Schrödinger operators on the Wiener space

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1. Essential self-adjointness

\( (B, H, \mu) \): an abstract Wiener space

- **B**: a Banach space
- **H**: a Hilbert space \( \subset B \)
- **\mu**: the Wiener measure with

\[
\int_B e^{\sqrt{-1}\langle x, \varphi \rangle} \mu(dx) = \exp\left\{ -\frac{1}{2} |\varphi|_{H^*}^2 \right\},
\]

\[ \varphi \in B^* \subset H^*. \]
$\mathcal{F}C_0^\infty$: $f: B \rightarrow \mathbb{R}$ such that

$$f(x) = F(\langle x, \varphi_1 \rangle, \ldots, \langle x, \varphi_n \rangle),$$

$$F \in C_0^\infty(\mathbb{R}^n), \varphi_1, \ldots, \varphi_n \in B^*.$$

$L - V$: Schrödinger operator on $L^2(\mu)$

$L$: the Ornstein-Uhlenbeck operator

$V$: a scalar potential

**Question:**

Is $L - V$ essentially self-adjoint on $\mathcal{F}C_0^\infty$?
\[ \| \|_2: \ L^2\text{-norm} \]

\[ V_+ := \max\{V, 0\} \text{ (the positive part)} \]

\[ V_- := \max\{-V, 0\} \text{ (the negative part)} \]

**Proposition 1.1.** Assume

- \( V_+ \in L^{2+} = \bigcup_{p > 2} L^p \),

- there exist \( 0 < a < 1, \ b > 0 \) such that

\[ \| V_- f \|_2 \leq a \| Lf \|_2 + b \| f \|_2. \]

Then \( L - V \) is essentially selfadjoint on \( \mathcal{FC}_0^\infty \).
What is sufficient for

$$\|V\!\!\!f\|_2 \leq a\|Lf\|_2 + b\|f\|_2 ?$$

(Defective) logarithmic Sobolev inequality

$$\int_B |f|^2 \log(\|f\|/\|f\|_2) \, d\mu \leq \alpha \mathcal{E}(f, f) + \beta \|f\|_2^2.$$ 

- \((B, \mu)\): a probability space
- \(\mathcal{E}\): a Dirichlet form
- \(L\): the associated generator
We assume

- $\mathcal{E}$ admits a \textbf{square field operator $\Gamma$.}

- $\mathcal{E}$ has a \textbf{local property.}

Hence $\mathcal{E}$ has the following form

\begin{equation}
\mathcal{E}(f, g) = \int_B \Gamma(f, g) \, d\mu
\end{equation}

and $\Gamma$ has the derivation property.

\textbf{E.g.} On an abstract Wiener space:

- $\Gamma(f, g) = \nabla f \cdot \nabla g$, \quad $\nabla$: the gradient operator
Theorem 1.2. Assume
\[ \int_B |f|^2 \log(\|f\|/\|f\|_2) \, d\mu \leq \alpha \mathcal{E}(f, f) + \beta \|f\|_2^2. \]

Then, for any \( \varepsilon > 0 \), there exist positive constants \( K_1 \), \( K_2 \) such that
\[ \int_B f^2 \log_+ f \, d\mu \leq \alpha^2 (1 + \varepsilon) \|L f\|_2^2 + K_1 + K_2 \|f\|_2^6. \]

Hausdorff-Young inequality

Set

$$
\Phi(x) = x \log^2_+ x, \quad \psi^{-1}(x) = \Phi'(x),
$$

$$
\psi(x) = e^{\sqrt{x+1}-1}.
$$

Define the complimentary function

$$
\Psi(x) = \int_0^x \psi(y) dy.
$$

Hausdorff-Young inequality:

$$
xy \leq \Phi(x) + \Psi(y) \leq x \log^2_+ x + 2\sqrt{ye^{\sqrt{y}}}
$$
Theorem 1.3. Assume the logarithmic inequality
\[ \int_B |f|^2 \log(\|f\|/\|f\|_2) \, d\mu \leq \alpha \mathcal{E}(f, f) + \beta \|f\|_2^2 \]
and \( \nu \geq 0 \),
\[ e^\nu \in L^{2\alpha^+} = \bigcup_{p>2\alpha} L^p. \]
Then, there exist constants \( 0 < a < 1 \) and \( b \geq 0 \) such that
\[ (1.2) \quad \|\nu f\|_2 \leq a\|Lf\|_2 + b\|f\|_2. \]
We now return to an abstract Wiener space.

**Gross’ logarithmic Sobolev inequality**

\[
\int_B |f|^2 \log(|f|/\|f\|_2) \, d\mu \leq \int_B |\nabla f|^2 \, d\mu
\]

\Rightarrow \int_B f^2 \log^2 f \, d\mu \leq (1 + \varepsilon)\|Lf\|_2^2 + K_1 + K_2\|f\|_2^6.

