Schrödinger operators on the Wiener space

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August 10, 2004

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Contents

1. Essential self-adjointness

$(\boldsymbol{B}, \boldsymbol{H}, \boldsymbol{\mu})$: an **abstract Wiener space**

- $\boldsymbol{B}$: a Banach space
- $\boldsymbol{H}$: a Hilbert space $\subset \boldsymbol{B}$
- $\boldsymbol{\mu}$: the Wiener measure with

$$\int_B e^{\sqrt{-1}\langle x,\varphi\rangle} \boldsymbol{\mu}(d\boldsymbol{x}) = \exp\left\{-\frac{1}{2}|\varphi|_{H^*}^2\right\},$$

$$\varphi \in \boldsymbol{B}^* \subset \boldsymbol{H}^*.$$

$\mathcal{F}C_0^\infty$: $f: B \to \mathbb{R}$ such that

$$f(x) = F(\langle x, \varphi_1 \rangle, \ldots, \langle x, \varphi_n \rangle),$$

$$F \in C_0^\infty(\mathbb{R}^n),\ \varphi_1, \ldots, \varphi_n \in B^*.$$

$L - V$: **Schrödinger operator** on $L^2(\mu)$

L: the **Ornstein-Uhlenbeck operator**

V: a scalar potential

Question:

Is $L - V$ essentially self-adjoin on $\mathcal{F}C_0^\infty$?

$\| \ \|_2$: L^2-norm

$V_+ := \max\{V, 0\}$ (the positive part)

$V_- := \max\{-V, 0\}$ (the negative part)

Proposition 1.1. Assume

- $V_+ \in L^{2+} = \bigcup_{p>2} L^p$,
- there exist $0 < a < 1$, $b > 0$ such that

$$\|V_- f\|_2 \le a\|Lf\|_2 + b\|f\|_2.$$

Then $L - V$ is essentially selfadjoint on $\mathcal{F}C_0^\infty$.

What is sufficient for

$$\|V_- f\|_2 \le a\|Lf\|_2 + b\|f\|_2 \ ?$$

(Defective) logarithmic Sobolev inequality

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \le \alpha \mathcal{E}(f,f) + \beta \|f\|_2^2.$$

- (B,μ): a probability space
- $\mathcal{E}$: a Dirichlet form
- L: the associated generator

We assume

- $\boldsymbol{\mathcal{E}}$ admits a **square field operator** $\boldsymbol{\Gamma}$.
- $\boldsymbol{\mathcal{E}}$ has a **local property**.

Hence $\boldsymbol{\mathcal{E}}$ has the following form

$$\boldsymbol{\mathcal{E}}(\boldsymbol{f},\boldsymbol{g}) = \int_{\boldsymbol{B}} \boldsymbol{\Gamma}(\boldsymbol{f},\boldsymbol{g})\, \boldsymbol{d\mu} \tag{1.1}$$

and $\boldsymbol{\Gamma}$ has the derivation property.

E.g. On an abstract Wiener space:

- $\boldsymbol{\Gamma}(\boldsymbol{f},\boldsymbol{g}) = \boldsymbol{\nabla f} \cdot \boldsymbol{\nabla g}$, $\quad \boldsymbol{\nabla}$: the gradient operator

Theorem 1.2. Assume

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \leq \alpha \mathcal{E}(f,f) + \beta \|f\|_2^2.$$

Then, for any $\varepsilon > 0$, there exist positive constants K_1, K_2 such that

$$\int_B f^2 \log_+^2 f\, d\mu$$

$$\leq \alpha^2 (1+\varepsilon) \|Lf\|_2^2 + K_1 + K_2 \|f\|_2^6.$$

cf. Feissner(1975), Bakry-Meyer(1982)

Hausdorff-Young inequality

Set

$$\Phi(x) = x \log_+^2 x, \quad \psi^{-1}(x) = \Phi'(x),$$

$$\psi(x) = e^{\sqrt{x+1}-1}.$$

Define the complimentary function

$$\Psi(x) = \int_0^x \psi(y)dy.$$

Hausdorff-Young inequality:

$$xy \leq \Phi(x) + \Psi(y) \leq x \log_+^2 x + 2\sqrt{y}e^{\sqrt{y}}$$

Theorem 1.3. Assume the logarithmic inequality

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \le \alpha \mathcal{E}(f,f) + \beta \|f\|_2^2$$

and $v \ge 0$,

$$e^v \in L^{2\alpha+} = \bigcup_{p>2\alpha} L^p.$$

Then, there exist constants $0 < a < 1$ and $b \ge 0$ such that

$$\|vf\|_2 \le a\|Lf\|_2 + b\|f\|_2. \tag{1.2}$$

We now return to an abstract Wiener space.

Gross' logarithmic Sobolev inequality

$$\int_B |f|^2 \log(|f|/\|f\|_2)\, d\mu \le \int_B |\nabla f|^2\, d\mu$$

$$\Rightarrow \int_B f^2 \log_+^2 f\, d\mu \le (1+\varepsilon)\|Lf\|_2^2 + K_1 + K_2\|f\|_2^6.$$

Theorem 1.4. Assume

- $V_+,\ e^{V_-} \in L^{2+}$.

