak{F}_{S,\xi}(\tau)$ the (S, ξ)-component of the Fourier-Jacobi expansion of $\mathfrak{F}(Z)$. Since the set $\{\theta_{[\xi]}(S; \tau, z_1, z_2) \mid \xi \in \Xi\}$ are linearly independent, $\mathfrak{F}_S(\tau, z_1, z_2)$ is a Jacobi form of index S and weight $2l$ if and only if

$$\mathfrak{F}_{S,\xi}(\gamma\langle\tau\rangle) = (c\tau + d)^{l-m+1} \sum_{\eta \in \Xi} u_S(\gamma)_{\eta\xi}\, \mathfrak{F}_{S,\eta}(\tau)$$

for any $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and any $\xi \in \Xi$.

7. **Vector-valued modular forms**

Let $\mathcal{K}_p = \mathrm{SL}_2(\mathbb{Z}_p)$ be the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{Q}_p)$. We put $\mathcal{K} = \prod_p \mathcal{K}_p$. Let (u, V) be a finite-dimensional continuous representation of $\mathcal{K}$. A V-valued modular form $\vec{h}(\tau)$ with type u is a holomorphic function of $\tau \in \mathfrak{H}_1$ with values in V which satisfies the following conditions (1) and (2):

(1) $\vec{h}(g\langle\tau\rangle) = (c\tau + d)^{\kappa}u(g)^{-1}\vec{h}(\tau)$ for any $g \in \mathrm{SL}_2(\mathbb{Z})$.

(2) $\vec{h}(\tau)$ has a Fourier expansion of the form

$$\vec{h}(\tau) = \sum_{N=0}^{\infty} \vec{c}(N) q^{N/M}$$

for some positive integer M.

We define vector-valued cusp forms similarly.

For each prime p, let $\mathcal{R}_p = \mathbb{C}[X_p, X_p^{-1}]$ be a copy of the Laurent polynomial ring. Put $\mathcal{R} = \otimes_p \mathcal{R}_p$. Then $\mathcal{R}$ is the ring of Laurent polynomials $\Phi(\mathbb{X}) = \Phi(X_2, X_3, \ldots) \in \mathbb{C}[X_2, X_2^{-1}, X_3, X_3^{-1}, \ldots]$. Let a_2, a_3,..., a_p, ... be non-zero complex numbers. Then the value of $\Phi(\mathbb{X})$ at $(X_2, X_3, \ldots, X_p, \ldots) = (a_2, a_3, \ldots, a_p, \ldots)$ is denoted by $\Phi(\{a_p\})$.

Lemma 7.1. *Let $\Phi(\mathbb{X})$ be an element of $\mathcal{R}$. Assume that $\Phi(\{p^{-s}\}) = 0$ for an infinite number of $s \in \mathbb{R}$. Then $\Phi(\mathbb{X})$ is identically 0.*

Proof. Write $\Phi(\mathbb{X})$ as a sum of monomials:

$$\Phi(\mathbb{X}) = \sum_{i=1}^{r} a_i \prod_p X_p^{e_{i,p}}.$$

Here $e_{i,p} = 0$ for almost all p. Put $N_i = \prod_p p^{e_{i,p}}$. Then our assumption implies $\sum a_i N_i^{-s} = 0$ for infinitely many real number s. Since $N_1, \ldots, N_r$ are mutually distinct, we have $a_1 = \cdots = a_r = 0$. □

Definition 7.2. Let $h(\tau)$ be a modular form of weight κ for some congruence subgroup Γ. Then we denote by $\mathcal{V}(h)$ the $\mathbb{C}$-vector space spanned by $\{h|\gamma \mid \gamma \in \mathrm{GL}_2(\mathbb{Q})^+\}$. If $u : \mathcal{K} \to \mathrm{GL}_d(\mathbb{C})$ is a representation of rank d, then $\mathcal{I}(\mathcal{V}(h)^d, u)$ is the space of $\mathbb{C}^d$-valued modular forms of type u whose entries belong to $\mathcal{V}(h)$.

We first consider Case E. For Case E, we put

$$E_{2k+1,\chi}(\tau) = -\frac{B_{2k+1,\chi}}{4k+2} + \sum_{N=1}^{\infty} \left(\sum_{d|N} \chi(d) d^{2k} \right) q^N \in M_{2k+1}(\Gamma_0(D), \chi).$$

Note that $L(s, E_{2k+1,\chi}) = \zeta(s) L(s-2k, \chi)$. In particular, the Satake parameter of $E_{2k+1,\chi}(\tau)$ is $\{p^{-k}, \chi(p)p^k\}$ for $p \nmid D$ and $\{q^{-k}\}$ for $q|D$.

Definition 7.3. (Case E) We define a compatible family of Eisenstein series $\{\mathcal{F}_{2k'+1}(\tau)\}_{k' \geq k'_0}$ as follows: A compatible family of Eisenstein series is a family of modular forms

$$\mathcal{F}_{2k'+1}(\tau) = b(2k'+1; 0) + \sum_{N \in \mathbb{Q}_+^\times} N^{k'} b(2k'+1; N) q^N$$

satisfying the following conditions (1), (2), and (3).

(1) $\mathcal{F}_{2k'+1} \in \mathcal{V}(E_{2k'+1,\chi})$ for any integer $k' \geq k'_0$.
(2) For each $N \in \mathbb{Q}_+^\times$, there exists an element $\Phi_N(\mathbb{X}) \in \mathcal{R}$ such that

$$b(2k'+1; N) = \Phi_N(\{p^{-k'}\}).$$

(3) There exists a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that $\mathcal{F}_{2k'+1} \in M_{2k'+1}(\Gamma)$ for all $k' \geq k'_0$.

As in [9], we have to prove the following lemma.

Lemma 7.4. *Let*

$$f(\tau) = \sum_{N>0} a(N) q^N \in S_{2k+1}(\Gamma_0(D), \chi)$$

be a primitive form and α_p a Satake parameter of $f(\tau)$. Assume that there are a finite-dimensional representation $(u, \mathbb{C}^d)$ of $\mathcal{K}$ and $\vec{\Phi}_N(\mathbb{X}) = {}^t(\Phi_{1,N}(\mathbb{X}), \dots, \Phi_{d,N}(\mathbb{X})) \in \mathcal{R}^d$, $(N \in \mathbb{Q}_+^\times)$ satisfying the following conditions (1) and (2):

(1) *For each sufficiently large integer k', there exists a vector-valued modular form*

$$\vec{\mathcal{F}}_{2k'+1}(\tau) = \vec{b}(2k'+1; 0) + \sum_{N \in \mathbb{Q}_+^\times} N^{k'} \vec{b}(2k'+1; N) q^N$$

of type u.

(2) *For each i $(1 \leq i \leq d)$, the i-th component $\mathcal{F}_{i,2k'+1}(\tau)$ of $\vec{\mathcal{F}}_{2k'+1}(\tau)$ is a compatible family of Eisenstein series such that*

$$b_i(2k'+1; N) = \Phi_{i,N}(\{p^{-k'}\}).$$

Here $\vec{b}(2k'+1; N) = {}^t(b_1(2k'+1; N), \dots, b_d(2k'+1; N))$.

Put

$$\vec{h}(\tau) = \sum_{N \in \mathbb{Q}_+^\times} N^k \vec{\Phi}_N(\{\alpha_p\}) q^N.$$

Then we have $\vec{h}(\tau) \in \mathcal{I}(\mathcal{V}(f)^d; u)$.

8. Behavior of the Whittaker functions

In this section, we will investigate the behavior of Whittaker functions on GL_2, which will be used to prove Lemma 7.4.

In this section, $F = F_v$ will denote a non-archimedean local field. We fix a non-trivial additive character ψ of F. The maximal order of F, the prime ideal, and the order of the residue field are denoted by $\mathfrak{o}$, $\mathfrak{p}$, and q, respectively. We put $\mathcal{R}_v = \mathbb{C}[q^s, q^{-s}]$ and $\tilde{\mathcal{K}} = \mathrm{GL}_2(\mathfrak{o})$. The Borel subgroup of GL_2 which consists of all upper triangular matrices are denoted by $\tilde{B}$. We put $B = \tilde{B} \cap \mathrm{SL}_2$, and $\mathcal{K} = \tilde{\mathcal{K}} \cap \mathrm{SL}_2$.

Let $\underline{\chi}$, $\underline{\chi}_1$ and $\underline{\chi}_2$ be characters of $F^\times$. The principal series representation $\tilde{I}(\underline{\chi}_1 \boxtimes \underline{\chi}_2, s) = \mathrm{Ind}_{\tilde{B}}^{\mathrm{GL}_2}(\underline{\chi}_1|\ |^s \boxtimes \underline{\chi}_2|\ |^{-s})$ is the representation of GL_2 which is induced from the character of $\tilde{B}$ given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto \underline{\chi}_1(a)\underline{\chi}_2(d)|ad^{-1}|^s.$$

Similarly, the principal series representation $I(\underline{\chi}, s) = \mathrm{Ind}_B^{\mathrm{SL}_2}(\underline{\chi}|\ |^s)$ is the representation of SL_2 which is induced from the character of B given by

$$\begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \mapsto \underline{\chi}(a)|a|^s.$$

Then the restriction of $\tilde{I}(\underline{\chi}_1 \boxtimes \underline{\chi}_2, s)$ to SL_2 is canonically isomorphic to $I(\underline{\chi}_1\underline{\chi}_2^{-1}, 2s)$.

In this section, we assume $\underline{\chi}$ is a unitary character.

Lemma 8.1. *Let $\underline{\chi}$ be a character of $F^\times$ such that $\underline{\chi}^2 = 1$. Any irreducible representation of $\mathcal{K}$ is multiplicity free in $\tilde{I}(\mathbf{1} \boxtimes \underline{\chi}^{-1}, s)$. Moreover, the set of irreducible representations of $\mathcal{K}$ which occur in $\tilde{I}(\mathbf{1} \boxtimes \underline{\chi}^{-1}, s)$ is independent of s.*

Proof. It is enough to consider the restriction of $\tilde{I}(\mathbf{1} \boxtimes \underline{\chi}^{-1}, s)$ to SL_2. Note that the restriction is isomorphic to $I(\underline{\chi}, 2s)$. Put $\mathcal{B} = B \cap \mathcal{K}$ and $\mathcal{N} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \middle| \, n \in \mathfrak{o} \right\}$. We are going to prove that the induced representation $\mathrm{Ind}_{\mathcal{B}}^{\mathcal{K}}\underline{\chi}$ is multiplicity-free. It is enough to prove that the Hecke algebra

$$\begin{aligned} &\mathcal{H}\left(\mathcal{B}\backslash\mathcal{K}/\mathcal{B}; \underline{\chi}\right) \\ &= \left\{ \phi \in C^\infty(\mathcal{K}) \mid \phi(b_1 k b_2) = \underline{\chi}(b_1 b_2)\phi(k), b_1, b_2 \in \mathcal{B}, k \in \mathcal{K} \right\} \end{aligned}$$

is commutative.

We consider the anti-involution ${}^\tau\!\begin{pmatrix} x & y \\ z & w \end{pmatrix} = \begin{pmatrix} w & y \\ z & x \end{pmatrix}$ of $\mathcal{K}$. We shall prove that for any $k = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \in \mathcal{K}$, there exist $b_1, b_2 \in \mathcal{B}$ such that ${}^\tau k = b_1 k b_2$ and $\underline{\chi}(b_1 b_2) = 1$. If $z \in \mathfrak{o}^\times$, then one can easily find $n_1, n_2 \in \mathcal{N}$ such that ${}^\tau k = n_1 k n_2$. If $z \in \mathfrak{p}$, then $k = b \begin{pmatrix} 1 & 0 \\ z' & 1 \end{pmatrix}$ for some $z' \in \mathfrak{p}$ and $b \in \mathcal{B}$. In this case one can choose $b_1 = b^{-1}$, $b_2 = {}^\tau b$. This completes the proof of the first assertion. The last assertion of the lemma is clear. $\square$

We need to investigate the behavior of Whittaker functions at the points of reducibility of $I(\underline{\chi}, 2s)$. Since $I(\underline{\chi}|\ |^{2s'}, 2s) \simeq I(\underline{\chi}, 2(s+s'))$, we have only to consider real s by changing $\underline{\chi}$ if necessary. If $\underline{\chi}^2 \neq 1$, then there are no real points of reducibility of $I(\underline{\chi}, 2s)$. When $\underline{\chi} = 1$, the real points of reducibility of $I(\mathbf{1}, 2s)$ are $s = \pm\frac{1}{2}$. $I(\mathbf{1}, 1)$ contains the Steinberg representation St, and the quotient of $I(\mathbf{1}, 1)/\mathrm{St}$ is the trivial representation $\mathbf{1}$. $I(\mathbf{1}, -1)$ contains $\mathbf{1}$, and $I(\mathbf{1}, -1)/\mathbf{1} \simeq \mathrm{St}$. When $\underline{\chi}^2 = 1$, $\underline{\chi} \neq 1$, then the real point of reducibility of $I(\underline{\chi}, 2s)$ is $s = 0$. In this case, $I(\underline{\chi}, 0)$ is the direct sum of two irreducible representations $I(\underline{\chi}, 0)^+ \oplus I(\underline{\chi}, 0)^-$. This decomposition is described in terms of the normalized intertwining operator $M^*(2s, \underline{\chi}) = \varepsilon(2s, \underline{\chi}, \psi)M(2s, \underline{\chi})$. The irreducible constituents $I(\underline{\chi}, 0)^+$ and $I(\underline{\chi}, 0)^-$ are the spaces of elements of $I(\underline{\chi}, 0)$ on which $M^*(0, \underline{\chi})$ acts by 1 and by -1, respectively.

An $\mathcal{R}_v$-valued function on GL_2 can be regarded as a function $f(g, s)$ on $\mathrm{GL}_2 \times \mathbb{C}$ such that $f(g, s) \in \mathcal{R}_v = \mathbb{C}[q^s, q^{-s}]$ for each $g \in \mathrm{GL}_2$. We say that a function on $\mathrm{GL}_2 \times \mathbb{C}$ or on $\mathrm{SL}_2 \times \mathbb{C}$ is right $\mathcal{K}$-finite if there is a open subgroup $\mathcal{K}_c$ of $\mathcal{K}$ such that $f(g, s)$ is right $\mathcal{K}_c$-invariant for any s. We also consider $\mathcal{R}_v$-valued functions and right $\mathcal{K}$-finiteness for SL_2.

Definition 8.2. A right $\mathcal{K}$-finite $\mathcal{R}_v$-valued function $f(g, s)$ on GL_2 is called a holomorphic section of $\tilde{I}(\underline{\chi}_1 \boxtimes \underline{\chi}_2, s)$, if $f(g, s) \in \tilde{I}(\underline{\chi}_1 \boxtimes \underline{\chi}_2, s)$ for each $s \in \mathbb{C}$. A holomorphic section $f(g, s)$ of $\tilde{I}(\underline{\chi}_1 \boxtimes \underline{\chi}_2, s)$ is called a standard section if the restriction of $f(g, s)$ to $\mathcal{K} \times \mathbb{C}$ does not depend on $s \in \mathbb{C}$. We define holomorphic sections and standard sections of $I(\underline{\chi}, 2s)$ similarly.

For $f \in I(\underline{\chi}, 2s)$ or $f \in \tilde{I}(\underline{\chi}_1 \boxtimes \underline{\chi}_2, s)$, we put

$$\mathrm{Wh}_\psi(f)(g) = \int_F f\left(w \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\psi(x)} dx,$$

where $w = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$. This integral is absolutely convergent if $\mathrm{Re}(s) > 0$. If $f(g, s)$ is a holomorphic section of $I(\underline{\chi}, 2s)$, then $\mathrm{Wh}_\psi(f(g, s))$ is an $\mathcal{R}_v$-valued function on SL_2. Thus $\mathrm{Wh}_\psi(f)$ is meaningful for any $s \in \mathbb{C}$. We denote $\mathcal{W}_\psi(I(\underline{\chi}, 2s)) = \{\mathrm{Wh}_\psi(f) \big| f \in I(\underline{\chi}, 2s)\}$ for each $s \in \mathbb{C}$. It is known that $\mathcal{W}_\psi(I(\underline{\chi}, 2s)) \neq (0)$ for any $s \in \mathbb{C}$. If $I(\underline{\chi}, 2s)$ is irreducible, then $\mathcal{W}_\psi(I(\underline{\chi}, 2s))$ is equal to the Whittaker space of $I(\underline{\chi}, 2s)$. Conversely we can show the following lemma.

Lemma 8.3. *Let $\underline{\chi}$ be a character of $F^\times$. Let $W(g, s)$ be a right $\mathcal{K}$-finite $\mathcal{R}_v$-valued function on SL_2 such that $W(g, s) \in \mathcal{W}_\psi(I(\underline{\chi}, 2s))$ for* *each $s \in \mathbb{C}$. Then there exists a holomorphic section $f(g,s)$ of $I(\underline{\chi}, 2s)$ such that $W(g,s) = \mathrm{Wh}_\psi(f)(g,s)$.*

Proof. First notice that the validity of the lemma does not depend on the choice of ψ. If s is not a point of reducibility of $I(\underline{\chi}, 2s)$, then there exists a unique $f(g,s)$ such that $\mathrm{Wh}_\psi(f)(g,s) = W(g,s)$. Then $f(g,s)$ is meromorphic in the sense that there exists a holomorphic function $\phi(s) \in \mathcal{R}_v$ such that $\phi(s)f(g,s)$ is a holomorphic section. We have to prove that $f(g,s)$ is holomorphic everywhere. Obviously, if $\mathrm{Ker}[\mathrm{Wh}_\psi : I(\underline{\chi}, 2s_0) \to \mathcal{W}(I(\underline{\chi}, 2s_0))] = 0$, then $f(g,s)$ is holomorphic at $s = s_0$.

Fix $s_0 \in \mathbb{C}$ such that $\mathrm{Ker}[\mathrm{Wh}_\psi : I(\underline{\chi}, 2s_0) \to \mathcal{W}_\psi(I(\underline{\chi}, 2s_0))] \neq 0$. It is known that $\mathrm{Ker}[\mathrm{Wh}_\psi : I(\underline{\chi}, 2s_0) \to \mathcal{W}_\psi(I(\underline{\chi}, 2s_0))]$ is an irreducible representation of SL_2. Let X (resp. X_0) be the set of irreducible $\mathcal{K}$-types which occur in $I(\underline{\chi}, 2s_0)$ (resp. $\mathrm{Ker}[\mathrm{Wh}_\psi : I(\underline{\chi}, 2s_0) \to \mathcal{W}_\psi(I(\underline{\chi}, 2s_0))]$). For each standard section f, we put

$$\alpha(f) = \min_{g \in \mathrm{SL}_2} \left(\mathrm{ord}_{s=s_0} \mathrm{Wh}_\psi(f)(g,s)\right).$$

Let f_u be a standard section with $\mathcal{K}$-type u. Then $\alpha(f_u)$ does not depend on the choice of f_u. Obviously, $\alpha(f_u) = 0$ for $u \notin X_0$ and $\alpha(f_u) \geq 1$ for $u \in X_0$. In fact, $\alpha(f_u)$ depends only on u. Let $u, u' \in X_0$ be irreducible $\mathcal{K}$-types and $f_u(g,s)$ and $f_{u'}(g,s)$ be standard sections with $\mathcal{K}$-types u and u', respectively. Then there exists a Hecke operator $T \in C_c(\mathrm{SL}_2)$ such that $f_u * T = t(s) f_{u'}$ for some $t(s) \in \mathcal{R}_v$, $t(s_0) \neq 0$. It follows that $\alpha(f_u) \geq \alpha(f_{u'})$. Changing the roles of u and u', we have $\alpha(f_u) = \alpha(f_{u'})$. Thus to prove the lemma, it is enough to find a standard section f such that $\alpha(f) = 1$.

We have to consider the following cases:

(1) $\underline{\chi} = 1$ and $s_0 = -\frac{1}{2}$.
(2) $\underline{\chi}$ is the unramified character such that $\underline{\chi}(\varpi) = -1$ and $s_0 = 0$.
(3) $\underline{\chi}$ is a ramified character such that $\underline{\chi}^2 = 1$ and $s_0 = 0$.

In the case (1) and (2), assume that the order of ψ is 1, i.e., the largest ideal of $\mathfrak{o}$ on which ψ is trivial is $\mathfrak{p}$. It is well-known that the trivial $\mathcal{K}$-type $u = 1 \in X_0$ in these cases. Let f_u be the standard section whose restriction of $\mathcal{K}$ is identically 1. A direct calculation shows that the Whittaker function of f_u satisfies

$$\mathrm{Wh}_\psi(f_u)\left(\begin{pmatrix} \varpi^n & 0 \\ 0 & \varpi^{-n} \end{pmatrix}, s\right) = q^{1/2}(\epsilon q^{-2s-1})^n(\epsilon q^{2s} - q^{-1})\frac{(1-q^{4ns})}{1-\epsilon q^{2s}}$$

for $n \geq 0$. Here $\epsilon = \underline{\chi}(\varpi)$. It follows that $\alpha(f_u) = 1$ in the case (1) and (2).