**Theorem 1.4.** Assume

- \(V_+, e^{V-} \in L^{2+}\).

Then \(L - V\) is essentially self-adjoint on \(\mathcal{F}C_0^{\infty}\).

2. Domain of Schrödinger operator

We consider a Schrödinger operator $\mathcal{A} = L - V + W$ on an abstract Wiener space $(B, H, \mu)$.

Basic assumptions

(A.1) $V \geq 1$, $V \in L^{2+}$.

(A.2) $W \geq 0$ and there exists a constant $0 < \alpha < 1$ such that $e^W \in L^{2/\alpha}$.

$\Rightarrow \mathcal{A} = L - V + W$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$
Aim: To determine the domain,
i.e., $\text{Dom}(A) = \text{Dom}(L) \cap \text{Dom}(V)$

Main tools

- The Lax-Milgram theorem.
- The intertwining property, i.e.,

$$\sqrt{V}A = A\sqrt{V}.$$
How to define an operator $A$?

We define a vector field $b$ by

$$b = \frac{\nabla V}{2V} = \frac{1}{2} \nabla \log V.$$ 

and a bilinear form $\mathcal{E}_A$ by

$$\mathcal{E}_A(f, g) = (\nabla f, \nabla g) + (b \cdot \nabla f, g)$$

$$- (f, b \cdot \nabla g) + ((V - W - |b|^2)f, g).$$

By a formal computation, the associated generator is given by

$$A = L - 2b \cdot \nabla + (\nabla^* b - V + W + |b|^2).$$
Decompose $\mathcal{E}_A$ as

$$\mathcal{E}_A(f, g) = \hat{\mathcal{E}}_A(f, g) + \tilde{\mathcal{E}}_A(f, g)$$

where

$$\hat{\mathcal{E}}_A(f, g) = (\nabla f, \nabla g) + ((V - W - |b|^2)f, g),$$

$$\tilde{\mathcal{E}}_A(f, g) = (b \cdot \nabla f, g) - (f, b \cdot \nabla g).$$

Moreover, we set

$$\hat{\mathcal{E}}_{A-\lambda}(f, g) = \hat{\mathcal{E}}_A(f, g) + \lambda(f, g).$$

The bilinear form associated to $L - V$ is

$$\mathcal{E}_{L-V}(f, g) = (\nabla f, \nabla g) + (V f, g).$$
Clearly

\[ \text{Dom}(\mathcal{E}_{L-V}) = \text{Dom}(\nabla) \cap \text{Dom}(\sqrt{V}). \]

We will show that \( \text{Dom}(\hat{\mathcal{E}}_A) = \text{Dom}(\mathcal{E}_{L-V}). \)

**Additional assumptions**

We assume either

(B.1) \[ e^{W+|b|^2} \in L^{2/\alpha} \]

or there exists a constant \( C > 0 \) such that

(B.2) \[ |b|^2 \leq \alpha V + C. \]
Proposition 2.1. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then there exists a constant $\beta$ such that

$$(W + |b|^2 f, f) \leq \alpha \mathcal{E}_{L-V}(f, f) + \beta(f, f)$$

appeared in (A.2)

and hence

$$(1 - \alpha) \mathcal{E}_{L-V}(f, f) \leq \hat{\mathcal{E}}_A(f, f) + \beta(f, f)$$

$$\leq (1 + \alpha) \mathcal{E}_{L-V}(f, f) + \beta(f, f).$$

Therefore

$\text{Dom}(\hat{\mathcal{E}}_A) = \text{Dom}(\mathcal{E}_{L-V}) = \text{Dom}(\nabla) \cap \text{Dom}(\sqrt{V}).$
Estimate of $\tilde{\mathcal{E}}_A$

**Proposition 2.2.** Assume (A.1), (A.2) and one of (B.1) and (B.2). Then, for sufficiently large $\lambda$, there exists a constant $K > 0$ such that

$$|\tilde{\mathcal{E}}_A(f, g)| \leq K \tilde{\mathcal{E}}_{A-\lambda}(f, f)^{1/2} \tilde{\mathcal{E}}_{A-\lambda}(g, g)^{1/2}.$$ 

Therefore $\mathcal{E}_A$ satisfies the sector condition.