Then $L - V$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$.

cf. Segal(1969), Glimm & Jaffe(1970), Simon(1973), Simon & Høegh-Krohn(1972)

2. Domain of Schrödinger operator

We consider a Schrödinger operator $\mathfrak{A} = L - V + W$ on an abstract Wiener space (B, H, μ).

Basic assumptions

(A.1) $V \geq 1$, $V \in L^{2+}$.

(A.2) $W \geq 0$ and there exists a constant $0 < \alpha < 1$ such that $e^W \in L^{2/\alpha}$.

$\Rightarrow \mathfrak{A} = L - V + W$ is essentially self-adjoint on $\mathcal{F}C_0^\infty$

Aim : To determine the domain,

i.e., $\mathbf{Dom}(\mathfrak{A}) = \mathbf{Dom}(\boldsymbol{L}) \cap \mathbf{Dom}(\boldsymbol{V})$

Main tools

- The **Lax-Milgram theorem**.
- The **intertwining property**, i.e.,

$$\sqrt{\boldsymbol{V}}\,\mathfrak{A} = \boldsymbol{A}\sqrt{\boldsymbol{V}}.$$

How to define an operator A?

We define a vector field $\boldsymbol{b}$ by

$$\boldsymbol{b} = \frac{\nabla V}{2V} = \frac{1}{2}\nabla \log V.$$

and a bilinear form $\mathcal{E}_A$ by

$$\begin{aligned}\mathcal{E}_A(f,g) &= (\nabla f, \nabla g) + (\boldsymbol{b}\cdot\nabla f, g) \\ &\quad - (f, \boldsymbol{b}\cdot\nabla g) + ((V - W - |\boldsymbol{b}|^2)f, g).\end{aligned}$$

By a formal computation, the associated generator is given by

$$A = L - 2\boldsymbol{b}\cdot\nabla + (\nabla^* \boldsymbol{b} - V + W + |\boldsymbol{b}|^2). \tag{2.1}$$

Decompose $\boldsymbol{\mathcal{E}_A}$ as

$$\boldsymbol{\mathcal{E}_A(f,g) = \underbrace{\hat{\mathcal{E}}_A(f,g)}_{\text{symmetric}} + \underbrace{\check{\mathcal{E}}_A(f,g)}_{\text{skew-symmetric}}}$$

where

$$\boldsymbol{\hat{\mathcal{E}}_A(f,g) = (\nabla f, \nabla g) + ((V - W - |b|^2)f, g),}$$
$$\boldsymbol{\check{\mathcal{E}}_A(f,g) = (b \cdot \nabla f, g) - (f, b \cdot \nabla g).}$$

Moreover, we set

$$\boldsymbol{\hat{\mathcal{E}}_{A-\lambda}(f,g) = \hat{\mathcal{E}}_A(f,g) + \lambda(f,g).}$$

The bilinear form associated to $\boldsymbol{L - V}$ is

$$\boldsymbol{\mathcal{E}_{L-V}(f,g) = (\nabla f, \nabla g) + (Vf, g).}$$

Clearly

$$\mathbf{Dom}(\mathcal{E}_{L-V}) = \mathbf{Dom}(\nabla) \cap \mathbf{Dom}(\sqrt{V}).$$

We will show that $\mathbf{Dom}(\hat{\mathcal{E}}_A) = \mathbf{Dom}(\mathcal{E}_{L-V})$.

Additional assumptions

We assume either

(B.1) $$e^{W+|b|^2} \in L^{2/\alpha}$$

or there exists a constant $C > 0$ such that

(B.2) $$|b|^2 \leq \alpha V + C.$$

Proposition 2.1. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then there exists a constant β such that

$$(W + |b|^2 f, f) \leq \alpha \mathcal{E}_{L-V}(f, f) + \beta(f, f)$$

appeared in (A.2)

and hence

$$\begin{aligned}(1 - \alpha)\mathcal{E}_{L-V}(f, f) &\leq \hat{\mathcal{E}}_A(f, f) + \beta(f, f) \\ &\leq (1 + \alpha)\mathcal{E}_{L-V}(f, f) + \beta(f, f).\end{aligned}$$

Therefore

$$\mathbf{Dom}(\hat{\mathcal{E}}_A) = \mathbf{Dom}(\mathcal{E}_{L-V}) = \mathbf{Dom}(\nabla) \cap \mathbf{Dom}(\sqrt{V}).$$

Estimate of $\check{\mathcal{E}}_A$

Proposition 2.2. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then, for sufficiently large λ, there exists a constant $K > 0$ such that

$$|\check{\mathcal{E}}_A(f,g)| \le K \hat{\mathcal{E}}_{A-\lambda}(f,f)^{1/2} \hat{\mathcal{E}}_{A-\lambda}(g,g)^{1/2}.$$

Therefore $\mathcal{E}_A$ satisfies the sector condition.

$\mathcal{E}_A = \hat{\mathcal{E}}_A + \check{\mathcal{E}}_A$ is a closed bilinear form.