Now we consider the case (3). We assume that the order of ψ is 0. In this case, $\mathrm{Ker}[\mathrm{Wh}_\psi : I(\underline{\chi}, 0) \to \mathcal{W}_\psi(I(\underline{\chi}, 0))] = I(\underline{\chi}, 0)^-$. Let $\mathfrak{d} = \mathfrak{p}^f$ be the conductor of $\underline{\chi}$ and $\tilde{\mathcal{K}}_0$ be the compact open subgroup of $\tilde{\mathcal{K}}$ given by

$$\tilde{\mathcal{K}}_0 = \left\{ k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\mathcal{K}} \,\middle|\, c \in \mathfrak{d} \right\}.$$

We put $\mathcal{K}_0 = \mathcal{K} \cap \tilde{\mathcal{K}}_0$.

Let $\tilde{f}_1(g,s)$ and $\tilde{f}_2(g,s)$ be standard sections of $\tilde{I}(\mathbf{1} \boxtimes \underline{\chi}, s)$, whose restrictions to $\tilde{\mathcal{K}}$ are as follows:

$$\tilde{f}_1(k,s) = \begin{cases} \underline{\chi}(d) & k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\mathcal{K}}_0 \\ 0 & k \notin \tilde{\mathcal{K}}_0, \end{cases}$$

$$\tilde{f}_2(nwk,s) = \begin{cases} \underline{\chi}(d) & n \in \mathcal{N}, k = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\mathcal{K}}_0 \\ 0 & k \notin \mathcal{N} w \tilde{\mathcal{K}}_0. \end{cases}$$

Here, $\mathcal{N} = \left\{ \begin{pmatrix} 1 & n \\ 0 & 1 \end{pmatrix} \,\middle|\, n \in \mathfrak{o} \right\}$. We denote the restriction of $\tilde{f}_1(g,s)$ and $\tilde{f}_2(g,s)$ to SL_2 by $f_1(g,s)$ and $f_2(g,s)$, respectively. By direct calculation, one can show

$$\mathrm{Wh}_\psi(\tilde{f}_1)\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, s \right) = \begin{cases} \underline{\chi}(t)|t|^{-s+(1/2)} & |t| \le 1 \\ 0 & |t| > 1 \end{cases}$$

$$\mathrm{Wh}_\psi(\tilde{f}_2)\left(\begin{pmatrix} t & 0 \\ 0 & 1 \end{pmatrix}, s \right) = \begin{cases} \bar{\kappa}|\mathfrak{d}|^{2s+1}|t|^{s+(1/2)} & |t| \le 1 \\ 0 & |t| > 1, \end{cases}$$

where

$$\kappa = \int_{\varpi^{-f}\mathfrak{o}^\times} \underline{\chi}(x)\psi(x)\,dx.$$

We define the normalized intertwining operator $M^*(2s, \underline{\chi})$ by

$$M^*(2s, \underline{\chi}) = \varepsilon(2s, \underline{\chi}, \psi) M(2s, \underline{\chi}).$$

Then $M^*(-2s, \underline{\chi}) \circ M^*(2s, \underline{\chi}) = \mathrm{id}$ by Proposition 3.1. The calculation above shows

$$\underline{\chi}(\det g)\mathrm{Wh}_\psi(\tilde{f}_2)(g, -s) = \bar{\kappa}|\mathfrak{d}|^{2s+1}\mathrm{Wh}_\psi \circ M^*(2s, \underline{\chi})(\tilde{f}_1)(g,s).$$

Note that a Whittaker function is determined by its restriction to the diagonal set by the uniqueness of the Kirillov model. It follows that

$$\underline{\chi}(\det g)\tilde{f}_2(g, -s) = \bar{\kappa}|\mathfrak{d}|^{2s+1} M^*(2s, \underline{\chi})(\tilde{f}_1)(g,s).$$

Applying $M^*(-2s,\underline{\chi})$ both sides and changing s by $-s$, we have

$$\underline{\chi}(\det g)M^*(2s,\underline{\chi})\tilde{f}_2(g,s)=\bar{\kappa}|\mathfrak{d}|^{-2s+1}\tilde{f}_1(g,-s).$$

Put

$$\tilde{f}(g,s)=\tilde{f}_1(g,s)-\bar{\kappa}^{-1}|\mathfrak{d}|^{-1}\tilde{f}_2(g,s).$$

Then $\tilde{f}(g,s)$ is a standard section of $\tilde{I}(\mathbf{1}\boxtimes\underline{\chi},s)$ and $M^*(2s,\underline{\chi})\tilde{f}(g,0)=-\underline{\chi}(\det g)\tilde{f}(g,0)$. Denote the restriction of $\tilde{f}(g,0)$ to SL_2 by $f(g)$. Then we have

$$f\in I(\underline{\chi},0)^-=\mathrm{Ker}[\mathrm{Wh}_\psi\colon I(\underline{\chi},0)\to\mathcal{W}_\psi(I(\underline{\chi},0))].$$

Moreover, the calculation of the Whittaker function shows $\alpha(f)=1$. □

Lemma 8.4. *Let $\underline{\chi}$ be a character of $F^\times$. Let $W(g,s)$ be a right $\mathcal{K}$-finite function on $\mathrm{GL}_2\times\mathbb{C}$ satisfying the following conditions (1) and (2):*

(1) *$W(g,s)\in\mathcal{W}_\psi(\tilde{I}(\mathbf{1}\boxtimes\underline{\chi},s))$ for each $s\in\mathbb{C}$,*
(2) *The restriction of $W(g,s)$ to SL_2 is an $\mathcal{R}_v$-valued function.*

Then $W(g,s)$ is a right $\tilde{\mathcal{K}}$-finite $\mathcal{R}_v$-valued function on GL_2.

Proof. By Lemma 8.3, there is a holomorphic section $f(g,s)$ on SL_2 such that $W(g,s)=\mathrm{Wh}_\psi(f)(g,s)$ for any $g\in\mathrm{SL}_2$. The holomorphic section $f(g,s)$ of $I(\underline{\chi},2s)$ can be uniquely extended to a holomorphic section $\tilde{f}(g,s)$ of $\tilde{I}(\mathbf{1}\boxtimes\underline{\chi},s)$. Then $W(g,s)=\mathrm{Wh}_\psi(\tilde{f})(g,s)$ for any $g\in\mathrm{GL}_2$. We should prove that $\mathrm{Wh}_\psi(\tilde{f})(g,s)$ is $\mathcal{R}_v$-valued on $\{g\in\mathrm{GL}_2|\det g=a\}$ for any $a\in F^\times$. In fact, for $g\in\mathrm{SL}_2$, we have

$$\begin{aligned}\mathrm{Wh}_\psi(f)\left(\begin{pmatrix}a&0\\0&1\end{pmatrix}g,s\right)&=\int_F f\left(w\begin{pmatrix}1&x\\0&1\end{pmatrix}\begin{pmatrix}a&0\\0&1\end{pmatrix}g,s\right)\overline{\psi(x)}dx\\&=\int_F f\left(\begin{pmatrix}1&0\\0&a\end{pmatrix}w\begin{pmatrix}1&a^{-1}x\\0&1\end{pmatrix}g,s\right)\overline{\psi(x)}dx\\&=\underline{\chi}(a)|a|^{-s}\mathrm{Wh}_{\psi_a}(f)(g,s),\end{aligned}$$

where $\psi_a(x)=\psi(ax)$. This is an $\mathcal{R}_v$-valued function. □

9. **Adelic compatible family**

Let $\tilde{B}=\left\{\begin{pmatrix}*&*\\0&*\end{pmatrix}\in\mathrm{GL}_2\right\}$ be the standard Borel subgroup of GL_2. Put $\tilde{\mathcal{K}}=\mathrm{GL}_2(\hat{\mathbb{Z}})$ and $\tilde{\mathcal{K}}_0=\left\{\begin{pmatrix}a&b\\c&d\end{pmatrix}\in\mathrm{GL}_2(\hat{\mathbb{Z}})\middle|c\in D\mathbb{Z}\right\}$.

Recall that given $h \in M_{2k+1}(\Gamma_0(D), \chi)$, one can define an adelic cusp form $h^\sharp$ on $\mathrm{GL}_2(\mathbb{A})$ by the formula

$$h^\sharp(g) = \underline{\chi}(d)(f|g_\infty)(\sqrt{-1})$$

for $g = \gamma g_\infty g_0 \in \mathrm{GL}_2(\mathbb{A})$ with $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2^+(\mathbb{R})$, and $g_0 = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \tilde{\mathcal{K}}_0$. For a primitive form $f \in S_{2k+1}(\Gamma_0(D), \chi)$, we put $\mathbf{f} = f^\sharp$. We also put $\mathbf{E}_{2k'+1,\chi} = (E_{2k'+1,\chi})^\sharp$. Let $\pi \simeq \otimes'_v \pi_v$ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ generated by $\mathbf{f}$. Then the central character of π is $\underline{\chi}$, and π_∞ is the (limit of) discrete series representation of $\mathrm{GL}_2(\mathbb{R})$ with minimal weight $\pm(2k+1)$. The p-component π_p is the principal series representation

$$\tilde{I}(\mathbf{1} \boxtimes \underline{\chi}_p, s_{0,p}) = \mathrm{Ind}_{\tilde{B}(\mathbb{Q}_p)}^{\mathrm{GL}_2(\mathbb{Q}_p)}(|\cdot|^{s_{0,p}} \boxtimes \underline{\chi}_p|\cdot|^{-s_{0,p}})$$

induced from the character of $\tilde{B}(\mathbb{Q}_p)$ given by

$$\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mapsto |a|^{s_{0,p}} \underline{\chi}_p(d)|d|^{-s_{0,p}}.$$

Here $s_{0,p} \in \mathbb{C}$ is a complex number such that $e^{-s_{0,p}\cdot \log p} = \alpha_p$. Note that $\mathrm{Re}(s_{0,p}) = 0$ by the Ramanujan conjecture.

The Whittaker function $W_{\mathbf{f}}$ of $\mathbf{f}$ is defined by

$$W_{\mathbf{f}}(g) = \int_{\mathbb{Q}\backslash\mathbb{A}} \mathbf{f}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) \overline{\mathbf{e}_{\mathbb{A}}(x)}\, dx.$$

Then $W_{\mathbf{f}}$ has a product expansion

$$W_{\mathbf{f}} = \left(\prod_{p<\infty} W_{p,\mathbf{f}}\right) W_{\infty,2k+1}.$$

Here, the infinite part $W_{\infty,2k+1}$ of $W_{\mathbf{f}}$ is given by

$$\begin{aligned} &W_{\infty,2k+1}\left(\begin{pmatrix} a & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\right) \\ &\qquad = \begin{cases} e^{2\pi\sqrt{-1}x} a^{k+(1/2)} e^{-2\pi a} e^{\sqrt{-1}(2k+1)\theta} & \text{if } a > 0, \\ 0 & \text{if } a < 0. \end{cases} \end{aligned}$$

The finite part $W_{\mathbf{f}}^0 = \prod_{p<\infty} W_{p,\mathbf{f}}$ of the Whittaker function satisfies

$$W_{\mathbf{f}}^0\left(\begin{pmatrix} Nu & 0 \\ 0 & 1 \end{pmatrix}\right) = N^{-k-(1/2)} a_f(N), \qquad N \in \mathbb{Q}_+^\times, u \in \hat{\mathbb{Z}}^\times.$$

Here we think of $a_f(N) = 0$ when $N \notin \mathbb{Z}$. See also [7] section 6.

Let $\mathcal{V}(\mathbf{f})$ be the $\mathbb{C}$-vector space spanned by the right translates of $\mathbf{f}$ by $\mathrm{GL}_2(\mathbb{A}_{\mathrm{f}})$. Then $\mathcal{V}(\mathbf{f})$ can be regarded as the representation space of

$\otimes'_{p<\infty}\pi_p$. We define $\mathcal{V}(\mathbf{E}_{2k'+1,\chi})$ similarly. $\mathcal{V}(\mathbf{E}_{2k'+1,\chi})$ is isomorphic to $\otimes'_{p<\infty}\tilde{I}(\mathbf{1}\boxtimes\underline{\chi}_p, k')$.

For each $\mathbf{h}\in\mathcal{V}(\mathbf{f})$ or each $\mathbf{h}\in\mathcal{V}(\mathbf{E}_{2k'+1,\chi})$, we can associate a function $\xi(\mathbf{h})$ on $\mathfrak{H}$ by

$$\xi(\mathbf{h})(\tau)=\mathbf{h}(g_\infty)j(g_\infty,\sqrt{-1})^{2k+1}$$

for $g_\infty\in\mathrm{GL}_2(\mathbb{R})^+\subset\mathrm{GL}_2(\mathbb{A})$, $g_\infty(\sqrt{-1})=\tau$.

Lemma 9.1. *The map ξ defines surjections $\mathcal{V}(\mathbf{E}_{2k'+1,\chi})\to\mathcal{V}(E_{2k'+1,\chi})$ and $\mathcal{V}(\mathbf{f})\to\mathcal{V}(f)$.*

Proof. This follows from the equality $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})=\mathrm{GL}_2(\mathbb{Q})^+\cdot\tilde{\mathcal{K}}_0$, which follows from the strong approximation of SL_2. $\square$

One can define an action ρ of $\mathrm{SL}_2(\mathbb{A}_\mathrm{f})$ on $\mathcal{V}(f)$ or on $\mathcal{V}(E_{2k'+1,\chi})$ by $\rho(g)h=h|\gamma^{-1}$, for $\gamma\in\mathrm{SL}_2(\mathbb{Q})$ sufficiently close to $g\in\mathrm{SL}_2(\mathbb{A}_\mathrm{f})$. Note that $\mathrm{SL}_2(\mathbb{Q})$ is dense in $\mathrm{SL}_2(\mathbb{A}_\mathrm{f})$ by the strong approximation theorem.

The maps $\xi:\mathcal{V}(\mathbf{E}_{2k'+1,\chi})\to\mathcal{V}(E_{2k'+1,\chi})$ and $\xi:\mathcal{V}(\mathbf{f})\to\mathcal{V}(f)$ are $\mathrm{SL}_2(\mathbb{A}_\mathrm{f})$ equivariant. In particular, $\xi:\mathcal{V}(\mathbf{E}_{2k'+1,\chi})\to\mathcal{V}(E_{2k'+1,\chi})$ is an isomorphism for sufficiently large k', since the restriction of $\tilde{I}(\mathbf{1}\boxtimes\underline{\chi}_p, s)$ to $\mathrm{SL}_2(\mathbb{Q}_p)$ is irreducible for $\mathrm{Re}(s)>1/2$.

Definition 9.2. When k' extends over sufficiently large integers, the family $\{\mathbf{F}_{2k'+1}\}_{k'}$ is an adelic compatible family of Eisenstein series if the following conditions (1) and (2) are satisfied:

(1) $\mathbf{F}_{2k'+1}\in\mathcal{V}(\mathbf{E}_{2k'+1,\chi})$ for each k'.
(2) There is an $\mathcal{R}$-valued function $\mathbb{W}(g;\mathbb{X})$ on $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ such that $W_{\mathbf{F}_{2k'+1}}(g)=\mathbb{W}(g;\{p^{-k'}\})$ for each k'.

We call $\mathbb{W}(g;\mathbb{X})$ the $\mathcal{R}$-valued Whittaker function associated to the family $\{\mathbf{F}_{2k'+1}\}$. It is easily seen that $\{\mathbf{E}_{2k'+1,\chi}\}_{k'}$ is an adelic compatible family of Eisenstein series.

Lemma 9.3. *Let $\{\mathcal{F}_{2k'+1}\}_{k'}$ be a family such that, for any $\gamma\in\mathrm{SL}_2(\mathbb{Z})$, $\{\mathcal{F}_{2k'+1}|\gamma\}_{k'}$ is a compatible family of Eisenstein series. Then there exists an adelic compatible family $\{\mathbf{F}_{2k'+1}\}_{k'}$ such that $\xi(\mathbf{F}_{2k'+1})=\mathcal{F}_{2k'+1}$.*

Proof. Put $\mathbf{F}_{2k'+1}=\xi^{-1}(\mathcal{F}_{2k'+1})$. Let $W^0_{\mathbf{F}_{2k'+1}}$ be the finite part of the Whittaker function of $\mathbf{F}_{2k'+1}$. We have to prove that $W^0_{\mathbf{F}_{2k'+1}}$ is an $\mathcal{R}$-valued function, i.e., there exists an element $\varphi_g(\mathbb{X})\in\mathcal{R}$ such that $W^0_{\mathbf{F}_{2k'+1}}(g)=\varphi_g(\{p^{-k'}\})$ for each $g\in\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$ and sufficiently large k'. First assume that $g\in\mathrm{SL}_2(\mathbb{A}_\mathrm{f})$. Then $\{\rho(g)\mathcal{F}_{2k'+1}\}_{k'}$ is a compatible family of Eisenstein series, because any element of $\mathrm{SL}_2(\mathbb{Q})$ is a product of upper triangular matrix and an element of $\mathrm{SL}_2(\mathbb{Z})$ (cf. the proof of

Lemma 10.3 of [9]). Then $W^0_{\mathbf{F}_{2k'+1}}(g) = W^0_{\rho(g)\mathbf{F}_{2k'+1}}(\mathbf{1}_2)$ is essentially the first Fourier coefficient of $\rho(\mathbf{F}_{2k'+1})$. It follows that the restriction of $W^0_{\mathbf{F}_{2k'+1}}$ to $\mathrm{SL}_2(\mathbb{A}_\mathrm{f})$ is an $\mathcal{R}$-valued Whittaker function. By Lemma 8.4, $W^0_{\mathbf{F}_{2k'+1}}$ is an $\mathcal{R}$-valued function on $\mathrm{GL}_2(\mathbb{A}_\mathrm{f})$. □

Proof of Lemma 7.4. Put $\mathbf{F}_{i,2k'+1} = \xi^{-1}(\mathcal{F}_{i,2k'+1})$. Then by Lemma 9.3, $\{\mathbf{F}_{i,2k'+1}\}_{k'}$ is an adelic compatible family. Let

$$\vec{\mathbb{W}}(g,\mathbb{X}) = {}^t(\mathbb{W}_1(g,\mathbb{X}),\dots,\mathbb{W}_d(g,\mathbb{X}))$$

be the $\mathcal{R}^d$-valued Whittaker function associated to $\{\vec{\mathbf{F}}_{2k'+1}\}_{k'}$. Put $W_i(g) = \mathbb{W}_i(g_\mathrm{f},\{\alpha_p\})W_{\infty,2k+1}(g_\infty)$ for $g = g_\mathrm{f}g_\infty$, $g_\mathrm{f} \in \mathrm{GL}_2(\mathbb{A}_\mathrm{f})$, $g_\infty \in \mathrm{GL}_2(\mathbb{R})$. Note that $\mathbb{W}_i(g_\mathrm{f},\{\alpha_p\})$ is meaningful by Lemma 8.4. Then

$$\mathbf{h}_i(g) = \sum_{t\in\mathbb{Q}^\times} W_i\left(\begin{pmatrix} t & 0\\ 0 & 1\end{pmatrix} g\right) \in \mathcal{V}(\mathbf{f})$$

for $i = 1,\dots,d$, since π is an irreducible cuspidal automorphic representation. Put $\vec{\mathbf{h}} = {}^t(\mathbf{h}_1,\dots,\mathbf{h}_d)$. Then obviously $\vec{h} = \xi(\vec{\mathbf{h}}) \in \mathcal{I}(\mathcal{V}(f)^d;u)$. □

10. **Proof of Theorem 5.1 and Theorem 5.2.**

Now we can prove Theorem 5.1. Let k' be a sufficiently large integer. Then there is a vector-valued modular form of weight $2k'+1$ with values in $\mathbb{C}^\Xi = \mathrm{Map}(\Xi,\mathbb{C})$ whose ξ-th component is the (S,ξ)-component of Fourier-Jacobi coefficient of the Eisenstein series

$$b_{S,\xi,2k'+1} + \sum_{N\in\mathbb{Z}_+} N^{k'}\left[\prod_{p\mid N}\tilde{F}_p(H_{S,\xi}(N);p^{-k'})\right] q^{N/\Delta}.$$

Here $b_{S,\xi,2k'+1}$ is some rational number and

$$\begin{aligned}\Delta =&[(\mathcal{O}^\sharp)^{m-1} : S\mathcal{O}^{m-1}] = D^{m-1}(\det S)^2,\\ H_{S,\xi}(N) =& \begin{pmatrix} S & S\xi\\ {}^t\overline{\xi}S & \frac{N}{\Delta} + {}^t\overline{\xi}S\xi\end{pmatrix}.\end{aligned}$$

As we have seen in §6, this vector-valued automorphic form is of type u_S. Note that the type u_S does not depend on k'.