$\mathcal{E}_A = \hat{\mathcal{E}}_A + \tilde{\mathcal{E}}_A$ is a closed bilinear form.
Intertwining property

Instead of

$$\sqrt{V} A = A \sqrt{V},$$

we show

$$\mathcal{E}_2(f, \sqrt{V} g) = \mathcal{E}_A(\sqrt{V} f, g).$$  \hspace{1cm} (2.2)

**Proposition 2.3.** Assume (A.1), (A.2) and one of (B.1) and (B.2). Then (2.2) holds for $f, g \in \mathcal{F}C_0^\infty$. Moreover, we have, for $f \in \text{Dom}(A), g \in \text{Dom}(A^*)$,

$$\mathcal{A} f, \sqrt{V} g) = (\sqrt{V} f, A^* g).$$  \hspace{1cm} (2.3)
Domain of the Schrödinger operator

**Theorem 2.4.** Assume (A.1), (A.2) and one of (B.1) and (B.2). Then $\text{Dom}(\mathcal{A}) = \text{Dom}(L) \cap \text{Dom}(V)$. Moreover, for sufficiently large $\lambda$, there exist positive constants $K_1$, $K_2$ such that

$$K_1 \| (\mathcal{A} - \lambda)f \|_2 \leq \|Lf\|_2 + \|Vf\|_2 \leq K_2 \| (\mathcal{A} - \lambda)f \|_2.$$

**Remark.** $K_1$, $K_2$ depend only on constants in (A.1), (A.2), (B1), (B.2).
3. Spectral gap of Schrödinger operator

A Schrödinger operator $\mathcal{A} = L - V + W$ on an abstract Wiener space $(B, H, \mu)$.

$\sigma(\mathcal{A})$: the spectrum of $\mathcal{A} = L - V + W$.

**Bounded potential**

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**Theorem 3.1.** Assume $V$ is bounded and $W = 0$. Then $l = \sup \sigma(\mathcal{A})$ is a point spectrum of multiplicity one and the associated eigenfunction can be chosen to be positive. Moreover, the spectrum is discrete on $(l - 1, l]$, i.e., it consists of point spectrums of finite multiplicity.
General potential

**Theorem 3.2.** Assume (A.1), (A.2) and one of (B.1) and (B.2). Then $l = \sup \sigma(\mathcal{A})$ is a point spectrum of multiplicity one and the associated eigenfunction can be chosen to be positive. Moreover, the spectrum is discrete on $(l - 1, l]$, i.e., it consists of point spectrums of finite multiplicity.
Proof of Theorem 3.1

Approximation method

\( \{ \varphi_i \}_{i=1}^{\infty} \subseteq B^* : \) a c.o.n.s of \( H^* \).

\( \mathcal{F}_n := \sigma(\varphi_1, \varphi_2, \ldots, \varphi_n) \).

\( V_n = E[V|\mathcal{F}_n] \).

\( \Rightarrow \begin{cases} 
\sigma(L - V_n) \text{ is discrete on } (\lambda(V_n) - 1, \lambda(V_n)) \\
\text{where } \lambda(V_n) = \sup \sigma(L - V_n). 
\end{cases} \)
We set

\[ G^{(n)} = (\lambda - L + V_n)^{-1}, \]
\[ G = (\lambda - L + V)^{-1}. \]

**Claim:** \( G^{(n)} \rightarrow G \) in norm sense

\[ G - G^{(n)} = G^{(n)}(V - V_n)G. \]

We show \( \|(V - V_n)G\|_{\text{op}} \rightarrow 0. \)

By the logarithmic Sobolev inequality and the Hausdorff-Young inequality \( xy \leq x \log x - x + e^y \)
\[
\| (V - V_n) Gf \|_2^2 \\
= E[ (V - V_n)^2 (Gf)^2 ] \\
= \frac{1}{N} E[ N (V - V_n)^2 (Gf)^2 ] \\
\leq \frac{1}{N} E[ (Gf)^2 \log (Gf)^2 - (Gf)^2 + e^{N(V - V_n)^2} ] \\
\leq \frac{1}{N} \left\{ 2E[|\nabla Gf|^2] + \|Gf\|_2^2 \log \|Gf\|_2^2 \\
- \|Gf\|_2^2 + E[e^{N(V - V_n)^2}] \right\}.
\]
Now replacing $f$ with $f/\|Gf\|_2$, 
\[
\|(V - V_n)Gf\|_2^2 \\
\leq \frac{1}{N} \left\{ 2E[|\nabla Gf|^2] + E[e^{N(V-V_n)^2} - 1]\|Gf\|_2^2 \right\} \\
\leq \frac{1}{N} \left\{ E[f^2] + E[(Gf)^2] + E[|V|(Gf)^2] \\
+ E[e^{N(V-V_n)^2} - 1]\|f\|_2^2 \right\} \\
\leq \frac{1}{N} \left\{ (2 + \|V\|_\infty)\|f\|_2^2 + E[e^{N(V-V_n)^2} - 1]\|f\|_2^2 \right\}.
\]
Hence 
\[
\|(V - V_n)G\|_{op}^2 \leq \frac{1}{N} \left\{ 2 + \|V\|_\infty + E[e^{N(V-V_n)^2} - 1] \right\}.
\]
Now letting \( n \to \infty \) and then letting \( N \to \infty \), we have

\[
\lim_{n \to \infty} \| (V - V_n) G \|_{\text{op}} = 0.
\]

This completes the proof.