By Theorem 3.2 of [8], $\mathcal{E}_{S,\xi}$ belongs to the space $\mathcal{V}(E_{2k'+1,\chi})$ for each sufficiently large integer k'. It follows that when k' extends over sufficiently large integers, $(\mathcal{E}^{(2n)}_{k'+n})_{S,\xi}(\tau)$ make up a compatible family of

Eisenstein series. By Lemma 7.4,

$$\sum_{N=1}^{\infty} N^k \left(\prod_{p|N} \tilde{F}_p(H_{S,\xi}(N);\alpha_p) \right) q^{N/\Delta}$$

is a vector-valued automorphic form with type u_S. It follows that

$$\sum_{\xi\in\Xi} \theta_{[\xi]}(S;\tau,z) \sum_{N=1}^{\infty} N^k \left(\prod_{p|N} \tilde{F}_p(H_{S,\xi}(N);\alpha_p) \right) q^{N/\Delta}$$

is a Jacobi form with index S. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, we put

$$\bar{\gamma} = \begin{pmatrix} \mathbf{1}_{m-1} & 0 & \mathbf{0}_{m-1} & 0 \\ 0 & a & 0 & b \\ \mathbf{0}_{m-1} & 0 & \mathbf{1}_{m-1} & 0 \\ 0 & c & 0 & d \end{pmatrix}.$$

Then we have $F(Z)|_{2k+2n}\bar{\gamma} = F(Z)$ for any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$. It is proved by Klingen [11] that $\Gamma_K^{(m)}$ is generated by $\Gamma_{K,\infty}^{(m)}$ and $\begin{pmatrix} \mathbf{0}_m & -\mathbf{1}_m \\ \mathbf{1}_m & \mathbf{0}_m \end{pmatrix}$ (See also §12). Therefore $F|_{2k+2n}\, g = (\det g)^{-k-n} F$ for any $g \in \Gamma_K^{(m)}$. This completes the proof of Theorem 5.1.

Now we consider Case O. Since the proof for Case O is almost the same as Case E, we just describe an outline. Let

$$E_{2k}(\tau) = -\frac{B_{2k}}{4k} + \sum_{N=1}^{\infty} \left(\sum_{d|N} d^{2k-1} \right) q^N \in M_{2k}(\mathrm{SL}_2(\mathbb{Z}))$$

be the usual Eisenstein series. Note that the Satake parameter of E_{2k} is $\{p^{k-(1/2)}, p^{-k+(1/2)}\}$.

Definition 10.1. (Case O) We define a compatible family of Eisenstein series $\{\mathcal{F}_{2k'}(\tau)\}_{k'\geq k'_0}$ as follows: A compatible family of Eisenstein series is a family $\{\mathcal{F}_{2k'}(\tau)\}_{k'}$ of modular forms

$$\mathcal{F}_{2k'}(\tau) = b(2k';0) + \sum_{N\in\mathbb{Q}_+^\times} N^{k'-(1/2)} b(2k';N) q^N$$

satisfying the following conditions (1), (2), and (3).

(1) $\mathcal{F}_{2k'} \in \mathcal{V}(E_{2k'})$ for any integer $k' \geq k'_0$.
(2) For each $N \in \mathbb{Q}_+^\times$, there exists an element $\Phi_N(\mathbb{X}) \in \mathcal{R}$ such that

$$b(2k';N) = \Phi_N(\{p^{-k'+(1/2)}\}).$$

(3) There exists a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ such that $\mathcal{F}_{2k'} \in M_{2k'}(\Gamma)$ for all $k' \geq k'_0$.

As in Case E, one can prove the following lemma.

Lemma 10.2. *Let*

$$f(\tau) = \sum_{N>0} a(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$$

be a normalized Hecke eigenform and α_p a Satake parameter of $f(\tau)$. Assume that there are a finite dimensional representation $(u, \mathbb{C}^d)$ of $\mathcal{K}$, and $\vec{\Phi}_N(\mathbb{X}) = {}^t(\Phi_{1,N}(\mathbb{X}), \dots, \Phi_{d,N}(\mathbb{X})) \in \mathcal{R}^d$, $(N \in \mathbb{Q}_+^\times)$ satisfying the following conditions (1) and (2):

(1) *For each sufficiently large integer k', there exists a vector-valued modular form*

$$\vec{\mathcal{F}}_{2k'}(\tau) = \vec{b}(2k';0) + \sum_{N\in\mathbb{Q}_+^\times} N^{k'-(1/2)}\vec{b}(2k';N)q^N$$

of type u.

(2) *For each i $(1 \leq i \leq d)$, the i-th component $\mathcal{F}_{i,2k'}(\tau)$ of $\vec{\mathcal{F}}_{2k'}(\tau)$ is a compatible family of Eisenstein series such that*

$$b_i(2k';N) = \Phi_{i,N}(\{p^{-k'+(1/2)}\}).$$

Here $\vec{b}(2k';N) = {}^t(b_1(2k';N), \dots, b_d(2k';N))$.

Put

$$\vec{h}(\tau) = \sum_{N\in\mathbb{Q}_+^\times} N^{k-(1/2)}\vec{\Phi}_N(\{\alpha_p\})q^N.$$

Then we have $\vec{h}(\tau) \in \mathcal{I}(\mathcal{V}(f)^d; u)$.

The proof of Lemma 10.2 and Theorem 5.2 are the same as Case E.

11. Existence of some hermitian matrices

Lemma 11.1. *If m is divisible by 4, then there exists an element $H_1 \in \Lambda_m(\mathcal{O})^+$ such that $\gamma(H_1) = 1$.*

Proof. We may assume that $m = 4$. We define hermitian matrices H_s and H_d by

$$H_s = \sqrt{-D}^{-1}\begin{pmatrix} 0 & -\mathbf{1}_2 \\ \mathbf{1}_2 & 0 \end{pmatrix}, \quad H_d = \begin{pmatrix} D^{-2} & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix}.$$

Then $H_s \in \Lambda(\mathcal{O})$, $H_d > 0$ and $\det H_s = \det H_d$. Since $\det H_s = \det H_d$, there exists an element $X_p \in \mathrm{GL}_4(K_p)$ such that $H_s = X_p H_d {}^t\bar{X}_p$ for each prime p (cf. Scharlau [22]). Replacing X_p by $X_p \begin{pmatrix} \det X_p^{-1} & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix}$, we may assume that $X_p \in \mathrm{SL}_4(K_p)$. We may also assume that $X_p \in \mathrm{SL}_4(\mathcal{O}_p)$ for almost all p. By the strong approximation theorem for SL_4, there exists an element $X \in \mathrm{SL}_4(K)$ such that $X^{-1}X_p \in \mathrm{SL}_4(\mathcal{O}_p)$ for any p. Put $H_1 = XH_d{}^t\bar{X}$. Then we have $H_1 \in \Lambda_4(\mathcal{O})^+$ and $\gamma(H_1) = 1$. □

Lemma 11.2. *Assume that m is odd. For any integer $N > 0$, there exists an element $H_N \in \Lambda_m(\mathcal{O})^+$ such that $|\gamma(H_N)| = N$.*

Proof. By Lemma 11.1, we may assume $m = 1$ or 3. If $m = 1$, we can take $H_N = (N)$. The proof in the case $m = 3$ is similar to Lemma 11.1. One can take

$$H_s = \begin{pmatrix} -N & 0 & 0 \\ 0 & 0 & -\sqrt{-D}^{-1} \\ 0 & \sqrt{-D}^{-1} & 0 \end{pmatrix}, \quad H_d = \begin{pmatrix} ND^{-1} & 0 \\ 0 & \mathbf{1}_2 \end{pmatrix}.$$

□

Lemma 11.3. *Assume that $m = 2n$ is even. Let $N > 0$ be a rational integer such that there exists an element $y \in \mathcal{O}^\sharp$ such that $Dy\bar{y} \equiv (-1)^n N \bmod D$. Then there exists an element $H_N \in \Lambda_m(\mathcal{O})^+$ such that $|\gamma(H_N)| = N$.*

Proof. We may assume $m = 2$ or 4. Put $x = (Dy\bar{y} - (-1)^n N)/D$. In the case $m = 2$, we put

$$H_s = \begin{pmatrix} x & y \\ \bar{y} & 1 \end{pmatrix}, \quad H_d = \begin{pmatrix} ND^{-1} & 0 \\ 0 & 1 \end{pmatrix}.$$

In the case $m = 4$, we put

$$H_s = \begin{pmatrix} x & y & 0 & 0 \\ \bar{y} & 1 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{-D}^{-1} \\ 0 & 0 & -\sqrt{-D}^{-1} & 0 \end{pmatrix}, \quad H_d = \begin{pmatrix} ND^{-2} & 0 \\ 0 & \mathbf{1}_3 \end{pmatrix}.$$

Then one can find a desired element $H_N \in \Lambda_m(\mathcal{O})^+$ as in Lemma 11.1. □

Lemma 11.4. *If $m = 2n$ is even, then there are an infinite number of primes p such that $|\gamma(H)| = p$ for some $H \in \Lambda_m(\mathcal{O})^+$.*

Proof. If $p \equiv (-1)^n \bmod D$, then there exists $H \in \Lambda_m(\mathcal{O})$ such that $|\gamma(H)| = p$ by Lemma 11.3. Hence the lemma. □

12. **Hermitian modular groups of general type**

In this section, we prove some facts about hermitian modular groups. We follow the argument of Klingen [11].

For a fractional ideal $\mathfrak{a}$ of K, the absolute norm of $\mathfrak{a}$ is denoted by $\mathrm{N}(\mathfrak{a})$. For an integral ideal β of K, the set of prime divisors of β is denoted by $\mathrm{Supp}(\beta)$.

Lemma 12.1. *Let $\mathfrak{a}$ be a fractional ideal and $\mathfrak{b}$ an integral ideal of K. Let $a \in \mathfrak{a}^{-1}$ and $b \in \mathcal{O}$ be elements such that $(a\mathfrak{a}, \mathfrak{b}, b) = 1$. Then there exists an element $x \in \mathfrak{a}^{-1}$ such that $((a + xb)\mathfrak{a}, \mathfrak{b}) = 1$.*

Proof. Decompose the integral ideals $a\mathfrak{a}$ and $\mathfrak{b}$ into products of integral ideals

$$a\mathfrak{a} = \beta_1\gamma_1, \qquad \mathfrak{b} = \beta_2\gamma_2, \qquad (\beta_1, \gamma_1) = (\beta_2, \gamma_2) = (\gamma_1, \gamma_2) = 1$$

such that $\mathrm{Supp}(\beta_1) = \mathrm{Supp}(\beta_2)$. Let γ_3 be an integral ideal which belongs to the same ideal class as $\mathfrak{a}\gamma_2^{-1}$, such that $(\gamma_3, \beta_1\gamma_1) = 1$. Let x be an element such that $(x) = \mathfrak{a}^{-1}\gamma_2\gamma_3$. Assume that there is a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} | ((a + xb)\mathfrak{a}, \mathfrak{b})$. Then $\mathfrak{p}|\beta_2$ or $\mathfrak{p}|\gamma_2$. If $\mathfrak{p}|\beta_2$, then $\mathfrak{p}|\beta_1$, which is impossible, since $(\beta_1, x\mathfrak{a}) = (\beta_1, \gamma_2\gamma_3) = 1$ and $(\beta_1, b) = 1$. Now assume $\mathfrak{p}|\gamma_2$. This is also impossible, since $\mathfrak{p}|xb\mathfrak{a}$ and $\mathfrak{p} \nmid a\mathfrak{a}$. Hence the lemma. □

Lemma 12.2. *Let α, $\mathfrak{a}_1$, $\mathfrak{a}_2$, and $\mathfrak{m}$ be fractional ideals of K. Then there exist elements y_1, $y_2 \in K$ such that $y_1 \in \alpha$, $\mathfrak{m} = y_1\mathfrak{a}_1 + y_2\mathfrak{a}_2$.*

Proof. Choose a non-zero element $y_1 \in \mathfrak{m}\mathfrak{a}_1^{-1} \cap \alpha$. Put $\mathfrak{n} = y_1\mathfrak{a}_1\mathfrak{m}^{-1}$. Choose an integral ideal $\mathfrak{b}$ which belongs to the same ideal class as $\mathfrak{m}^{-1}\mathfrak{a}_2$, such that $(\mathfrak{b}, \mathfrak{n}) = 1$. Put $(y_2) = \mathfrak{m}\mathfrak{a}_2^{-1}\mathfrak{b}$. Then we have $y_1\mathfrak{a}_1 + y_2\mathfrak{a}_2 = \mathfrak{m}\mathfrak{n} + \mathfrak{m}\mathfrak{b} = \mathfrak{m}$. □

We fix fractional ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_m$ of $\mathcal{O}$. We define an $\mathcal{O}$-module $\mathfrak{M}$ by

$$\mathfrak{M} = \{{}^t(a_1, \ldots, a_m) \in K^m \,|\, a_i \in \mathfrak{a}_i \ (i = 1, \ldots, m)\}.$$

Note that the isomorphism class of $\mathfrak{M}$ is determined by the ideal class of $\mathfrak{a}_1 \cdots \mathfrak{a}_m$. We define a group $\mathcal{U}$ by

$$\mathcal{U} = \{g \in \mathrm{GL}_m(K) \,|\, g\mathfrak{M} = \mathfrak{M}\}.$$

Lemma 12.3. *Let $\mathbf{x} = {}^t(x_1, \ldots, x_m)$ and $\mathbf{y} = {}^t(y_1, \ldots, y_m)$ be column vectors in K^m. Then there exists an element $g \in \mathcal{U}$ such that $g\mathbf{x} = \mathbf{y}$ if and only if $x_1\mathfrak{a}_1^{-1} + \cdots + x_m\mathfrak{a}_m^{-1} = y_1\mathfrak{a}_1^{-1} + \cdots + y_m\mathfrak{a}_m^{-1}$.*

Proof. "Only if" part is clear from the definition. We may assume $m > 1$. Assume $x_1\mathfrak{a}_1^{-1} + \cdots + x_m\mathfrak{a}_m^{-1} = y_1\mathfrak{a}_1^{-1} + \cdots + y_m\mathfrak{a}_m^{-1} \neq \{0\}$. Let G be the algebraic group defined by $G = \{g \in \mathrm{SL}_m(K) \,|\, g\mathbf{x} = \mathbf{x}\}$. Then the strong approximation theorem holds for $G \simeq k^{m-1} \rtimes \mathrm{SL}_{m-1}(k)$. Put $\mathcal{K}_\mathfrak{p} = \{g \in \mathrm{GL}_m(K_\mathfrak{p}) \mid g\mathfrak{M}_\mathfrak{p} = \mathfrak{M}_\mathfrak{p}\}$ for each prime $\mathfrak{p}$ of K. Here, $\mathfrak{M}_\mathfrak{p}$ is the closure of $\mathfrak{M}$ in $K_\mathfrak{p}^m$. Then $\prod_\mathfrak{p} \mathcal{K}_\mathfrak{p} \times \mathrm{GL}_m(\mathbb{C})$ is an open subgroup of $\mathrm{GL}_m(\mathbb{A}_K)$. Choose an element $h \in \mathrm{GL}_m(K)$ such that $h\mathbf{x} = \mathbf{y}$. The set

$$h^{-1}\left(\prod_\mathfrak{p} \mathcal{K}_\mathfrak{p} \times \mathrm{GL}_m(\mathbb{C})\right) \cap G(\mathbb{A}_K)$$

is a non-empty open subset of $G(\mathbb{A}_K)$ by our assumption. By the strong approximation theorem for G, there exists an element

$$g \in h^{-1}\left(\prod_\mathfrak{p} \mathcal{K}_\mathfrak{p} \times \mathrm{GL}_m(\mathbb{C})\right) \cap G(K).$$

Then we have $hg \in \mathcal{U}$ and $hg\mathbf{x} = \mathbf{y}$. □

We define a group $\mathbf{\Gamma}$ by

$$\mathbf{\Gamma} = \left\{ g \in \mathrm{U}(m,m)(\mathbb{Q}) \,\middle|\, g\begin{pmatrix}\mathfrak{M}\\ \mathfrak{M}'\end{pmatrix} = \begin{pmatrix}\mathfrak{M}\\ \mathfrak{M}'\end{pmatrix}\right\},$$

where $\mathfrak{M}' = \{{}^t(a_1,\dots,a_m) \in K^m \mid a_i \in \bar{\mathfrak{a}}_i^{-1}\ (i = 1,\dots,m)\}$. We temporarily call $\mathbf{\Gamma}$ a hermitian modular group of general type associated to $\mathfrak{M}$. If $\mathfrak{M}_1$ and $\mathfrak{M}_2$ are isomorphic $\mathcal{O}$-modules, then the corresponding groups $\mathbf{\Gamma}_1$ and $\mathbf{\Gamma}_2$ are conjugate by an element of $\mathrm{U}(m,m)(\mathbb{Q})$. We define the subgroups of $M_\mathbf{\Gamma}$, $N_\mathbf{\Gamma}$, and $\tilde{N}_\mathbf{\Gamma}$ of $\mathbf{\Gamma}$ by

$$\begin{aligned}
M_\mathbf{\Gamma} &= \left\{\begin{pmatrix} g & 0\\ 0 & {}^t\bar{g}^{-1}\end{pmatrix} \,\middle|\, g \in \mathcal{U}\right\}\\
N_\mathbf{\Gamma} &= \left\{\begin{pmatrix} \mathbf{1}_m & B\\ 0 & \mathbf{1}_m\end{pmatrix} \,\middle|\, {}^t\bar{B} = B = (b_{ij}),\ b_{ij} \in \mathfrak{a}_i\bar{\mathfrak{a}}_j\right\},\\
\tilde{N}_\mathbf{\Gamma} &= \left\{\begin{pmatrix} \mathbf{1}_m & 0\\ B & \mathbf{1}_m\end{pmatrix} \,\middle|\, {}^t\bar{B} = B = (b_{ij}),\ b_{ij} \in \bar{\mathfrak{a}}_i^{-1}\mathfrak{a}_j^{-1}\right\}.
\end{aligned}$$

Let us denote the subgroup generated by $M_\mathbf{\Gamma}$, $N_\mathbf{\Gamma}$, and $\tilde{N}_\mathbf{\Gamma}$ by $\mathbf{\Gamma}'$. We shall prove $\mathbf{\Gamma}' = \mathbf{\Gamma}$. To prove this, we may assume $\mathfrak{a}_1 = \mathcal{O}$, by replacing $\mathfrak{a}_i$ by $\mathfrak{a}_1^{-1}\mathfrak{a}_i$, and $\mathbf{\Gamma}$ by $\begin{pmatrix} \mathbf{1}_m & 0\\ 0 & \mathrm{N}(\mathfrak{a}_1)\mathbf{1}_m\end{pmatrix} \mathbf{\Gamma} \begin{pmatrix} \mathbf{1}_m & 0\\ 0 & \mathrm{N}(\mathfrak{a}_1)^{-1}\mathbf{1}_m\end{pmatrix}$. Since it is easy to prove $\mathbf{\Gamma}' = \mathbf{\Gamma}$ for $m = 1$, we consider the case $m \geq 2$.

Lemma 12.4. *Assume that $m \geq 2$ and that $\mathfrak{a}_1 = \mathcal{O}$. Let $b_1,\dots,b_m$, $c_1,\dots,c_m$ be elements of K satisfying the following properties (1) and (2):*

(1) $\sum_{i=1}^m b_i\mathfrak{a}_i^{-1} + c_i\bar{\mathfrak{a}}_i = \mathcal{O}$.
(2) $b_1\bar{c}_1 + \cdots + b_m\bar{c}_m \in \mathbb{Z}$.

Put $\mathbf{e}_1 = {}^t(1,0,\ldots,0) \in K^{2m}$. *Then there exists an element* $g \in \mathbf{\Gamma}'$ *such that*

$$g\mathbf{e}_1 = {}^t(b_1,\ldots,b_m,c_1,\ldots,c_m).$$

Proof. Note that $b_1, c_1 \in \mathcal{O}$, since $\mathfrak{a}_1 = \mathcal{O}$. Note also that the properties (1) and (2) are preserved by changing ${}^t(b_1,\ldots,b_m,c_1,\ldots,c_m)$ by $g \cdot {}^t(b_1,\ldots,b_m,c_1,\ldots,c_m)$ with $g \in \mathbf{\Gamma}$. By using Lemma 12.2 and Lemma 12.3, we may assume $c_3 = \ldots c_m = 0$. If $c_1 = c_2 = 0$, then ${}^t(b_1,\ldots,b_m)$ is a first row of an element of $\mathcal{U}$ by Lemma 12.3, which case is done. Assume that $(c_1,c_2) \neq (0,0)$. Put $\alpha = \sum_{i=3}^m b_i\mathfrak{a}_i^{-1}$, $\alpha' = \sum_{i=1}^m b_i\mathfrak{a}_i^{-1}$. By Lemma 12.2 and Lemma 12.3, we may assume

$$c_1c_2 \neq 0, \quad \alpha'|c_2\bar{\mathfrak{a}}_2.$$

If $b_1 = 0$, we replace b_1 by $b_1 + c_1$. Thus we may assume

$$(b_1, b_1\mathfrak{a}_2^{-1}, \alpha, c_1) = 1, \quad b_1c_1c_2 \neq 0, \quad c_3 = \cdots = c_m = 0, \quad \alpha'|c_2\bar{\mathfrak{a}}_2.$$

Let ρ be the product of all prime ideals $\mathfrak{p}$ such that $\bar{\mathfrak{p}}|b_1$, $\mathfrak{p} \nmid b_1$. Then we have $(b_2\mathfrak{a}_2^{-1}, \alpha, \rho, b_1) = 1$. By Lemma 12.1, there exists $x \in \mathfrak{a}_2$ such that $((b_2 + b_1x)\mathfrak{a}_2^{-1}, \alpha, \rho) = 1$. Note that $(b_1, (b_2 + b_1x)\mathfrak{a}_2^{-1}, \alpha, c_1 - \bar{x}c_2) = 1$. Decompose the integral ideals $((b_2 + b_1x)\mathfrak{a}_2^{-1}, \alpha)$ and b_1 into products of integral ideals

$$((b_2 + b_1x)\mathfrak{a}_2^{-1}, \alpha) = \beta_1\gamma_1, \qquad b_1 = \beta_2\gamma_2,$$
$$(\beta_1,\gamma_1) = (\beta_2,\gamma_2) = (\gamma_1,\gamma_2) = 1$$

such that $\mathrm{Supp}(\beta_1) = \mathrm{Supp}(\beta_2)$. Put $y = \mathrm{N}(\gamma_1)$. We shall show that

$$(b_1 + y(c_1 - \bar{x}c_2), (b_2 + b_1x)\mathfrak{a}_2^{-1}, \alpha) = 1.$$

Let $\mathfrak{p}$ be a prime divisor of the left hand side. Then we have either $\mathfrak{p}|\beta_1$ or $\mathfrak{p}|\gamma_1$. If $\mathfrak{p}|\gamma_1$, then we have $\mathfrak{p}|y$, and so $\mathfrak{p}|b_1$, which is impossible. Now we assume $\mathfrak{p}|\beta_1$. Then we have $\mathfrak{p}|\beta_2$, and so $\mathfrak{p}|b_1$. It follows that $\mathfrak{p}|y(c_1 - \bar{x}c_2)$. Note that $\mathfrak{p}|(c_1 - \bar{x}c_2)$ contradicts to $(b_1, (b_2 + b_1x)\mathfrak{a}_2^{-1}, \alpha, c_1 - \bar{x}c_2) = 1$. If $\mathfrak{p}|y$, then we have $\mathfrak{p}|\bar{\gamma}_1$, since $\mathfrak{p}|\gamma_1\bar{\gamma}_1$ and $\mathfrak{p}|\beta_1$. This implies $\bar{\mathfrak{p}} \nmid b_1$, and so $\bar{\mathfrak{p}}|\rho$, which contradicts to $((b_2+b_1x)\mathfrak{a}_2^{-1}, \alpha, \rho) = 1$. Hence we have $(b_1 + y(c_1 - \bar{x}c_2), (b_2 + b_1x)\mathfrak{a}_2^{-1}, \alpha) = 1$. We replace ${}^t(b_1,\ldots,b_m,c_1,\ldots,c_m)$ by

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{m'} & 0 & 0 & 0 \\ \hline y & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{m'} \end{array}\right) \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ x & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{1}_{m'} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & -\bar{x} & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{m'} \end{array}\right) \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ c_1 \\ c_2 \\ \vdots \end{pmatrix},$$

where $m' = m-2$. Then we have $b_1\mathfrak{a}_1^{-1} + b_2\mathfrak{a}_2^{-1} + \cdots + b_m\mathfrak{a}_m = \mathcal{O}$. In this case, the proposition can be easily prove by using Lemma 12.3. □

Now we prove that $\mathbf{\Gamma}' = \mathbf{\Gamma}$ by induction with respect to m. Let g be any element of $\mathbf{\Gamma}$ and put ${}^t(b_1, \dots, b_m, c_1, \dots, c_m) = g\mathbf{e}_1$. Then it is easy to see $b_1, \dots, b_m$, $c_1, \dots, c_m$ satisfy the properties (1) and (2). By Lemma 12.4, there exists an element $h \in \mathbf{\Gamma}'$ such that $g\mathbf{e}_1 = h\mathbf{e}_1$. Put

$$h^{-1}g = \left(\begin{array}{cc|cc} 1 & * & * & * \\ 0 & A & * & B \\ \hline 0 & 0 & 1 & 0 \\ 0 & C & * & D \end{array}\right), \qquad g_1 = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & A & 0 & B \\ \hline 0 & 0 & 1 & 0 \\ 0 & C & 0 & D \end{array}\right).$$

By induction hypothesis, we have $g_1 \in \mathbf{\Gamma}'$. Clearly, we have $g_1^{-1}h^{-1}g \in \mathbf{\Gamma}'$. Thus we have $g \in hg_1\mathbf{\Gamma}' = \mathbf{\Gamma}'$.

Proposition 12.5. *Put $a_i = \mathrm{N}(\mathfrak{a}_i)$, $(i = 1, \dots, m)$. The hermitian modular group $\mathbf{\Gamma}$ is generated $M_{\mathbf{\Gamma}} N_{\mathbf{\Gamma}}$ and*

$$w_j = \left(\begin{array}{ccc|ccc} \mathbf{1}_{j-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_j & 0 \\ 0 & 0 & \mathbf{1}_{m-j} & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & \mathbf{1}_{j-1} & 0 & 0 \\ 0 & a_j^{-1} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathbf{1}_{m-j} \end{array}\right), \qquad (1 \le j \le m).$$

Proof. Put $A = \mathrm{diag}(a_1, \dots, a_m)$. Then we have $\left(\begin{smallmatrix} 0 & A \\ A^{-1} & 0 \end{smallmatrix}\right) N_{\mathbf{\Gamma}} \left(\begin{smallmatrix} 0 & A \\ A^{-1} & 0 \end{smallmatrix}\right) = \tilde{N}_{\mathbf{\Gamma}}$. Thus the proposition follows from Lemma 12.4. □

Proposition 12.6. *Let P be the Siegel parabolic subgroup of $\mathcal{G} = \mathrm{U}(m,m)$. Then we have $\mathcal{G}(\mathbb{Q}) = P(\mathbb{Q}) \cdot \mathbf{\Gamma}$.*

Proof. Put $\mathcal{G}_1 = \mathrm{SU}(m,m)$, $P_1 = P \cap \mathcal{G}_1$, and $\mathbf{\Gamma}_1 = \mathbf{\Gamma} \cap \mathcal{G}_1(\mathbb{Q})$. It is enough to prove $\mathcal{G}_1(\mathbb{Q}) = P_1(\mathbb{Q}) \cdot \mathbf{\Gamma}_1$.

Let $\mathcal{K}_1$ be the maximal compact subgroup of $\mathcal{G}_1(\mathbb{A}_\mathrm{f})$ defined by $\mathcal{K}_1 = \mathcal{G}_1(\mathbb{A}_\mathrm{f}) \cap (\prod_p \mathrm{GL}_{2m}(\mathcal{O}_p))$. For $j = 1, \dots, m$, we choose elements $t_j \in \mathbb{A}_K^\times$ such that $\mathrm{ord}_\mathfrak{p} t_j = \mathrm{ord}_\mathfrak{p} \mathfrak{a}_j$, $(t_j)_\infty = 1$. Put

$$\begin{aligned} t =& \mathrm{diag}(t_1, \dots, t_m, \bar{t}_i^{-1}, \dots, \bar{t}_m^{-1}) \in \mathcal{G}(\mathbb{A}_\mathbf{f}), \\ \mathcal{K}_{\mathbf{\Gamma}} =& t\mathcal{K}_1 t^{-1}. \end{aligned}$$

Then we have $\mathcal{G}_1(\mathbb{Q}) \cap \mathcal{K}_{\mathbf{\Gamma}} \mathcal{G}_1(\mathbb{R}) = \mathbf{\Gamma}_1$. Note that the Levi factor of P_1 is isomorphic to $\{m \in \mathrm{GL}_m(K) \mid \det m \in \mathbb{Q}^\times\}$. In particular, the class number of P_1 is one. Thus we have

$$P_1(\mathbb{Q})\mathcal{K}_{\mathbf{\Gamma}}\mathcal{G}_1(\mathbb{R}) = P_1(\mathbb{A})\mathcal{K}_{\mathbf{\Gamma}}\mathcal{G}_1(\mathbb{R}) = \mathcal{G}_1(\mathbb{A}).$$

Hence we have

$$\mathcal{G}_1(\mathbb{Q}) = P_1(\mathbb{Q}) \cdot (\mathcal{G}_1(\mathbb{Q}) \cap \mathcal{K}_{\mathbf{\Gamma}}\mathcal{G}_1(\mathbb{R})) = P_1(\mathbb{Q}) \cdot \mathbf{\Gamma}_1.$$

□

13. Extension to the unitary group

In this section, we discuss the extension of $\mathrm{lift}^{(m)}(f)$ to an automorphic form on the adele group of the unitary group $\mathcal{G} = \mathrm{U}(m,m)$. Since $\mathcal{G}$ need not have class number 1, we need to consider several congruence subgroups. Let T be the maximal torus of $\mathcal{G}$, which consists of all diagonal elements. Let $\mathcal{K}$ be the maximal compact subgroup of $\mathcal{G}(\mathbb{A}_\mathrm{f})$ defined by $\mathcal{K} = \mathcal{G}(\mathbb{A}_\mathrm{f}) \cap (\prod_p \mathrm{GL}_{2m}(\mathcal{O}_p))$.

Lemma 13.1. *There is a natural bijection between the double coset $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\mathcal{K}\mathcal{G}(\mathbb{R})$ and the ideal class group $\mathcal{C}_K$ of K.*

Proof. Put $H = \mathrm{U}(1)$. Then by strong approximation of $\mathrm{SU}(m,m)$, there is a bijection $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\mathcal{K}\mathcal{G}(\mathbb{R}) \to H(\mathbb{Q})\backslash H(\mathbb{A})/\mathcal{I}H(\mathbb{R})$, where $\mathcal{I}$ is the image of $\mathcal{K}$ in $H(\mathbb{A}_\mathrm{f})$. Consider the exact sequence

$$1 \to \mathbb{A}_\mathbb{Q}^\times \to \mathbb{A}_K^\times \to H(\mathbb{A}) \to 1,$$

where the map $\mathbb{A}_K^\times \to H(\mathbb{A})$ is given by $x \mapsto x\bar{x}^{-1}$. One can easily show that $\mathcal{I} \subset H(\mathbb{A}_\mathrm{f})$ is equal to the image of $\prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^\times \subset \mathbb{A}_{K,\mathrm{f}}^\times$. Then we have $H(\mathbb{Q})\backslash H(\mathbb{A})/\mathcal{I}H(\mathbb{R}) \simeq \mathbb{A}_K^\times/\mathbb{A}_\mathbb{Q}^\times \prod_\mathfrak{p} \mathcal{O}_\mathfrak{p}^\times \mathbb{C}^\times \simeq \mathcal{C}_K$. □

Fix a complete set of representatives $\{\gamma_1, \dots, \gamma_h\}$ for the double coset $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\mathcal{K}\mathcal{G}(\mathbb{R})$. We may assume $\gamma_1 = \mathbf{1}_{2m}$ and $\gamma_i = \begin{pmatrix} t_i & 0 \\ 0 & \bar{t}_i^{-1} \end{pmatrix} \in T(\mathbb{A}_\mathrm{f})$. Let $\mathfrak{c}_i$ be the ideal of K such that $\mathrm{ord}_\mathfrak{p}(\mathfrak{c}_i) = \mathrm{ord}_\mathfrak{p}((\det t_i)_\mathfrak{p})$, where $\det t_i$ is considered as an element of $\mathbb{A}_K^\times$. Then the bijection $\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A})/\mathcal{K}\mathcal{G}(\mathbb{R}) \to \mathcal{C}_K$ is given by

$$\mathcal{G}(\mathbb{Q})\gamma_i\mathcal{K}\mathcal{G}(\mathbb{R}) \mapsto \text{the ideal class of } \mathfrak{c}_i.$$

Put

$$\Gamma_i = \Gamma_i^{(m)} = \mathcal{G}(\mathbb{Q}) \cap (\gamma_i\mathcal{K}\gamma_i^{-1} \cdot \mathcal{G}(\mathbb{R})).$$

Then Γ_i is a hermitian modular group of general type considered in §12. Put

$$\Lambda_m(\mathcal{O})_i = \{H \in \mathcal{H}_m(K) \mid \bar{t}_{i,p}Ht_{i,p} \in \Lambda_m(\mathcal{O}_p) \text{ for any } p\}.$$

The set of positive definite elements of $\Lambda_m(\mathcal{O})_i$ is denoted by $\Lambda_m(\mathcal{O})_i^+$.

For a holomorphic function F on $\mathcal{H}_m$ and $g \in \mathcal{G}(\mathbb{R})$, we introduce a notation

$$F||_{2l}\, g = (\det g)^l \cdot F|_{2l}\, g.$$

When l is clear from the context, we drop it from the notation. For $m \geq 2$, put

$$M_{2l}(\Gamma_i, \det^{-l}) = \{F \mid F||_{2l}\, g = F \text{ for any } g \in \Gamma_i\}.$$

For $m = 1$, we require the usual holomorphy condition at cusps. Then $F \in M_{2l}(\Gamma_i, \det^{-l})$ has a Fourier expansion of the form

$$F(Z) = \sum_{\substack{H \in \Lambda_m(\mathcal{O})_i \\ H \geq 0}} A(H)\mathbf{e}(HZ).$$

The space of cusp forms $S_{2l}(\Gamma_i, \det^{-l})$ is defined by

$$\{F \in M_{2l}(\Gamma_i, \det^{-l}) \mid A(H) = 0 \text{ unless } H \in \Lambda_m(\mathcal{O})_i^+\}.$$

Put $\mathbf{i} = \sqrt{-1} \cdot \mathbf{1}_m \in \mathcal{H}_m$. For $(F_1, \ldots, F_h) \in \oplus_{i=1}^h M_{2l}(\Gamma_i, \det^{-2l})$, we put

$$(F_1, \ldots, F_h)^\sharp(g) = (F_i||_{2l}\, x)(\mathbf{i}) = F_i(x\langle \mathbf{i} \rangle)\, j(x, \mathbf{i})^{-2l}(\det x)^l$$

for $g = u\gamma_i x\kappa$, $u \in \mathcal{G}(\mathbb{Q})$, $x \in \mathcal{G}(\mathbb{R})$, $\kappa \in \mathcal{K}$. Then $(F_1, \ldots, F_h)^\sharp$ is an automorphic form on $\mathcal{G}(\mathbb{A})$. We denote by $\mathcal{M}_{2l}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}), \det^{-l})$ the space of automorphic forms obtained in this way. Similarly, we put $\mathcal{S}_{2l}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}), \det^{-l}) = \{(F_1, \ldots, F_h)^\sharp \mid F_i \in S_{2l}(\Gamma_i, \det^{-l})\}$.

Let $\mathbf{f}_{2l}$ be the function on $\mathcal{G}(\mathbb{A})$ defined by

$$\mathbf{f}_{2l}(g) = \prod_p |\det(d_p \bar{d}_p)|_p^{-l}\, j(g_\infty, \mathbf{i})^{-2l}(\det g_\infty)^l,$$

where $g = (g_v)_v \in \mathcal{G}(\mathbb{A})$, $g_p = \left(\begin{smallmatrix} a_p & b_p \\ 0 & d_p \end{smallmatrix}\right)\kappa_p$, $\left(\begin{smallmatrix} a_p & b_p \\ 0 & d_p \end{smallmatrix}\right) \in P(\mathbb{Q}_p)$, $\kappa_p \in \mathcal{K}_p$. We consider the normalized Eisenstein series

$$\mathbf{E}_{2l}^{(m)}(g) = 2^{-m} \prod_{i=1}^m L(i - 2l, \chi^{i-1}) \sum_{\gamma \in P(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q})} \mathbf{f}_{2l}(\gamma g).$$

Then we have $\mathbf{E}_{2l}^{(m)} \in \mathcal{M}_{2l}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}), \det^{-l})$. We denote the corresponding Eisenstein series for Γ_i by $\mathcal{E}_{i,2l}$. Thus $(\mathcal{E}_{i,2l}, \ldots, \mathcal{E}_{i,2l})^\sharp = \mathbf{E}_{2l}^{(m)}$ and

$$\mathcal{E}_{i,2l}(Z) = 2^{-m} \prod_{i=1}^m L(i - 2l, \chi^{i-1}) \sum_{g \in \Gamma_{i,\infty}\backslash\Gamma_i} (\det g)^l j(g, Z)^{-2l},$$

where $\Gamma_{i,\infty} = P(\mathbb{Q}) \cap \Gamma_i$. Note that $P(\mathbb{Q})\backslash\mathcal{G}(\mathbb{Q}) \simeq \Gamma_{i,\infty}\backslash\Gamma_i$ by Proposition 12.6. As in §4, one can show that the H-th Fourier coefficient of $\mathcal{E}_{i,2l}(Z)$ is equal to

$$|\gamma(H)|^{l-(m/2)} \prod_p |\det t_{i,p}\bar{t}_{i,p}|_p^{m/2} \tilde{F}_p(\bar{t}_{i,p} H t_{i,p}\,; p^{-l+(m/2)})$$

for any $H \in \Lambda_m(\mathcal{O})_i^+$ and any sufficiently large integer l. Set

$$\Phi_i(H, \mathbb{X}) = \prod_p |\det t_{i,p}\bar{t}_{i,p}|_p^{m/2} \tilde{F}_p(\bar{t}_{i,p} H t_{i,p}\,; X_p) \in \mathcal{R}.$$

Then, for sufficiently large k', the H-th Fourier coefficient of $\mathcal{E}_{i,2k'+2n}$ is equal to $|\gamma(H)|^{k'-(\varepsilon/2)}\Phi_i(H, \{p^{-k'+(\mathfrak{e}/2)}\})$, where $n = [m/2]$, $\mathfrak{e} = m - 2n$.

Theorem 13.2. *In Case E, let $f(\tau) \in S_{2k+1}(\Gamma_0(D), \chi)$ be a primitive form with Satake parameter $\{\alpha_p, \beta_p\}$. Then for $i = 1, \ldots, h$, the Fourier series*

$$F_i(Z) = \sum_{H \in \Lambda_{2n}(\mathcal{O})_i^+} |\gamma(H)|^k \Phi_i(H, \{\alpha_p\})\mathbf{e}(HZ)$$

belongs to $S_{2k+2n}(\Gamma_i^{(2n)}, \det^{-k-n})$. Put $\mathcal{F} = (F_1, \ldots, F_h)^\sharp$. Then $\mathcal{F}$ is independent of the choice of the representatives $\{\gamma_1, \ldots, \gamma_h\}$.

Theorem 13.3. *In Case O, let $f(\tau) \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform with Satake parameter $\{\alpha_p, \alpha_p^{-1}\}$. Then for $i = 1, \ldots, h$, the Fourier series*

$$F_i(Z) = \sum_{H \in \Lambda_{2n+1}(\mathcal{O})_i^+} |\gamma(H)|^{k-(1/2)} \Phi_i(H, \{\alpha_p\})\mathbf{e}(HZ)$$

belongs to $S_{2k+2n}(\Gamma_i^{(2n+1)}, \det^{-k-n})$. Put $\mathcal{F} = (F_i, \ldots, F_i)^\sharp$. Then $\mathcal{F}$ is independent of the choice of the representatives $\{\gamma_1, \ldots, \gamma_h\}$.

Proof. We treat only Theorem 13.3. The proof of Theorem 13.2 can be treated in the same way.

By Proposition 12.5, $\mathbf{\Gamma}$ is generated by $M_{\mathbf{\Gamma}} N_{\mathbf{\Gamma}}$ and w_j, $(j = 1, \ldots, m)$, where $M_{\mathbf{\Gamma}}$, $N_{\mathbf{\Gamma}}$ and w_j are as in §12. Clearly, $F_i(Z)$ is modular with respect to $M_{\mathbf{\Gamma}} N_{\mathbf{\Gamma}}$. As in §10, one can prove that $F_i(Z)$ is modular with respect to w_m. By permutation of coordinates, $F_i(Z)$ is also modular with respect to w_j $(j = 1, \ldots, m)$. Therefore $F_i(Z)$ is a hermitian modular form with respect to Γ_i. The modularity of F_i follows from this. That F_i is a cusp form follows from Proposition 12.6.

We prove that the definition of $\mathcal{F}$ is independent of the choice of $\{\gamma_1, \ldots, \gamma_h\}$. Let M be the Levi factor of the parabolic subgroup P such that

$$M(\mathbb{Q}) = \left\{ \begin{pmatrix} A & 0 \\ 0 & {}^t\bar{A}^{-1} \end{pmatrix} \,\middle|\, A \in \mathrm{GL}_m(K) \right\}.$$

Then the natural map

$$M(\mathbb{Q})\backslash M(\mathbb{A})/(\mathcal{K} \cap M(\mathbb{A}_\mathrm{f}))M(\mathbb{R}) \to \mathcal{G}(\mathbb{Q})\backslash \mathcal{G}(\mathbb{A})/\mathcal{K}\mathcal{G}(\mathbb{R})$$

is surjective. Let $\{\gamma'_1, \dots, \gamma'_h\}$, $\gamma'_i = \begin{pmatrix} t'_i & 0 \\ 0 & \bar{t}'^{-1}_i \end{pmatrix} \in T(\mathbb{A}_{\mathrm{f}})$ be another set of representatives. Then there exists $u_i \in M(\mathbb{Q})$ such that $t'_i \in u_i t_i(\mathcal{K} \cap M(\mathbb{A}_{\mathrm{f}}))M(\mathbb{R})$. By multiplying some element in $\mathcal{K}$ from the right, we may assume $t'_i = u_{i,\mathrm{f}} t_i$, where $u_i = u_{i,\mathrm{f}} u_{i,\infty}$, $u_{i,\mathrm{f}} \in M(\mathbb{A}_{\mathrm{f}})$, $u_{i,\infty} \in M(\mathbb{R})$. Note that

$$\gamma_i x = \begin{pmatrix} u_i^{-1} & 0 \\ 0 & {}^t\bar{u}_i \end{pmatrix} \gamma'_i \begin{pmatrix} u_{i,\infty} & 0 \\ 0 & {}^t\bar{u}_{i,\infty}^{-1} \end{pmatrix} x$$

for $x \in \mathcal{G}(\mathbb{R})$. We define $\mathcal{E}'_{i,2l}$ and F'_i by using the the representative γ'_i instead of γ_i. Then we have

$$\begin{aligned}(\mathcal{E}_{i,2l}||_{2l} x)(\mathbf{i}) &= \mathbf{E}_{2l}^{(m)}(\gamma_i x) \\ &= \left(\mathcal{E}'_{i,2l}||_{2l} \begin{pmatrix} u_{i,\infty} & 0 \\ 0 & {}^t\bar{u}_{i,\infty}^{-1} \end{pmatrix} x \right)(\mathbf{i}).\end{aligned}$$

It follows that $\mathcal{E}'_{i,2l} = \mathcal{E}_{i,2l}||_{2l} \begin{pmatrix} u_i^{-1} & 0 \\ 0 & {}^t\bar{u}_i \end{pmatrix}$. Comparing the Fourier expansion, we have $F'_i = F_i||_{2k+2n} \begin{pmatrix} u_i^{-1} & 0 \\ 0 & {}^t\bar{u}_i \end{pmatrix}$, which implies the desired independence. □

Definition 13.4. We call $\mathcal{F}$ in Theorem 13.2 or Theorem 13.3 the lift of $f(\tau)$ to $\mathcal{S}_{2k+2n}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}), \det^{-k-n})$ and denote it by $\mathit{Lift}^{(m)}(f)$.

Next, we shall show that $\mathit{Lift}^{(m)}(f)$ is a common Hecke eigenform for any Hecke operators if it is not identically zero. Recall that the Eisenstein series $\mathbf{E}_{2l}^{(m)}$ is a common Hecke eigenform for any Hecke operators for $(\mathcal{G}(\mathbb{A}_{\mathrm{f}}), \mathcal{K})$.

Recall that the action of a Hecke operator can be described as follows. Let $T = \mathcal{K}q\mathcal{K}$ be a double coset for $q \in \mathcal{G}(\mathbb{A}_{\mathrm{f}})$. Let

$$\mathcal{K}q\mathcal{K} = \coprod_{r \in I} \mathcal{K}q_r, \qquad q_r \in \mathcal{G}(\mathbb{A}_{\mathrm{f}})$$

be a decomposition into a sum of left cosets, where I is a finite index set. For each $r \in I$, and $j \in \{1, \dots, h\}$, choose $i_{r,j} \in \{1, \dots, h\}$ and $u_{r,j} \in \mathcal{G}(\mathbb{Q})$ such that $\gamma_j q_r \in u_{r,j}\gamma_{i_{r,j}}\mathcal{K}\mathcal{G}(\mathbb{R})$. Then the action of the Hecke operator T on $\mathcal{S}_{2l}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}), \det^{-l})$ is given by

$$(F_1, \dots, F_h)^\sharp | T = (\sum_{r \in I} F_{i_{r,1}}||_{2l}\, u_{r,1}^{-1}, \dots, \sum_{r \in I} F_{i_{r,h}}||_{2l}\, u_{r,h}^{-1})^\sharp.$$

Note that one can assume $u_{r,i} \in P(\mathbb{Q})$ by Proposition 12.6.

Proposition 13.5. *For each Hecke operator T for $(\mathcal{G}(\mathbb{A}_{\mathrm{f}}), \mathcal{K})$, there exists an element $\Phi_T \in \mathcal{R}$ such that*

$$\mathbf{E}_{2k'+2n}^{(m)} | T = \Phi_T(\{p^{-k'+(\varepsilon/2)}\})\mathbf{E}_{2k'+2n}^{(m)}.$$

where $\varepsilon = 0$ in Case E and $\varepsilon = 1$ in Case O.

Proof. Put $\mathbf{E}^{(m)}_{2k'+2n}|T = (E_{1,T}, \dots, E_{h,T})$. Then, by using Proposition 12.6, one can easily show that $E_{i,T}$ has an Fourier expansion

$$E_{i,T}(Z) = \sum_{H \in \Lambda_m(\mathcal{O})_i^+} |\gamma(H)|^{k'-(\varepsilon/2)} \Phi'_{i,T}(H, \{p^{-k'+(\varepsilon/2)}\}).$$

for some $\Phi'_{i,T}(\mathbb{X}) \in \mathcal{R}$ (cf. [9], p.664). If $m \not\equiv 2 \bmod 4$, then there exists an element $H \in \Lambda_m(\mathcal{O})$ such that $|\gamma(H)| = 1$ by Lemma 11.1 and Lemma 11.2. For such an H, we have $\tilde{F}_p(H, X) = 1$ for any prime p. Therefore the eigenvalue is equal to $\Phi'_{1,T}(H, \{p^{-k'+(\varepsilon/2)}\})$. If $m \equiv 2 \bmod 4$, there exist an infinite number of prime p such that $p = -\gamma(H)$ for some $H \in \Lambda_m(\mathcal{O})$ by Lemma 11.4. Choose such primes $p_1 \neq p_2$ with $p_1 = -\gamma(H_1)$ and $p_2 = -\gamma(H_2)$. Note that $\mathrm{ord}_{p_1} F_{p_1}(H_1, X) = \mathrm{ord}_{p_2} F_{p_2}(H_2, X) = 1$. By the functional equation, we have $F_{p_1}(H_1, X) = F_{p_2}(H_2, X) = 1 + X$. It follows that

$$\prod_p \tilde{F}_p(H_1, X_p) = X_{p_1} + X_{p_1}^{-1},$$
$$\prod_p \tilde{F}_p(H_2, X_p) = X_{p_2} + X_{p_2}^{-1}.$$

By Lemma 7.1, we have

$$(X_{p_1} + X_{p_1}^{-1}) \Phi'_{1,T}(H_2, \mathbb{X}) = (X_{p_2} + X_{p_2}^{-1}) \Phi'_{1,T}(H_1, \mathbb{X}).$$

By unique factorization of $\mathcal{R}$, $\Phi'_{1,T}(H_1, \mathbb{X})/(X_{p_1} + X_{p_1}^{-1}) \in \mathcal{R}$. Hence the proposition. □

Theorem 13.6. *The automorphic form $Lift^{(m)}(f)$ is a common eigenform of all Hecke operators for $(\mathcal{G}(\mathbb{A}_\mathrm{f}), \mathcal{K})$. Moreover, for each Hecke operator T for $(\mathcal{G}(\mathbb{A}_\mathrm{f}), \mathcal{K})$, we have*

$$Lift^{(m)}(f)|T = \Phi_T(\{\alpha_p\}) Lift^{(m)}(f).$$

Proof. As in §11 of [9], the theorem follows from Proposition 13.5 and Lemma 7.1.

□

Let $\mathfrak{c}$ be an integral ideal of K. We assume $\mathfrak{c}$ is prime to D. Set $C = \mathrm{N}(\mathfrak{c})$. We choose a finite idele $t = (t_\mathfrak{p})_\mathfrak{p} \in \mathbb{A}_{K,\mathrm{f}}$ such that $\mathrm{ord}_\mathfrak{p} t_\mathfrak{p} = \mathrm{ord}_\mathfrak{p} \mathfrak{c}$ for each prime ideal $\mathfrak{p}$ of K. Set

$$\mathbf{t} = \begin{pmatrix} \mathbf{1}_{m-1} & 0 \\ 0 & t \end{pmatrix}, \qquad \gamma = \begin{pmatrix} \mathbf{t} & 0 \\ 0 & \bar{\mathbf{t}}^{-1} \end{pmatrix}.$$

Definition 13.7. We set

$$\begin{aligned}\Gamma_K^{(m)}[\mathfrak{c}] =&\mathcal{G}(\mathbb{Q}) \cap (\gamma\mathcal{K}\gamma^{-1} \cdot \mathcal{G}(\mathbb{R})),\\ \Lambda_m^{\mathfrak{c}}(\mathcal{O})^+ =&\{H \in \mathcal{H}_m(K) \big| H > 0,\ \bar{\mathbf{t}}_p H \mathbf{t}_p \in \Lambda_m(\mathcal{O}_p) \text{ for any } p\}.\end{aligned}$$

If $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized Hecke eigenform, then the lift of f in $S_{2k+2n}(\Gamma_K^{(2n+1)}[\mathfrak{c}], \det^{-k-n})$ is denoted by $\mathrm{Lift}_{\mathfrak{c}}^{(2n+1)}(f)$. Similarly, if $f \in S_{2k+1}(\Gamma_0(D), \chi)$ is a primitive form, then the lift of f in $S_{2k+2n}(\Gamma_K^{(2n)}[\mathfrak{c}], \det^{-k-n})$ is denoted by $\mathrm{Lift}_{\mathfrak{c}}^{(2n)}(f)$. When $\mathfrak{c} = \mathcal{O}$, we simply drop $\mathfrak{c}$ from the notation.

Note that $\Gamma_K^{(m)}[\mathfrak{c}]$ is a hermitian modular group of general type considered in §12. We have

$$\mathrm{Lift}_{\mathfrak{c}}^{(m)}(f) = C^{-k-n} \sum_{H \in \Lambda_m^{\mathfrak{c}}(\mathcal{O})^+} |C\gamma(H)|^{k-(\mathfrak{e}/2)} \prod_p \tilde{F}_p(\bar{\mathbf{t}}_p H \mathbf{t}_p\,; \alpha_p)\mathbf{e}(HZ),$$

where $\mathfrak{e} = m - 2[m/2]$. If $\mathfrak{c}_1 = \mathcal{O}, \mathfrak{c}_2, \ldots, \mathfrak{c}_h$ is a representative for the ideal class group, then we have

$$\mathit{Lift}^{(m)}(f) = (\mathrm{Lift}(f), \mathrm{Lift}_{\mathfrak{c}_2}(f), \ldots, \mathrm{Lift}_{\mathfrak{c}_h}(f))^\sharp.$$

The proof of the following lemmas are the same as Lemma 11.2 and Lemma 11.3. We omit the detail.

Lemma 13.8. *Assume that m is odd. For any integer $N > 0$, there exists an element $H_N \in \Lambda_m^{\mathfrak{c}}(\mathcal{O})^+$ such that $C|\gamma(H_N)| = N$.*

Lemma 13.9. *Assume that $m = 2n$ is even. Let $N > 0$ be a rational integer such that there exists an element $y \in \mathfrak{c}^{-1}\mathcal{O}^\sharp$ such that $CDy\bar{y} \equiv (-1)^n N \bmod D$. Then there exists an element $H_N \in \Lambda_m^{\mathfrak{c}}(\mathcal{O})^+$ such that $C|\gamma(H_N)| = N$.*

14. **Linearization of the lifting (Case O)**

In this section, we consider only Case O. Put $m = 2n+1$. We fix an integral ideal $\mathfrak{c}$ of K such that $C = \mathrm{N}(\mathfrak{c})$ is prime to D. We choose $\mathbf{t}$ as in the last section. Let $f(\tau) \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. For each $N > 0$, we put

$$\begin{aligned}\Psi_p(N; X) =& p^{-e/2}\frac{X^{e+1} - X^{-e-1}}{X - X^{-1}}, \quad e = \mathrm{ord}_p N,\\ \Psi(N; \mathbb{X}) =& \prod_{p|N} \Psi_p(N; X_p) \in \mathcal{R}.\end{aligned}$$

Then the N-th Fourier coefficient $a_f(N)$ of f is equal to $N^k \Psi(N; \{\alpha_p\})$.

Fix an element $H \in \Lambda_{2n+1}^{\mathfrak{c}}(\mathcal{O})^+$. Then $\tilde{F}_p(\bar{\mathbf{t}}_p H \mathbf{t}_p; X)$ belongs to the $\mathbb{C}$-vector space

$$\{\Phi \in X^{-\operatorname{ord}_p C\gamma(H)}\mathbb{C}[X^2] \mid \Phi(X^{-1}) = \Phi(X)\}.$$

Note that $\{\Psi_p(\frac{C|\gamma(H)|}{p^{2i}}; X) \mid 0 \le 2i \le \operatorname{ord}_p C\gamma(H)\}$ is a basis of this vector space. It follows that there exists $\phi(a, H) \in \mathbb{C}$ for each $a^2 | C\gamma(H)$ such that

$$|C\gamma(H)|^{-1/2} \prod_{p|C\gamma(H)} \tilde{F}_p(\bar{\mathbf{t}}_p H \mathbf{t}_p; X_p) = \sum_{a^2|C\gamma(H)} \phi(a, H)\Psi(\tfrac{C|\gamma(H)|}{a^2}; \mathbb{X}).$$

One can easily show that $\phi(a, H) \in \mathbb{Q}$. Moreover, $\phi(1, H) = 1$ for any $H \in \Lambda_{2n+1}^{\mathfrak{c}}(\mathcal{O})^+$, as the constant term of $F_p(\bar{\mathbf{t}}_p H \mathbf{t}_p; X)$ is 1.

For each $f_0(\tau) = \sum_{N>0} a_{f_0}(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, we put

$$\iota(f_0)(Z) = \sum_{H \in \Lambda_{2n+1}^{\mathfrak{c}}(\mathcal{O})^+} \sum_{a^2|C\gamma(H)} a^{2k}\phi(a, H)a_{f_0}(\tfrac{C|\gamma(H)|}{a^2})\mathbf{e}(HZ).$$

If f is a normalized Hecke eigenform, then $\iota(f) = C^{k+n}\mathrm{Lift}_{\mathfrak{c}}^{(2n+1)}(f)$. Since normalized Hecke eigenforms span $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, the image of ι is contained in $S_{2k+2n}(\Gamma_K^{(2n+1)}[\mathfrak{c}], \det^{-k-n})$.

We shall show that ι is injective. Assume that $\iota(f_0) = 0$ for some $f_0 \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$. We show that all Fourier coefficients of f_0 are 0. By Lemma 13.8, there exists an element $H_N \in \Lambda_{2n+1}^{\mathfrak{c}}(\mathcal{O})^+$ such that $C|\gamma(H_N)| = N$. Then the H_N-th Fourier coefficient of $\iota(f_0)$ is $a_{f_0}(N)+$ (lower terms). It follows that $a_{f_0}(N) = 0$ by induction. Now we have proved the following theorem.

Theorem 14.1. *There exists an injective linear map*

$$\iota : S_{2k}(\mathrm{SL}_2(\mathbb{Z})) \to S_{2k+2n}(\Gamma_K^{(2n+1)}[\mathfrak{c}], \det^{-k-n})$$

satisfying the following properties:

(1) *For each* $f_0 = \sum_{N>0} a_{f_0}(N)q^N \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$,

$$\iota(f_0)(Z) = \sum_{H \in \Lambda_{2n+1}^{\mathfrak{c}}(\mathcal{O})^+} \sum_{a^2|C\gamma(H)} a^{2k}\phi(a, H)a_{f_0}(\tfrac{C|\gamma(H)|}{a^2})\mathbf{e}(HZ).$$

(2) *If* $f(\tau) \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ *is a normalized Hecke eigenform, then* $\iota(f) = C^{k+n}\mathrm{Lift}_{\mathfrak{c}}^{(2n+1)}(f)$.

Corollary 14.2. *Let* $f(\tau) \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ *be a normalized Hecke eigenform. Then* $\mathrm{Lift}_{\mathfrak{c}}^{(2n+1)}(f)$ *is not identically zero.*

Obviously, Corollary 14.2 implies that $\mathit{Lift}^{(2n+1)}(f) \neq 0$ for any normalized Hecke eigenform $f \in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$.

15. **Linearization of the lifting (Case E)**

In this section, we consider Case E. Put $m = 2n$. We fix an integral ideal $\mathfrak{c}$ of K such that $C = \mathrm{N}(\mathfrak{c})$ is prime to D. We are going to show that the lifting can be described by a linear map

$$S^{\varepsilon *}_{2k+1}(\Gamma_0(D), \chi) \to S_{2k+2n}(\Gamma_K^{(2n)}[\mathfrak{c}], \det{}^{-k-n})$$

where $S^{\varepsilon *}_{2k+1}(\Gamma_0(D), \chi)$ is a certain subspace of $S_{2k+1}(\Gamma_0(D), \chi)$. Unlike Case O, the subspace $S^{\varepsilon *}_{2k+1}(\Gamma_0(D), \chi)$ depends on the ideal class (more precisely, the genus) of $\mathfrak{c}$.

Let Q_D be the set of all primes which divides D. For each prime $q \in Q_D$, we put $D_q = q^{\mathrm{ord}_q D}$. We define a primitive Dirichlet character χ_q by

$$\chi_q(N) = \begin{cases} \chi(N') & \text{if } (N, q) = 1 \\ 0 & \text{if } q | N, \end{cases}$$

where N' is an integer such that

$$N' \equiv \begin{cases} N & \mod D_q \\ 1 & \mod D_q^{-1} D. \end{cases}$$

Then we have $\chi = \prod_{q|D} \chi_q$. Note that D_q is the conductor of χ_q and that χ_q corresponds to the quadratic field with discriminant $\chi_q(-1)D_q$. One should not confuse χ_q with $\underline{\chi}_q$.

Lemma 15.1. *If $q \nmid N$, then $\chi_q(N) = \underline{\chi}_q(N)$.*

Proof. Since χ_q corresponds to $\mathbb{Q}(\sqrt{\chi_q(-1)D_q})/\mathbb{Q}$, we have

$$\begin{aligned} \chi_q(N) &= \left(\frac{\chi_q(-1)D_q, N}{\mathbb{Q}_q} \right) \\ &= \left(\frac{-D, N}{\mathbb{Q}_q} \right) \times \prod_{\substack{q' \in Q_D \\ q' \neq q}} \left(\frac{\chi_{q'}(-1)D_{q'}, N}{\mathbb{Q}_q} \right). \end{aligned}$$

In this equation, the second factor is trivial since both $\chi_{q'}(-1)D_{q'}$ and N are units in $\mathbb{Q}_q$, and $\chi_{q'}(-1)D_{q'} \equiv 1 \mod 4$ if $q = 2$. $\square$

When Q is a subset of Q_D, we set

$$\chi_Q = \prod_{q \in Q} \chi_q, \qquad \chi'_Q = \prod_{\substack{q \in Q_D \\ q \notin Q}} \chi_q, \qquad D_Q = \prod_{q \in Q} D_q.$$

When $Q = \{q\}$, $\chi'_{\{q\}}$ is simply denoted by χ'_q.

As in §13, let $\mathfrak{c}$ be an integral ideal of K such that $C = \mathrm{N}(\mathfrak{c})$ is prime to D. Put $\varepsilon(Q) = \chi_Q(C)$. Then we have

$$\varepsilon(\emptyset) = \varepsilon(Q_D) = 1,$$
$$\varepsilon(Q)\varepsilon(Q') = \varepsilon(Q \cup Q')\varepsilon(Q \cap Q')$$

for any $Q, Q' \subset Q_D$. By genus theory (see e.g., Hecke [6] §48), any function $\varepsilon : \{Q \mid Q \subset Q_D\} \to \{\pm 1\}$ with these properties is obtained in this way. Moreover, two integral ideals $\mathfrak{c}$ and $\mathfrak{c}'$ give the same function ε if and only if $\mathfrak{c}$ and $\mathfrak{c}'$ belong to the same genus. If $Q = \{q\}$, we denote $\varepsilon(\{q\})$ simply by $\varepsilon(q)$.

We fix a primitive form $f = \sum a_f(N)q^N \in S_{2k+1}(\Gamma_0(D), \chi)$. Recall that for each subset $Q \subset Q_D$, there exists a primitive form

$$f_Q = \sum_{N>0} b(N)q^N \in S_{2k+1}(\Gamma_0(D), \chi)$$

such that

$$\begin{cases} b(p) = \chi_Q(p)a_f(p) & \text{if } p \notin Q \\ b(q) = \chi'_Q(q)\overline{a_f(q)} & \text{if } q \in Q. \end{cases}$$

(See Miyake [19] Theorem 4.6.16.) Note that $(f_Q)_{Q'} = f_{Q''}$ where $Q'' = (Q \cup Q') - (Q \cap Q')$.

Definition 15.2. If $f \in S_{2k+1}(\Gamma_0(D), \chi)$ is a primitive form, we put

$$f^{\varepsilon *} = \sum_{Q \subset Q_D} \varepsilon(Q)\chi_Q(-1)^n f_Q.$$

When $\varepsilon = 1$, $f^{\varepsilon *}$ is simply denoted by f^*.

Obviously, $(f_Q)^{\varepsilon *} = \varepsilon(Q)\chi_Q(-1)^n f^{\varepsilon *}$.

Definition 15.3. Following Krieg [16], we define

$$\mathbf{a}_D^{\varepsilon}(N) = \prod_{q \in Q_D} (1 + \varepsilon(q)\chi_q((-1)^n N)) = \prod_{\substack{q \in Q_D \\ q \nmid N}} (1 + \underline{\chi}_q((-1)^n CN)).$$

Here, $\varepsilon(q) = \varepsilon(\{q\})$. When $\varepsilon = 1$, $\mathbf{a}_D^{\varepsilon}(N)$ is simply denoted by $\mathbf{a}_D(N)$. Note that $\mathbf{a}_D^{\varepsilon}(N) = \mathbf{a}_D(CN)$.

Lemma 15.4. *The N-th Fourier coefficient of f_Q is equal to*

$$a_{f_Q}(N) = a_f(N'N'_Q)\overline{a_f(N_Q)} \prod_{q \in Q} \underline{\chi}_q(N),$$

where

$$N' = \prod_{p \nmid D} p^{\mathrm{ord}_p N}, \quad N'_Q = \prod_{\substack{q \in Q_D \\ q \notin Q}} q^{\mathrm{ord}_q N}, \quad N_Q = \prod_{q \in Q} q^{\mathrm{ord}_q N}.$$

Proof. It is enough to consider the case when $Q = \{q\}$ and $q|N$. In this case, we have

$$\begin{aligned} a_{f_Q}(N) =& a_{f_Q}(N')a_{f_Q}(N'_Q)a_{f_Q}(N_Q) \\ =& a_f(N'N'_Q)\overline{a_f(N_Q)}\chi_q(N'N'_Q)\chi'_q(N_Q). \end{aligned}$$

By Lemma 15.1, we have $\chi_q(N'N'_Q) = \underline{\chi}_q(N'N'_Q)$. We shall show that

$$\chi'_q(N_Q) = \left(\frac{-D, N_Q}{\mathbb{Q}_q}\right) = \underline{\chi}_q(N_Q).$$

It is enough to show that $\chi'_q(q) = \underline{\chi}_q(q)$, since N_Q is a power of q. By Lemma 15.1, we have

$$\chi'_q(q) = \prod_{\substack{q' \in Q_D \\ q' \neq q}} \chi_{q'}(q) = \prod_{\substack{q' \in Q_D \\ q' \neq q}} \left(\frac{-D, q}{\mathbb{Q}_{q'}}\right).$$

By Hilbert product formula,

$$\chi'_q(q)\underline{\chi}_q(q) = \left(\frac{-D, q}{\mathbb{R}}\right) \prod_{p \notin Q_D} \left(\frac{-D, q}{\mathbb{Q}_p}\right).$$

The first factor of the right hand side is 1 since $q > 0$. The second factor is 1 because both $-D$ and q are units in $\mathbb{Q}_p$, and $-D \equiv 1 \mod 4$ if $2 \notin Q_D$. Hence the lemma. □

Corollary 15.5. *The N-th Fourier coefficient of $f^{\varepsilon *}$ is given by*

$$\begin{aligned} a_{f^{\varepsilon *}}(N) =& a_f(N') \prod_{q|D} (a_f(N_q) + \underline{\chi}_q((-1)^n CN)\overline{a_f(N_q)}) \\ =& \mathbf{a}_D^{\varepsilon}(N) a_f(N') \prod_{q|(D,N)} (a_f(N_q) + \underline{\chi}_q((-1)^n CN)\overline{a_f(N_q)}). \end{aligned}$$

Here $N_q = N_{\{q\}}$. In particular, we have $a_{f^{\varepsilon *}}(N) = \mathbf{a}_D^{\varepsilon}(N)a_f(N)$ for $(N, D) = 1$. Note that $f^{\varepsilon *}$ is characterized as the unique element of $S_{2k+1}(\Gamma_0(D), \chi)$ with this property.

Corollary 15.6. *$f^{\varepsilon *}(\tau) = 0$ if and only if $f(\tau)$ comes from a Hecke character of $\mathbb{Q}(\sqrt{\chi_Q(-1)D_Q})$, for some $Q \subset Q_D$, $\varepsilon(Q)\chi_Q(-1)^n = -1$. In particular, $f^* = 0$ if and only if n is odd and $f(\tau)$ comes from a Hecke character $\mathbb{Q}(\sqrt{-D_Q})$ for some $Q \subset Q_D$, $\chi_Q(-1) = -1$.*

Proof. If $f(\tau)$ comes from a Hecke character of $\mathbb{Q}(\sqrt{\chi_Q(-1)D_Q})$ with $\varepsilon(Q)\chi_Q(-1)^n = -1$, then obviously $f^{\varepsilon *} = 0$, since $f^{\varepsilon *} = (f_Q)^{\varepsilon *} = -f^{\varepsilon *}$. Conversely, assume that $f^{\varepsilon *} = \sum_{Q \subset Q_D} \varepsilon(Q)\chi_Q(-1)^n f_Q = 0$. As the primitive forms are linearly independent, there exists a subset $Q \subset Q_D$

such that $f_Q = f$ and $\varepsilon(Q)\chi_Q(-1)^n = -1$. It follows that f comes from a Hecke character of $\mathbb{Q}(\sqrt{\chi_Q(-1)D_Q})$ by Labesse and Langlands [18]. ☐

Definition 15.7. For each primitive form $f \in S_{2k+1}(\Gamma_0(D), \chi)$, put

$$\eta_n^\varepsilon(f) = \sum_{\substack{Q \subset Q_D \\ f_Q = f}} \varepsilon(Q)\chi_Q(-1)^n.$$

Lemma 15.8. *If $k > 0$, then $0 \leq \eta_n^\varepsilon(f) \leq 2$. When $k = 0$, we have $0 \leq \eta_n^\varepsilon(f) \leq 4$.*

Proof. That $\eta_n^\varepsilon(f) \leq 4$ follows from Labesse-Langlands [18]. To prove the first part, it is enough to prove that if $k > 0$, then there is no non-empty $Q \subset Q_D$ such that $f = f_Q$, $\chi_Q(-1) = 1$. Assume $f = f_Q$, $Q \neq \emptyset$, $\chi_Q(-1) = 1$. Let K_Q be the quadratic field corresponding to χ_Q. Since $\chi_Q(-1) = 1$, the quadratic field K_Q is real. Then $f(\tau)$ come from a Hecke character of K_Q. Comparing the gamma factor, it is impossible if $k > 2$. ☐

Lemma 15.9. *$f^{\varepsilon *} = 0$ if and only if $\eta_n^\varepsilon(f) = 0$.*

Proof. This lemma follows from Corollary 15.6. ☐

Recall that the Petersson inner product of cusp forms $f_1, f_2 \in S_l(\Gamma')$ for a congruence subgroup $\Gamma' \subset \mathrm{SL}_2(\mathbb{Z})$ is given by

$$\langle f_1, f_2 \rangle = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma' \cdot \{\pm 1\}]^{-1} \int_{\Gamma' \backslash \mathfrak{H}_1} f_1(\tau)\overline{f_2(\tau)} y^{l-2} dx\, dy.$$

The complete adjoint L-function $\Lambda(s, f, \mathrm{Ad})$ is defined by

$$\begin{aligned}
\Lambda(s, f, \mathrm{Ad}) =& \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{C}}(s + 2k)L(s, f, \mathrm{Ad}), \\
\Gamma_{\mathbb{R}}(s) =& \pi^{-s/2}\Gamma(s/2), \qquad \Gamma_{\mathbb{C}}(s) = 2(2\pi)^{-s}\Gamma(s), \\
L(s, f, \mathrm{Ad}) =& \prod_{p \nmid D} \left[(1 - \alpha_p\beta_p^{-1}p^{-s})(1 - p^{-s})(1 - \alpha_p^{-1}\beta_p p^{-s})\right]^{-1} \\
& \times \prod_{q | D} (1 - q^{-s})^{-1}.
\end{aligned}$$

The following lemma is well-known.

Lemma 15.10. *Let $f \in S_{2k+1}(\Gamma_0(D), \chi)$ be a primitive form. Then we have*

$$\langle f, f \rangle = 2^{-2k-1}\Lambda(1, f, \mathrm{Ad}) \prod_{q|D} (1 + q^{-1})^{-1},$$

In particular, we have $\langle f, f \rangle = \langle f_Q, f_Q \rangle$ for any $Q \subset Q_D$.

Lemma 15.11. *Put $t = \sharp Q_D$. Then we have*

$$\langle f^*, f^* \rangle = 2^t \eta_n(f) \langle f, f \rangle.$$

Proof. We may assume $f^* \neq 0$. Let $\{Q_1, Q_2, \dots, Q_l\}$ be a maximal subset such that $\{f_{Q_1}, f_{Q_2}, \dots, f_{Q_l}\}$ are linearly independent. Then $\eta_n(f) l = 2^t$ and $f^* = \eta_n(f) \sum_{i=1}^{l} f_{Q_i}$. Note that

$$\langle f_{Q_i}, f_{Q_j} \rangle = 0 \quad \text{for } 1 \le i, j \le l, i \neq j,$$

since f_{Q_1} and f_{Q_2} are different primitive forms. Therefore we have $\langle f^*, f^* \rangle = l \eta_n(f)^2 \langle f, f \rangle = 2^t \eta_n(f) \langle f, f \rangle$. □

Proposition 15.12. *$f^*(\tau)$ is identically zero if and only if n is odd and $f(\tau)$ comes from a Hecke character of some imaginary quadratic field.*

Proof. It is enough to prove that f comes from a Hecke character of some quadratic field K', then $K' = \mathbb{Q}(\sqrt{\chi_Q(-1) D_Q})$ for some $Q \subset Q_D$. Let $\rho = \otimes_v \rho_v$ be the Hecke character of $\mathbb{A}^\times / \mathbb{Q}^\times$ corresponding to $K'/\mathbb{Q}$. Then $\tilde{I}(\mathbf{1} \boxtimes \underline{\chi}_p, s_{0,p})$ is isomorphic to $\tilde{I}(\rho_p \boxtimes \rho_p \underline{\chi}_p, s_{0,p})$ for each prime p. Comparing the conductor, one can shows that either ρ_p or $\rho_p \underline{\chi}_p^{-1}$ is unramified. It follows that $K'K/K$ is unramified, and so $K' = \mathbb{Q}(\sqrt{\chi_Q(-1) D_Q})$ for some $Q \subset Q_D$ by genus theory. □

Lemma 15.13. *Let $f_0(\tau)$ be an element of $S_{2k+1}(\Gamma_0(D), \chi)$. Assume that N-th Fourier coefficient $a_{f_0}(N)$ is zero whenever $(N, D) = 1$. Then $f_0 = 0$.*

Proof. This is a special case of Miyake [19], Theorem 4.6.8. □

Lemma 15.14. *Let N be a rational integer. Then there exists an integer $y \in \mathfrak{c}^{-1} \mathcal{O}^\sharp$ such that $CDy\bar{y} \equiv (-1)^n N \bmod D$ if and only if $\mathbf{a}_D^\varepsilon(N) \neq 0$.*

Proof. As remarked in Krieg [16], p. 670, we have

$$\mathbf{a}_D(N) = \sharp\{u \in \mathcal{O}^\sharp / \mathcal{O} \mid Du\bar{u} \equiv (-1)^n N \bmod D\}.$$

Choose $\alpha \in \mathfrak{c}$ such that $(\alpha, D) = 1$. Put $\mathfrak{c}' = (\alpha)\mathfrak{c}^{-1}$, $C' = \mathrm{N}(\mathfrak{c}')$. Then $\mathfrak{c}$ and $\mathfrak{c}'$ are integral ideals which belong to the same genus. Note that $\chi_Q(\mathrm{N}(\alpha)) = \chi_Q(C) \chi_Q(C') = 1$ for any $Q \subset Q_D$. The map $y \mapsto u = \alpha y$ induces an isomorphism $\mathfrak{c}^{-1} \mathcal{O}^\sharp / \mathfrak{c}^{-1} \simeq \mathcal{O}^\sharp / \mathcal{O}$. Then we have

$$\begin{aligned} \mathbf{a}_D^\varepsilon(N) = \mathbf{a}_D(C'N) &= \sharp\{u \in \mathcal{O}^\sharp / \mathcal{O} \mid Du\bar{u} \equiv (-1)^n C'N\} \\ &= \sharp\{y \in \mathfrak{c}^{-1} \mathcal{O}^\sharp / \mathfrak{c}^{-1} \mid \mathrm{N}(\alpha) D y \bar{y} \equiv (-1)^n C'N\} \\ &= \sharp\{y \in \mathfrak{c}^{-1} \mathcal{O}^\sharp / \mathfrak{c}^{-1} \mid CDy\bar{y} \equiv (-1)^n N\} \end{aligned}$$

Hence the lemma. □

Lemma 15.15. *Let N be a positive integer. Then there exists an element $H \in \Lambda_{2n}^{\mathfrak{c}}(\mathcal{O})^+$ such that $C|\gamma(H)| = N$ if and only if $\mathbf{a}_D^{\varepsilon}(N) \neq 0$.*

Proof. Assume that $\mathbf{a}_D^{\varepsilon}(N) \neq 0$. Then by Lemma 13.9 and Lemma 15.14, there exists an element $H \in \Lambda_{2n}^{\mathfrak{c}}(\mathcal{O})^+$ such that $C|\gamma(H)| = N$. Conversely, assume that $N = C|\gamma(H)|$ for $H \in \Lambda_{2n}^{\mathfrak{c}}(\mathcal{O})^+$. It is enough to prove that $\underline{\chi}_q(N) = \underline{\chi}_q((-1)^n C)$ for any $q|D$, $q \nmid N$. Since $\gamma(H) = (-1)^n N C^{-1} \in \mathbb{Z}_q^{\times}$, we have $\tilde{F}_q(H, X) = 1$. Then $\underline{\chi}_q(\gamma(H)) = 1$ by Lemma 2.2. Hence the lemma. □

Definition 15.16. Let $S_{2k+1}^{\varepsilon *}(\Gamma_0(D), \chi)$ be the space of cusp forms

$$f_0(\tau) = \sum_{N>0} a_{f_0}(N) q^N \in S_{2k+1}(\Gamma_0(D), \chi)$$

whose N-th Fourier coefficient is zero whenever $\mathbf{a}_D^{\varepsilon}(N) = 0$. If $\varepsilon = 1$, then $S_{2k+1}^{\varepsilon *}(\Gamma_0(D), \chi)$ is simply denoted by $S_{2k+1}^{*}(\Gamma_0(D), \chi)$.

Let $\{f_i\}_{i \in I}$ be the set of primitive forms in $S_{2k+1}(\Gamma_0(D), \chi)$. By Corollary 15.5, $f_i^{\varepsilon *} \in S_{2k+1}^{\varepsilon *}(\Gamma_0(D), \chi)$. The following proposition is essentially Krieg [16], p.671, Proposition.

Proposition 15.17. *The space $S_{2k+1}^{\varepsilon *}(\Gamma_0(D), \chi)$ is spanned by $\{f_i^{\varepsilon *}\}_{i \in I}$.*

Proof. Let g be any element of $S_{2k+1}^{\varepsilon *}(\Gamma_0(D), \chi)$. Then g can be uniquely expressed as a linear combination of primitive forms $\{f_i\}_{i \in I}$:

$$g = \sum_{i \in I} a_i \cdot f_i.$$

For $Q \subset Q_D$, consider

$$g' = \sum_{i \in I} a_i \cdot (f_i)_Q.$$

If $(D_Q, N) = 1$, then the N-th Fourier coefficient of $g - \varepsilon(Q)\chi_Q(-1)^n g'$ vanishes. By Lemma 15.13, we have $g = \varepsilon(Q)\chi_Q(-1)^n g'$. Hence the lemma. □

Let $f \in S_{2k+1}(\Gamma_0(D), \chi)$ be a primitive form. In terms of Satake parameters, $a_{f^{\varepsilon *}}(N)$ can be expressed as follows. Put

$$\Psi_p^{\varepsilon}(N; X) = \begin{cases} \dfrac{X^{e_{p,N}+1} - (\xi_p X^{-1})^{e_{p,N}+1}}{X - \xi_p X^{-1}} & p \notin Q_D, \\ X^{e_{p,N}} + \underline{\chi}_p((-1)^n C N) X^{-e_{p,N}} & p \in Q_D \end{cases}$$

where $e_{p,N} = \mathrm{ord}_p N$ and $\xi_p = \chi(p)$. Put

$$\Psi^\varepsilon(N;\mathbb{X}) = \prod_{p|DN} \Psi^\varepsilon_p(N;X_p) \in \mathcal{R}.$$

Then we have $a_{f^{\varepsilon*}}(N) = N^k\Psi^\varepsilon(N;\{\alpha_p\})$ by Corollary 15.5. Note that $\Psi^\varepsilon(N;\mathbb{X}) = 0$ if and only if $\mathbf{a}^\varepsilon_D(N) = 0$.

Fix $H \in \Lambda^{\mathfrak{c}}_{2n}(\mathcal{O})^+$. Then $\tilde{F}_p(\bar{\mathbf{t}}_p H\mathbf{t}_p;X)$ belongs to the $\mathbb{Q}$-vector space

$$\mathcal{V} = \{\Phi \in X^{-\mathrm{ord}_p C\gamma(H)}\mathbb{Q}[X^2] \mid \Phi(X^{-1}) = \underline{\chi}_p(C\gamma(H))\Phi(X)\}.$$

We put $\mathcal{B}_p(H) = \{p^r \mid 0 \le 2r \le \mathrm{ord}_p(C\gamma(H)), \mathbf{a}^\varepsilon_D(\frac{C|\gamma(H)|}{p^{2r}}) \ne 0\}$. Since

$$\{\Psi^\varepsilon_p(\tfrac{C|\gamma(H)|}{p^{2r}};X) \mid p^r \in \mathcal{B}_p(H)\}$$

is a basis of $\mathcal{V}$,

$$F_p(\bar{\mathbf{t}}_p H\mathbf{t}_p;X) = \sum_{p^r\in\mathcal{B}_p(H)} \phi_p(p^r,H)\Psi^\varepsilon_p(\tfrac{C|\gamma(H)|}{p^{2r}};X)$$

for some $\phi_p(p^r,H) \in \mathbb{Q}$. Note that

$$\phi_p(1,H) = \begin{cases} \frac{1}{2} & p \in Q_D, p \nmid C\gamma(H), \\ 1 & \text{otherwise.} \end{cases}$$

The set $\mathcal{B}(H) = \prod_p \mathcal{B}_p(H)$ can be identified with the set of positive integers a such that $a^2|C\gamma(H)$ and $\mathbf{a}^\varepsilon_D(\frac{C|\gamma(H)|}{a^2}) \ne 0$. For each $a = \prod_{p|a} p^{r_p} \in \mathcal{B}(H)$, we put $\phi(a,H) = \prod_p \phi_p(p^{r_p},H)$. Then we have

$$\begin{aligned}\prod_p \tilde{F}_p(\bar{\mathbf{t}}_p H\mathbf{t}_p;X_p) &= \prod_p \left[\sum_{p^r\in\mathcal{B}_p(H)} \phi_p(p^r,H)\Psi^\varepsilon_p(\tfrac{C|\gamma(H)|}{p^{2r}};X_p)\right] \\ &= \sum_{a\in\mathcal{B}(H)} \phi(a,H)\Psi^\varepsilon(\tfrac{C|\gamma(H)|}{a^2};\mathbb{X}).\end{aligned}$$

Here we have used the fact that $\Psi^\varepsilon_p(a^2N;X) = \Psi^\varepsilon_p(N;X)$ for $p \nmid a$. Note that $1 \in \mathcal{B}(H)$ by Lemma 15.15. One can easily see that $\phi(1,H) = \mathbf{a}^\varepsilon_D(C|\gamma(H)|)^{-1} \ne 0$.

For each $f_0(\tau) = \sum_{N>0} a_{f_0}(N)q^N \in S^{\varepsilon*}_{2k+1}(\Gamma_0(D),\chi)$, we put

$$\iota(f_0)(Z) = \sum_{H\in\Lambda^{\mathfrak{c}}_{2n}(\mathcal{O})^+} \sum_{a\in\mathcal{B}(H)} a^{2k}\phi(a,H)a_{f_0}(\tfrac{C|\gamma(H)|}{a^2})\mathbf{e}(HZ).$$

If $f \in S_{2k+1}(\Gamma_0(D),\chi)$ is a primitive form, then

$$\iota(f^{\varepsilon*}) = C^{k+n}\mathrm{Lift}^{(2n)}_{\mathfrak{c}}(f) \in S_{2k+2n}(\Gamma^{(2n)}_K[\mathfrak{c}], \det{}^{-k-n}).$$

Since $\{f_i^{\varepsilon *}\}_{i\in I}$ spans $S^{\varepsilon *}_{2k+1}(\Gamma_0(D),\chi)$, the image of ι is contained in $S_{2k+2n}(\Gamma_K^{(2n)}[\mathfrak{c}],\det^{-k-n})$.

Theorem 15.18. *There exists an injective linear map*

$$\iota : S^{\varepsilon *}_{2k+1}(\Gamma_0(D),\chi) \to S_{2k+2n}(\Gamma_K^{(2n)}[\mathfrak{c}],\det^{-k-n})$$

satisfying the following properties:

(1) *For each* $f_0 \in S^{\varepsilon *}_{2k+1}(\Gamma_0(D),\chi)$,

$$\iota(f_0)(Z) = \sum_{H\in\Lambda_{2n}(\mathcal{O})^+} \sum_{a\in\mathcal{B}(H)} a^{2k}\phi(a,H)a_{f_0}(\tfrac{C|\gamma(H)|}{a^2})\mathbf{e}(HZ),$$

where $\mathcal{B}(H)=\{a\in\mathbb{Z}\,|\,a>0, a^2|C\gamma(H), \mathbf{a}_D(\frac{C|\gamma(H)|}{a^2})\neq 0\}$.

(2) *If* f *is a primitive form in* $S_{2k+1}(\Gamma_0(D),\det^{-k-n})$, *then* $\iota(f^{\varepsilon *}) = C^{k+n}\mathrm{Lift}_{\mathfrak{c}}^{(2n)}(f)$.

Proof. We need to prove the injectivity of ι. Assume that $\iota(f_0)=0$ for $f_0\in S^{\varepsilon *}_{2k+1}(\Gamma_0(D),\chi)$. We have to show that $a_{f_0}(N)=0$ for $\mathbf{a}^{\varepsilon}_D(N)\neq 0$. By Lemma 15.15, there exists an element $H_N\in\Lambda^{\mathfrak{c}}_m(\mathcal{O})^+$ such that $C|\gamma(H_N)|=N$. As in Case O, the H_N-th Fourier coefficient of $\iota(f_0)$ is equal to $\mathbf{a}^{\varepsilon}_D(N)^{-1}a_{f_0}(N)$ + (lower terms). By induction, we have $a_{f_0}(N)=0$. □

Corollary 15.19. *Let* $f(\tau)\in S_{2k+1}(\Gamma_0(D),\chi)$ *be a primitive form. Then* $\mathrm{Lift}_{\mathfrak{c}}(f)=0$ *if and only if* f *comes from a Hecke character of a field* $\mathbb{Q}(\sqrt{\chi_Q(-1)D_Q})$ *such that* $\varepsilon(Q)\chi_Q(-1)^n=-1$.

Corollary 15.20. *Let* $f(\tau)\in S_{2k+1}(\Gamma_0(D),\chi)$ *be a primitive form. Then* $\mathrm{Lift}^{(2n)}(f)$ *is identically zero if and only if* n *is odd and* f *comes from a Hecke character of some imaginary quadratic field.*

Corollary 15.21. *Let* $f(\tau)\in S_{2k+1}(\Gamma_0(D),\chi)$ *be a primitive form. Then Lift*$^{(2n)}(f)\in\mathcal{S}_{2k+2n}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}),\det^{-k-n})$ *is identically zero if and only if* n *is odd and* f *comes from a Hecke character of* K.

Proof. Note that *Lift*$^{(2n)}(f)=0$ if and only if $f^{\varepsilon *}=0$ for any ε. Assume that $f^{\varepsilon *}=0$ for any ε. Then n is odd, since otherwise $f^*\neq 0$. If n is odd, we have

$$f - f_{Q_D} = 2^{1-t}\sum_{\varepsilon} f^{\varepsilon *} = 0,$$

where $t=\sharp Q_D$. Conversely, if n is odd and $f=f_{Q_D}$, then $f^{\varepsilon *}=0$ for any ε. Hence the corollary. □

16. **An example: the case** $m=2$

The case $m=2$ was first considered by Kojima [15] for $K=\mathbb{Q}(\sqrt{-1})$ and later by Gritsenko [5]. Krieg [16] and Sugano [26] investigated the Maass spaces for arbitrary imaginary quadratic field. In this case, $\mathrm{Lift}^{(2)}(f)$ is called the Maass lift of f. Recently, Klosin [13] defined the Maass space for $\mathrm{U}(2,2)$ in the adelic setting, and constructed the extension of $\mathrm{Lift}^{(2)}(f)$ under the assumption that the class number h_K is odd. As we described in §15, there is an injective linear map $S^*_{2k+1}(\Gamma_0(D),\chi)^{h_K}\to\mathcal{S}_{2k+2}(\mathcal{G}(\mathbb{Q})\backslash\mathcal{G}(\mathbb{A}),\det^{-k-1})$ in this case.

We do not give a detailed proof for the results in this section, as most of the results are not new, and contained in the references above (at least when $\mathfrak{c}=\mathcal{O}$).

Let $\mathfrak{c}$ be an integral ideal of K such that $C=\mathrm{N}(\mathfrak{c})$ is prime to D.

Definition 16.1.

$$F(Z)=\sum_{H\in\Lambda_2^{\mathfrak{c}}(\mathcal{O})}A_F(H)\mathbf{e}(HZ)\in M_{2k+2}(\Gamma_K^{(2)}[\mathfrak{c}],\det{}^{-k-1})$$

satisfies the Maass relation if and only if there is a function

$$\alpha_F^*:\mathbb{Z}_{\geq 0}\to\mathbb{C}$$

such that

$$A_F(H)=\sum_{d|\varepsilon(H)}d^{2k+1}\alpha_F^*(\tfrac{C|\gamma(H)|}{d^2}).$$

Here

$$\varepsilon(H)=\max\{q\in\mathbb{Z}_{>0}\,|\,q^{-1}H\in\Lambda_2^{\mathfrak{c}}(\mathcal{O})\}.$$

Note that the values $\alpha_F^*(N)$ for $\mathbf{a}_D^{\varepsilon}(N)=0$ plays no role. We denote the space of elements of $M_{2k+2}(\Gamma_K^{(2)},\det^{-k-1})$ satisfying the Maass relation by $M_{2k+2}^{\mathrm{Maass}}(\Gamma_K^{(2)},\det^{-k-1})$. We set $S_{2k+2}^{\mathrm{Maass}}(\Gamma_K^{(2)},\det^{-k-1})=M_{2k+2}^{\mathrm{Maass}}(\Gamma_K^{(2)},\det^{-k-1})\cap S_{2k+2}(\Gamma_K^{(2)},\det^{-k-1})$.

It is known that the normalized hermitian Eisenstein series

$$F=\mathcal{E}_{2k+2}^{(2)}(Z)=\frac{B_{2k+2}B_{2k+1,\chi}}{8(k+1)(2k+1)}E_{2k+2}^{(2)}(Z),$$

satisfies the Maass relation for $\mathfrak{c}=\mathcal{O}$. The function α_F^* is given by

$$\alpha_F^*(N)=\begin{cases}0 & \mathbf{a}_D(N)=0\\ -B_{2k+1,\chi}/(4k+2) & N=0\\ \mathbf{a}_D(N)^{-1}\sum_{d|N}\sum_{Q\subset Q_D}\chi_Q(\frac{-N}{d})\chi'_Q(d)d^{2k} & N>0,\mathbf{a}_D(N)\neq 0.\end{cases}$$

(cf. Krieg [16], p. 679. Krieg [16] assumed $w_K|(2k+2)$, but the modification is easy.) Using these results, one can calculate the Laurent polynomial $\tilde{F}_p(H;X)$ as follows:

$$\tilde{F}_p(H;X) = \begin{cases} \sum_{i=0}^{b} p^i \sum_{j=0}^{a-2i} \underline{\chi}_p(p)^j X^{a-2i-2j} & \text{if } p \nmid D, \\ \sum_{i=0}^{b} p^i (X^{a-2i} + \underline{\chi}_p(\gamma(H)) X^{-a+2i}) & \text{if } p|D, 2b < a, \\ p^b + \sum_{i=0}^{b-1} p^i (X^{a-2i} + X^{-a+2i}) & \text{if } p|D, 2b = a. \end{cases}$$

Here $a = \operatorname{ord}_p \gamma(H)$, $b = \operatorname{ord}_p \varepsilon(H)$. When the class number of K is one, this has been already essentially calculated by Nagaoka ([20] Th.1.3.1). Using this formula, we have

$$\prod_p \tilde{F}_p(\bar{\mathbf{t}}_p H \mathbf{t}; X_p) = \sum_{d \in \varepsilon(H)} d\, \mathbf{a}_D^{\varepsilon}(\tfrac{C|\gamma(H)|}{d^2})^{-1} \Psi^{\varepsilon}(\tfrac{C|\gamma(H)|}{d^2}; \mathbb{X})$$

for $H \in \Lambda_2^{\mathfrak{c}}(\mathcal{O})^+$. It follows that the map

$$\iota : S_{2k+1}^{\varepsilon *}(\Gamma_0(D), \chi) \to S_{2k+2}(\Gamma_K^{(2)}[\mathfrak{c}], \det^{-k-1})$$

is given by

$$\iota(f)(Z) = \sum_{H \in \Lambda_2(\mathcal{O})^+} \sum_{d \in \varepsilon(H)} d^{2k+1} \mathbf{a}_D^{\varepsilon}(\tfrac{C|\gamma(H)|}{a^2})^{-1} a_f(\tfrac{C|\gamma(H)|}{a^2}) \mathbf{e}(HZ).$$

In particular, the image of ι is contained in $S_{2k+2}^{\mathrm{Maass}}(\Gamma_K^{(2)}[\mathfrak{c}], \det^{-k-1})$.

For $F \in S_{2k+2}(\Gamma_K^{(2)}[\mathfrak{c}], \det^{-k-1})$, consider the first Fourier-Jacobi coefficient

$$\phi_1(\tau, z_1, z_2) = \sum_{\substack{l \in C^{-1}\mathbb{Z} \\ t \in \mathfrak{c}^{-1}\mathcal{O}^{\sharp}}} A_F\left(\begin{pmatrix} 1 & t \\ \bar{t} & l \end{pmatrix}\right) \mathbf{e}(l\tau + \bar{t} z_1 + t z_2).$$

Then there exist functions $f_{[u]}(\tau)$ $(u \in \mathfrak{c}^{-1}\mathcal{O}^{\sharp}/\mathfrak{c}^{-1})$ such that

$$\phi_1(\tau, z_1, z_2) = \sum_{u \in \mathfrak{c}^{-1}\mathcal{O}^{\sharp}/\mathfrak{c}^{-1}} \theta_{[u]}(\tau, z_1, z_2) f_{[u]}(\tau/C),$$

where

$$\theta_{[u]}(\tau, z_1, z_2) = \sum_{a \in u + \mathfrak{c}^{-1}} \mathbf{e}(a\bar{a}\tau + \bar{a} z_1 + a z_2).$$

Then one can show that

(1) $f_{[u]} \in S_{2k+1}(\Gamma(D))$, where $\Gamma(D)$ is the principal congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ modulo D.
(2) $f_{[u]}(\tau + 1) = \mathbf{e}(-Cu\bar{u}) f_{[u]}$.
(3) $f_{[u]}|_{2k+1} \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right] = -(1/\sqrt{-D}) \sum_{v \in \mathcal{O}^{\sharp}/\mathcal{O}} \mathbf{e}(C(u\bar{v} + v\bar{u})) f_{[v]}$.
(4) $f_{[0]} \in S_{2k+1}(\Gamma_0(D), \chi)$.

If $\mathfrak{c}=\mathcal{O}$, this is proved in Krieg [16], p.669. One can easily treat the general case by using the theta transformation formula for hermitian theta function (see Shintani [25], Shimura [24], A7). Set

$$\Omega(F)(\tau)=\sum_{u\in\mathfrak{c}^{-1}\mathcal{O}^\sharp/\mathfrak{c}^{-1}} f_{[u]}(D\tau).$$

Then we have $\Omega(F)\in S^{\varepsilon *}_{2k+1}(\Gamma_0(D),\chi)$. If $F\in S^{\mathrm{Maass}}_{2k+2}(\Gamma_K^{(2)}[\mathfrak{c}],\det^{-k-1})$, then we have

$$f_{[u]}(\tau)=\sum_{N\equiv -CDu\bar{u} \bmod D}\alpha_F^*(N)\mathbf{e}(N\tau/D),$$

$$\Omega(F)(\tau)=\sum_{N>0}\mathbf{a}_D^{\varepsilon}(N)\alpha_F^*(N)\mathbf{e}(N\tau).$$

It follows that ι gives the isomorphism between $S^{\varepsilon *}_{2k+1}(\Gamma_0(D),\chi)$ and $S^{\mathrm{Maass}}_{2k+2}(\Gamma_K^{(2)}[\mathfrak{c}],\det^{-k-1})$.

17. **Petersson norms of** $\mathrm{Lift}^{(m)}(f)$.

We recall the definition of the Petersson inner product for hermitian modular forms. For $F_1,F_2\in S_l(\Gamma_K^{(m)},\sigma)$, the Pertersson inner product $\langle F_1,F_2\rangle$ is defined by

$$\langle F_1,F_2\rangle=\int_{\Gamma_K^{(m)}\backslash\mathcal{H}_m} F_1(Z)\overline{F_2(Z)}(\det Y)^{l-2m}dX\,dY,$$

where $X=(Z+{}^t\bar{Z})/2$, $Y=(Z-{}^t\bar{Z})/(2\sqrt{-1})$. The measure dX on the space of hermitian matrices is defined by $dX=\prod_{i\le j}dX_{ij}^{(r)}\prod_{i<j}dX_{ij}^{(i)}$, where $X=X^{(r)}+\sqrt{-1}X^{(i)}$, $X_{ij}^{(r)},X_{ij}^{(i)}\in\mathbb{R}$.

In this section, we investigate the Petersson norm of the lifts of f. For simplicity, we only consider the case case $\mathfrak{c}=\mathcal{O}$.

Let $f\in S_{2k+1}(\Gamma_0(D),\chi)$ be a primitive form. Put $F=\mathrm{Lift}^{(2)}(f)\in S_{2k+2}(\Gamma_K^{(2)},\det^{-k-1})$. As we have seen in §16, the first Fourier-Jacobi coefficient ϕ_1 of F has a decomposition

$$\phi_1(\tau,z_1,z_2)=\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} f_{[u]}(\tau)\theta_{[u]}(\tau,z_1,z_2),$$

$$\theta_{[u]}(\tau,z_1,z_2)=\sum_{a\equiv u \bmod \mathcal{O}}\mathbf{e}(a\bar{a}\tau+\bar{a}z_1+az_2),$$

where

$$f_{[u]}(\tau)=\sum_{N\equiv -Du\bar{u} \bmod D}\alpha_F^*(N)\mathbf{e}(N\tau/D)$$

for each $u\in\mathcal{O}^\sharp/\mathcal{O}$. Note that $f^*(\tau)=\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} f_{[u]}(D\tau)$.

The Petersson inner product $\langle \phi_1, \phi_1 \rangle$ is defined by

$$\int_{J\backslash(\mathfrak{h}_1\times\mathbb{C}^2)} \phi_1(\tau, z_1, z_2)\overline{\phi_1(\tau, z_1, z_2)} y^{l-4} e^{-\pi|z_1-\bar{z}_2|^2/y} dt_1\, dw_1\, dt_2\, dw_2\, dx\, dy.$$

Here, $J = J_{2,1}(\mathcal{O})$, $\tau = x + \sqrt{-1}y$, $z_1 = t_1 + \sqrt{-1}w_1$, and $z_2 = t_2 + \sqrt{-1}w_2$.

Proposition 17.1. *We have*

$$\langle \phi_1, \phi_1 \rangle = \frac{\sqrt{D}}{4} \sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} \langle f_{[u]}, f_{[u]} \rangle.$$

Proof. (cf. [4], Theorem 5.3.) Since the non-trivial element of the center of $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{C}^2/L_\tau$ by $(z_1, z_2) \mapsto (-z_1, -z_2)$, we have

$$J\backslash(\mathfrak{h}_1 \times \mathbb{C}^2) = \{(\tau, z_1, z_2) \mid \tau \in \mathrm{SL}_2(\mathbb{Z})\backslash\mathfrak{h}_1, (z_1, z_2) \in (\mathbb{C}^2/L_\tau)/\{\pm 1\}\}.$$

It follows that $\langle \phi_1, \phi_1 \rangle$ is equal to

$$\frac{1}{2}[\mathrm{SL}_2(\mathbb{Z}) : \Gamma(D)\{\pm 1\}]^{-1} \int_{\tau\in(\Gamma(D)\backslash\mathfrak{h}_1)} \sum_{u,v\in\mathcal{O}^\sharp/\mathcal{O}} f_{[u]}(\tau)\overline{f_{[v]}(\tau)} y^{2k-2}$$
$$\times \int_{\mathbb{C}^2/L_\tau} \theta_{[u]}(\tau, z_1, z_2)\overline{\theta_{[v]}(\tau, z_1, z_2)} e^{-\pi y|z_1-\bar{z}_2|^2} dt_1\, dw_1\, dt_2\, dw_2\, dx\, dy,$$

where $L_\tau = \{(\lambda\tau + \mu, \bar{\lambda}\tau + \bar{\mu}) \mid \lambda, \mu \in \mathcal{O}\}$. It is easy to show

$$\int_{\mathbb{C}^2/L_\tau} \theta_{[u]}(\tau, z_1, z_2)\overline{\theta_{[v]}(\tau, z_1, z_2)} e^{-\pi|z_1-\bar{z}_2|^2/y} dt_1\, dw_1\, dt_2\, dw_2 = \delta_{uv}\frac{\sqrt{D}}{2}y.$$

Hence the proposition. □

Lemma 17.2. *Set $N_u = -Du\bar{u} \in \mathbb{Z}/D\mathbb{Z}$ for each $u \in \mathcal{O}^\sharp/\mathcal{O}$. Then we have*

$$\langle f_{[u]}, f_{[u]} \rangle = \mathbf{a}_D(N_u)^{-1} \langle f_{[0]}, f_{[0]} \rangle.$$

Proof. Let $\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} \mathbb{C}f_{[u]}$ be the space generated by $\{f_{[u]} \mid u \in \mathcal{O}^\sharp/\mathcal{O}\}$. It is well-known that the space $\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} \mathbb{C}f_{[u]}$ can be naturally identified with a subrepresentation of the (finite) Weil representation (cf. Shintani [25].) Let $\mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}] = \bigoplus_{q|D} \mathbb{C}[\mathcal{O}_q^\sharp/\mathcal{O}_q]$ be the space of functions on $\mathcal{O}^\sharp/\mathcal{O} = \bigoplus_{q|D} \mathcal{O}_q^\sharp/\mathcal{O}_q$, where $\mathcal{O}_q^\sharp = \mathcal{O}^\sharp \otimes_{\mathcal{O}} \mathcal{O}_q$. Let $\varphi_u \in \mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}]$ be the characteristic function of $u \in \mathcal{O}^\sharp/\mathcal{O}$. Recall that there exists a representation, called the (finite) Weil representation ω of $\mathrm{SL}_2(\mathbb{Z}/D\mathbb{Z})$ on $\mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}]$, which is characterized by

(i) $\omega\left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)\varphi_u = \mathbf{e}(-u\bar{u})\varphi_u$.
(ii) $\omega\left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)\varphi_u = -(1/\sqrt{-D}) \sum_{v\in\mathcal{O}^\sharp/\mathcal{O}} \mathbf{e}(u\bar{v} + v\bar{u})\varphi_v$.

The Weil representation ω is a unitary representation with respect to the natural inner product on $\mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}] = L^2(\mathcal{O}^\sharp/\mathcal{O})$. For each prime $q \in Q_D$, let $\mathcal{U}_q$ be the kernel of the norm map $N_{K_q/Q_q} : K_q^\times \to \mathbb{Q}_q^\times$. Note that the group $\mathcal{U} = \prod_{q|D} \mathcal{U}_q$ acts on $\mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}]$. This action commutes withe the action ω of $\mathrm{SL}_2(\mathbb{Z}/D\mathbb{Z})$. We denote by $\mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}]^{\mathcal{U}}$ the space of $\mathcal{U}$-invariants in $\mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}]$. This is an irreducible subrepresentation of ω. Then the space $\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} \mathbb{C}f_{[u]}$ is isomorphic to $\mathbb{C}[\mathcal{O}^\sharp/\mathcal{O}]^{\mathcal{U}}$ as a representation of $\mathrm{SL}_2(\mathbb{Z}/D\mathbb{Z})$. Put

$$\tilde{\varphi}_u = \mathbf{a}_D(N_u)^{-1} \sum_{\substack{v\in\mathcal{O}^\sharp/\mathcal{O} \\ N_v = N_u}} \varphi_v.$$

Then $f_{[u]}$ corresponds to $\tilde{\varphi}_u$ under the isomorphism. Clearly we have $\mathbf{a}_D(N_u)\|\tilde{\varphi}_u\|^2 = \|\tilde{\varphi}_0\|^2$, since $\mathbf{a}_D(N_u) = \#\{v \in \mathcal{O}^\sharp/\mathcal{O} \mid N_v = N_u\}$.

□

Proposition 17.3. *We have*

$$\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} \langle f_{[u]}, f_{[u]}\rangle = D^{2k+1} \left(\prod_{q|D} \frac{1}{2}(1+q^{-1}) \right) \langle f^*, f^*\rangle$$

Proof. By Lemma 17.2, we have

$$\begin{aligned}\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} \langle f_{[u]}, f_{[u]}\rangle &= \left(\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} \mathbf{a}_D(N_u)^{-1} \right) \langle f_{[0]}, f_{[0]}\rangle \\ &= \sharp\{N \bmod D \mid \mathbf{a}_D(N) \neq 0\} \cdot \langle f_{[0]}, f_{[0]}\rangle \\ &= D \left(\prod_{q|D} \frac{1}{2}(1+q^{-1}) \right) \langle f_{[0]}, f_{[0]}\rangle.\end{aligned}$$

Since $\sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} f_{[u]} = -\sqrt{-D} f_{[0]}|_{2k+1} \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right]$, we have

$$D\langle f_{[0]}, f_{[0]}\rangle = \langle \sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} f_{[u]}, \sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} f_{[u]}\rangle.$$

Since $f^*(\tau) = \sum_{u\in\mathcal{O}^\sharp/\mathcal{O}} f_{[u]}(D\tau)$, the proposition follows. □

By Lemma 15.10, Lemma 15.11, Proposition 17.1, and Proposition 17.3, we have

$$\langle \phi_1, \phi_1\rangle = 2^{-2k-3} D^{2k+(3/2)} \eta_1(f) \Lambda(1, f, \mathrm{Ad}).$$

Sugano ([27], Corollary 8.3) has proved that

$$\langle F, F\rangle = 2^{-2k-5} D^{3/2} \pi^{-2} \xi(2) \Lambda(2, f, \mathrm{Ad}, \chi) \langle \phi_1, \phi_1\rangle,$$

where

$$\begin{aligned}\xi(s) =&\Gamma_{\mathbb{R}}(s)\zeta(s),\\ \Lambda(s,f,\mathrm{Ad},\chi) =&\Gamma_{\mathbb{R}}(s+1)\Gamma_{\mathbb{C}}(s+2k)L(s,f,\mathrm{Ad},\chi),\\ L(s,f,\mathrm{Ad},\chi) =&\prod_{p\nmid D}\left[(1-\alpha_p^2p^{-s})(1-\chi(p)p^{-s})(1-\beta_p^2p^{-s})\right]^{-1}\\ &\times\prod_{q\mid D}\left[(1-\alpha_q^2q^{-s})(1-\alpha_q^{-2}q^{-s})\right]^{-1}.\end{aligned}$$

Note that Sugano has formulated his theorems in terms of orthogonal groups. In particular, the normalization of the inner products are different from our normalization. By combining our calculation and Sugano's result, we obtain the following Proposition.

Proposition 17.4. *Let $f\in S_{2k+1}(\Gamma_0(D),\chi)$ be a primitive form, and put $F=\mathrm{Lift}^{(2)}(f)$. Then we have*

$$\langle F,F\rangle=\eta_1(f)2^{-4k-8}D^{2k+3}\pi^{-2}\xi(2)\Lambda(1,f,\mathrm{Ad})\Lambda(2,f,\mathrm{Ad},\chi).$$

As in Ikeda [10], we can give a conjecture on the Petersson norm of $\mathrm{Lift}^{(m)}(f)$. We define the modified complete L-functions as follows. Put

$$\tilde\Lambda(s,\chi^i)=\begin{cases}\Gamma_{\mathbb{C}}(s)\zeta(s) & \text{if } i \text{ is even},\\ \Gamma_{\mathbb{C}}(s)L(s,\chi) & \text{if } i \text{ is odd}.\end{cases}$$

For a normalized eigenform $f\in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$, we put

$$\begin{aligned}\tilde\Lambda(s,f,\mathrm{Ad},\chi^i) =&\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+2k-1)\times\begin{cases}L(s,f,\mathrm{Ad}) & \text{if } i \text{ is even},\\ L(s,f,\mathrm{Ad},\chi) & \text{if } i \text{ is odd},\end{cases}\\ L(s,f,\mathrm{Ad}) =&\prod_p\left[(1-\alpha_p^2p^{-s})(1-p^{-s})(1-\beta_p^2p^{-s})\right]^{-1},\\ L(s,f,\mathrm{Ad},\chi) =&\prod_p\left[(1-\alpha_p^2\chi(p)p^{-s})(1-\chi(p)p^{-s})(1-\beta_p^2\chi(p)p^{-s})\right]^{-1}.\end{aligned}$$

Similarly, for a primitive form $f\in S_{2k+1}(\Gamma_0(D),\chi)$, we put

$$\tilde\Lambda(s,f,\mathrm{Ad},\chi^i)=\Gamma_{\mathbb{C}}(s)\Gamma_{\mathbb{C}}(s+2k)\times\begin{cases}L(s,f,\mathrm{Ad}) & \text{if } i \text{ is even},\\ L(s,f,\mathrm{Ad},\chi) & \text{if } i \text{ is odd},\end{cases}$$

Conjecture 17.5. Let $f\in S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ be a normalized Hecke eigenform. Then the Petersson norm of $F=\mathrm{Lift}^{(2n+1)}(f)$ is given by

$$\langle F,F\rangle=2^{\alpha}D^{\beta}\tilde\Lambda(1,f,\mathrm{Ad})\prod_{i=2}^{2n+1}\tilde\Lambda(i,\chi^i)\tilde\Lambda(i,f,\mathrm{Ad},\chi^{i-1})$$

for some integers α and β depending only on n and k.

Conjecture 17.6. Let $f \in S_{2k+1}(\Gamma_0(D), \chi)$ be a primitive form. Then the Petersson norm of $F = \mathrm{Lift}^{(2n)}(f)$ is given by

$$\langle F, F\rangle = \eta_n(f) 2^{\gamma} D^{\delta} \tilde{\Lambda}(1, f, \mathrm{Ad}) \prod_{i=2}^{2n} \tilde{\Lambda}(i, \chi^i) \tilde{\Lambda}(i, f, \mathrm{Ad}, \chi^{i-1})$$

for some integers γ and δ depending only on n and k. For the definition of $\eta_n(f)$, see Definition 15.7.

18. **An interpretation in terms of automorphic representations**

Let f be as in Theorem 5.1 or as in Theorem 5.2. Now we consider the L-function of *Lift*$^{(m)}(f)$. Recall that the L-group of $\mathcal{G} = \mathrm{U}(m, m)$ is described as follows:

$${}^L\mathcal{G} = \mathrm{GL}_{2m}(\mathbb{C}) \rtimes W_{\mathbb{Q}},$$

where $W_{\mathbb{Q}}$ is the Weil group of $\mathbb{Q}$. The action of $W_{\mathbb{Q}}$ on $\mathrm{GL}_{2m}(\mathbb{C})$ factors through $W_{\mathbb{Q}} \to \mathrm{Gal}(K/\mathbb{Q})$ and the non-trivial element of $\mathrm{Gal}(K/\mathbb{Q})$ acts by $g \mapsto g^*$, where

$$g^* = \begin{pmatrix} 0 & -w_1^{-1} \\ w_1 & 0 \end{pmatrix} \cdot {}^t g^{-1} \cdot \begin{pmatrix} 0 & w_1^{-1} \\ -w_1 & 0 \end{pmatrix},$$

$$w_1 = \begin{pmatrix} & & & (-1)^{m-1} \\ & & \cdot^{\cdot^{\cdot}} & \\ & -1 & & \\ 1 & & & \end{pmatrix} \in \mathrm{GL}_m(\mathbb{C}).$$

The homomorphism $\mathrm{st} : {}^L\mathcal{G} \to \mathrm{GL}_{4m}(\mathbb{C})$ defined by

$$g \rtimes u \mapsto \begin{cases} \begin{pmatrix} g & 0 \\ 0 & g^* \end{pmatrix} & \text{if } u \in W_K, \\ \begin{pmatrix} 0 & g \\ g^* & 0 \end{pmatrix} & \text{otherwise.} \end{cases}$$

Here, W_K is Weil group of K. Let $L(s, f, \chi)$ be the twist of $L(s, f)$ by χ. In terms of Euler product, $L(s, f, \chi)$ is defined by

$$L(s, f, \chi) = \prod_{p \nmid D} (1 - \alpha_p \chi(p) p^{k-s})^{-1} (1 - \beta_p \chi(p) p^{k-s})^{-1} \prod_{q \mid D} (1 - \alpha_p^{-1} p^{k-s})^{-1}$$

if m is even, and

$$L(s, f, \chi) = \prod_{p \nmid D} (1 - \alpha_p \chi(p) p^{k-s})^{-1} (1 - \beta_p \chi(p) p^{k-s})^{-1},$$

if m is odd. As in §11 of [9], Theorem 13.6 implies the following theorem.

Theorem 18.1. *Let m, n, and f be as in Theorem 5.1 or as in Theorem 5.2. Assume that $Lift^{(m)}(f) \neq 0$. Let $L(s, Lift^{(m)}(f), \mathrm{st})$ be the L-function of $Lift^{(m)}(f)$ associated to $\mathrm{st} : {}^L\mathcal{G} \to \mathrm{GL}_{4m}(\mathbb{C})$. Then up to bad Euler factors, $L(s, Lift^{(2n)}(f), \mathrm{st})$ is equal to*

$$\prod_{i=1}^{m} L(s+k+n-i+(1/2), f)L(s+k+n-i+(1/2), f, \chi).$$

In terms of the Arthur conjecture, Theorem 18.1 can be interpreted as follows. From now on, we assume the Arthur conjecture. Let $\mathcal{L}_{\mathbb{Q}}$ be the hypothetical Langlands group for $\mathbb{Q}$. The canonical homomorphism $\mathcal{L}_{\mathbb{Q}} \to W_{\mathbb{Q}}$ is denoted by pr. Let τ be an irreducible cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ generated by f. Note that the central character ω_τ is equal to $\underline{\chi}^{m-1}$. We denote the Langlands parameter of τ by $\rho_\tau : \mathcal{L}_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C})$. We normalize the irreducible representation $\mathrm{Sym}^{m-1} : \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SL}_m(\mathbb{C})$ so that

$${}^t\mathrm{Sym}^{m-1}(x)^{-1} = w_1 \mathrm{Sym}^{m-1}(x) w_1^{-1}, \qquad x \in \mathrm{SL}_2(\mathbb{C}).$$

We put

$$\rho_\tau^{(m)}(u) = \begin{pmatrix} \omega_\tau(u)a \cdot \mathbf{1}_m & b \cdot \mathbf{1}_m \\ \omega_\tau(u)c \cdot \mathbf{1}_m & d \cdot \mathbf{1}_m \end{pmatrix} \rtimes \mathrm{pr}(u),$$

for $u \in \mathcal{L}_{\mathbb{Q}}$, $\rho_\tau(u) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, and put

$$\rho_\tau^{(m)}(x) = \begin{pmatrix} \mathrm{Sym}^{m-1}(x) & 0 \\ 0 & \mathrm{Sym}^{m-1}(x) \end{pmatrix} \rtimes 1.$$

for $x \in \mathrm{SL}_2(\mathbb{C})$. Then we get a homomorphism $\rho_\tau^{(m)} : \mathcal{L}_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{G}$. One can easily show that $L(s, \mathrm{st} \circ \rho_\tau^{(m)}) = L(s, Lift^{(m)}(f), \mathrm{st})$. Thus, if we admit the Arthur conjecture, the Arthur parameter of $Lift^{(m)}(f)$ should be $\rho_\tau^{(m)}$.

In Case O, τ can be considered as an automorphic representation of $\mathrm{PGL}_2(\mathbb{A})$. The automorphic representation generated by $Lift^{(m)}(f)$ can be considered as a functorial lift of τ by the L-homomorphism

$${}^L\mathrm{PGL}_2 \times \mathrm{SL}_2(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}) \times W_{\mathbb{Q}} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{G}$$

given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times u \times x \mapsto \begin{pmatrix} a \cdot \mathrm{Sym}^{m-1}(x) & b \cdot \mathrm{Sym}^{m-1}(x) \\ c \cdot \mathrm{Sym}^{m-1}(x) & d \cdot \mathrm{Sym}^{m-1}(x) \end{pmatrix} \rtimes u.$$

In case E, we take an auxiliary Hecke character $\underline{\hat{\chi}} : \mathbb{A}_K^\times / K^\times \to \mathbb{C}^\times$ such that $\underline{\hat{\chi}}|_{\mathbb{A}_\mathbb{Q}^\times} = \underline{\chi}$. Consider the algebraic groups G' and Z defined over $\mathbb{Q}$ such that

$$\begin{aligned} G'(\mathbb{Q}) =& \{(x, g) \in K^\times \times \mathrm{GL}_2(\mathbb{Q}) \mid \mathrm{N}_{K/\mathbb{Q}}(x) \det g = 1\}, \\ Z(\mathbb{Q}) =& \{(z, z^{-1}) \in G' \mid z \in \mathbb{Q}^\times\} \simeq \mathbb{Q}^\times. \end{aligned}$$

Then there is an exact sequence

$$1 \to Z \to G' \to \mathrm{U}(1,1) \to 1.$$

Here, the map $G' \to \mathrm{U}(1,1)$ is given by $(x, g) \mapsto xg$. Then $\underline{\hat{\chi}}^{-1} \boxtimes \tau$ induces an automorphic representation $\hat{\tau}$ of $\mathrm{U}(1,1)(\mathbb{A})$. Fix an element $u_0 \in W_\mathbb{Q}$, $u_0 \notin W_K$. Then one can define an L-homomorphism

$${}^L\mathrm{U}(1,1) \rtimes \mathrm{SL}_2(\mathbb{C}) = \mathrm{GL}_2(\mathbb{C}) \rtimes W_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{G}$$

by

$$\begin{aligned} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto& \begin{pmatrix} a \cdot \mathbf{1}_m & b \cdot \mathbf{1}_m \\ c \cdot \mathbf{1}_m & d \cdot \mathbf{1}_m \end{pmatrix} \rtimes 1, & \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C}), \\ u \mapsto& \underline{\hat{\chi}}(u) \cdot \mathbf{1}_{2m} \rtimes u, & u \in W_K, \\ u_0 \mapsto& \begin{pmatrix} -\mathbf{1}_m & 0 \\ 0 & \mathbf{1}_m \end{pmatrix} \rtimes u_0, & \\ x \mapsto& \begin{pmatrix} \mathrm{Sym}^{m-1}(x) & 0 \\ 0 & \mathrm{Sym}^{m-1}(x) \end{pmatrix} \rtimes 1, & x \in \mathrm{SL}_2(\mathbb{C}). \end{aligned}$$

The automorphic representation generated by $Lift^{(m)}(f)$ can be considered as a functorial lift of $\hat{\tau}$ by this L-homomorphism.

We recall Arthur's conjectural multiplicity formula. Let $\psi : \mathcal{L}_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to {}^LG = \hat{G} \rtimes W_\mathbb{Q}$ be an A-parameter for a quasi-split reductive algebraic group G. Let $\Pi(\psi)$ and $\Pi_v(\psi)$ be the global and local A-packet for ψ. Set $\mathcal{S} = \mathrm{Cent}_{\hat{G}}(\psi)/\mathrm{Cent}(\hat{G})^{W_\mathbb{Q}}$. The group $\mathcal{S}$ is closely related to the internal structure of the A-packet. Arthur conjectured there exist a pairing $\langle s, \pi_v \rangle_v : \mathcal{S} \times \Pi_v(\psi) \to \mathbb{C}$ and a "sign character" $\epsilon_\psi(s) \in \{\pm 1\}$ for each $s \in \mathcal{S}$ and $\pi_v \in \Pi_v(\psi)$. (In fact, Arthur treated these objects locally.) For each $\pi = \otimes_v' \pi_v$, $\pi_v \in \Pi_v(\psi)$, set

$$m_\psi(\pi) = \frac{1}{\sharp \mathcal{S}} \sum_{s \in \mathcal{S}} \epsilon_\psi(s) \prod_v \langle s, \pi_v \rangle_v.$$

Then Authur's conjectural multiplicity formula says the multiplicity of π in the space of square-integrable automorphic forms on $G(\mathbb{A})$ is equal to $\sum_{\pi \in \Pi(\psi)} m_\psi(\pi)$.

Now we consider the case $\psi = \rho_\tau^{(m)}$. In this case, the sign character $\varepsilon_\psi(s)$ must be trivial. One can easily show that

$$\mathcal{S} \simeq \begin{cases} \{\pm 1\} & \text{if } \tau \text{ comes from a Hecke character of } K, \\ \{1\} & \text{otherwise.} \end{cases}$$

Let $\pi = \otimes_v \pi_v$ be an element of the conjectural A-packet $\Pi(\rho_\tau^{(m)})$. If $\mathcal{S} = \{1\}$, then the Arthur conjectural multiplicity formula suggests that any element of global A-packet should be automorphic. This is compatible with Corollary 14.2.

Now assume that $\mathcal{S} \simeq \{\pm 1\}$. Note that m must be even, since a normalized Hecke eigenform of $S_{2k}(\mathrm{SL}_2(\mathbb{Z}))$ does not come from a Hecke character of a quadratic field. Let $s \in \mathcal{S}$ be the non-trivial element.

For each prime p, τ_p is a unramified principal series induced from

$$\begin{pmatrix} x & * \\ 0 & y \end{pmatrix} \mapsto \mu_p(x)\nu_p(y).$$

Here, μ_p and ν_p are characters of $\mathbb{Q}_p$ such that $\mu_p(x) = \alpha_p^{\mathrm{ord}_p x}$ and $\mu_p \nu_p = \underline{\chi}_p$. The local A-packet $\Pi_p(\rho_{\tau_p}^{(m)})$ should consist of the irreducible components of the degenerate principal series induced from the character

$$(\mu_p \circ \mathrm{N}_{K_p/\mathbb{Q}_p} \circ \det) : P(\mathbb{Q}_p) \to \mathbb{C}^\times.$$

Here, P is the Siegel parabolic subgroup of $\mathcal{G}$. If $\pi_p \in \Pi_p(\rho_{\tau_p}^{(m)})$ has a vector fixed by the maximal compact subgroup $\mathcal{G}(\mathbb{Q}_p) \cap \mathrm{GL}_{2m}(\mathcal{O}_p)$, then the character $\langle *, \pi_p \rangle_p$ should be trivial.

At infinity place, the local A-packet should consists of certain cohomologically induced modules (see Adams-Johnson [1]). If π_∞ is the lowest weight module of $\mathcal{G}(\mathbb{R})$ generated by $\mathit{Lift}^{(m)}(f)$, then $\langle s, \pi_\infty \rangle_\infty = (-1)^{m/2}$ by the result of [1]. Therefore the Arthur conjectural multiplicity formula is compatible with Corollary 15.21.

Next, we describe the multiplicity formula for $\mathcal{G}_1 = \mathrm{SU}(m,m)$. We consider only Case E. Let ψ be the Arthur parameter $\mathcal{L}_\mathbb{Q} \times \mathrm{SL}_2(\mathbb{C}) \to {}^L\mathcal{G}_1$ induced from $\rho_\tau^{(m)}$. In this case, the group $\mathcal{S}$ can be identified with the group

$$\{\chi_Q \mid f_Q = f\}.$$

For a prime p, the local A-packet $\Pi_p(\psi)$ should consist of the irreducible constituents of the degenerate principal series induced from the character

$$(\mu_p^2 \circ \det) : P_1(\mathbb{Q}_p) \to \mathbb{C}^\times.$$

Here, P_1 is the Siegel parabolic subgroup of $\mathcal{G}_1$. We denote the maximal compact subgroup $\mathrm{GL}_{2m}(\mathcal{O}_p) \cap \mathcal{G}_1(\mathbb{Q}_p)$ by $\mathcal{K}_{1,p}$. If π_p is the element of the packet $\Pi_p(\psi)$ with non-trivial $\mathcal{K}_{1,p}$-fixed vector, then the pairing $\langle *, \pi_p \rangle_p$ should be trivial. Let $\mathfrak{c}$ be an integral ideal of K such that $C = \mathrm{N}(\mathfrak{c})$ is prime to D_K. If $\pi_p \in \Pi_p(\psi)$ has a $\mathcal{K}_1[\mathfrak{c}]_p$-fixed vector, where $\mathcal{K}_1[\mathfrak{c}]_p$ is the closure of $\Gamma_K^{(m)}[\mathfrak{c}] \cap \mathcal{G}_1(\mathbb{Q})$ in $\mathcal{G}_1(\mathbb{Q}_p)$, then the pairing $\mathcal{S} \times \Pi_p(\rho_\tau^{(m)})$ should be given by

$$\langle \chi_Q, \pi_p \rangle_p = \chi_Q(p)^{\mathrm{ord}_p C}, \qquad p \nmid D_K.$$

For $v = \infty$, we have $\langle \chi_Q, \pi_\infty \rangle_\infty = \chi_Q(-1)^{m/2}$. Set $\pi = \otimes_{p<\infty} \pi_p \otimes \pi_\infty$, where $\pi_p \in \Pi_p(\psi)$ has a non-trivial $\mathcal{K}_1[\mathfrak{c}]_p$-fixed vector. Then $m_\psi(\pi) = 1$ if and only if $\varepsilon_Q = \chi_Q(-1)^{m/2}$ for any $Q \subset Q_D$ such that $f_Q = f$. This is compatible with Corollary 15.19.

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