Intertwining property

Instead of

$$\sqrt{V}\mathfrak{A} = A\sqrt{V},$$

we show

$$\mathcal{E}_{\mathfrak{A}}(f, \sqrt{V}g) = \mathcal{E}_A(\sqrt{V}f, g). \tag{2.2}$$

Proposition 2.3. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then (2.2) holds for $f, g \in \mathcal{F}C_0^\infty$. Moreover, we have, for $f \in \mathbf{Dom}(\mathfrak{A})$, $g \in \mathbf{Dom}(A^*)$,

$$(\mathfrak{A}f, \sqrt{V}g) = (\sqrt{V}f, A^*g). \tag{2.3}$$

Domain of the Schrödinger operator

Theorem 2.4. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then $\mathbf{Dom}(\mathfrak{A}) = \mathbf{Dom}(L) \cap \mathbf{Dom}(V)$. Moreover, for sufficiently large λ, there exist positive constants K_1, K_2 such that

$$\begin{aligned} K_1\|(\mathfrak{A}-\lambda)f\|_2 &\le \|Lf\|_2 + \|Vf\|_2 \\ &\le K_2\|(\mathfrak{A}-\lambda)f\|_2. \end{aligned}$$

Remark. K_1, K_2 depend only on constants in (A.1), (A.2), (B1), (B.2).

3. Spectral gap of Schrödinger operator

A Schrödinger operator $\mathfrak{A} = \boldsymbol{L} - \boldsymbol{V} + \boldsymbol{W}$ on an abstract Wiener space $(\boldsymbol{B}, \boldsymbol{H}, \boldsymbol{\mu})$.

$\boldsymbol{\sigma}(\mathfrak{A})$: the spectrum of $\mathfrak{A} = \boldsymbol{L} - \boldsymbol{V} + \boldsymbol{W}$.

Bounded potential

Theorem 3.1. Assume $\boldsymbol{V}$ is bounded and $\boldsymbol{W} = \mathbf{0}$. Then $\boldsymbol{l} = \sup \boldsymbol{\sigma}(\mathfrak{A})$ is a point spectrum of multiplicity one and the associated eigenfunction can be chosen to be positive. Moreover, the spectrum is discrete on $(\boldsymbol{l} - \mathbf{1}, \boldsymbol{l}]$, i.e., it consists of point spectrums of finite multiplicity.

General potential

Theorem 3.2. Assume (A.1), (A.2) and one of (B.1) and (B.2). Then $\boldsymbol{l = \sup \sigma(\mathfrak{A})}$ is a point spectrum of multiplicity one and the associated eigenfunction can be chosen to be positive. Moreover, the spectrum is discrete on $\boldsymbol{(l-1, l}]$, i.e., it consists of point spectrums of finite multiplicity.

Proof of Theorem 3.1

Approximation method

$\{\varphi_i\}_{i=1}^{\infty} \subseteq B^*$: a c.o.n.s of H^*.

$\mathcal{F}_n := \sigma(\varphi_1, \varphi_2, \ldots, \varphi_n)$.

$V_n = E[V|\mathcal{F}_n]$.

$$\Rightarrow \begin{cases} \sigma(L - V_n) \text{ is discrete on } (\lambda(V_n) - 1, \lambda(V_n)] \\ \text{where } \lambda(V_n) = \sup \sigma(L - V_n). \end{cases}$$

We set

$$G^{(n)} = (\lambda - L + V_n)^{-1},$$
$$G = (\lambda - L + V)^{-1}.$$

claim: $G^{(n)} \to G$ in norm sense

$$G - G^{(n)} = G^{(n)}(V - V_n)G.$$

We show $\|(V - V_n)G\|_{\mathrm{op}} \to 0$.

By the logarithmic Sobolev inequality and the Hausdorff-Young inequality $xy \leq x \log x - x + e^y$

$$
\begin{aligned}
&\|(V - V_n)Gf\|_2^2 \\
&= E[(V - V_n)^2 (Gf)^2] \\
&= \frac{1}{N} E[N(V - V_n)^2 (Gf)^2] \\
&\leq \frac{1}{N} E[(Gf)^2 \log(Gf)^2 - (Gf)^2 + e^{N(V - V_n)^2}] \\
&\leq \frac{1}{N}\{2E[|\nabla Gf|^2] + \|Gf\|_2^2 \log \|Gf\|_2^2 \\
&\qquad - \|Gf\|_2^2 + E[e^{N(V - V_n)^2}]\}.
\end{aligned}
$$

Now replacing $\boldsymbol{f}$ with $\boldsymbol{f}/\|\boldsymbol{G}\boldsymbol{f}\|_2$,

$$\begin{aligned}
&\|(\boldsymbol{V}-\boldsymbol{V}_n)\boldsymbol{G}\boldsymbol{f}\|_2^2\\
&\le \frac{1}{\boldsymbol{N}}\{2\boldsymbol{E}[|\nabla \boldsymbol{G}\boldsymbol{f}|^2]+\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_n)^2}-1]\|\boldsymbol{G}\boldsymbol{f}\|_2^2\}\\
&\le \frac{1}{\boldsymbol{N}}\{\boldsymbol{E}[\boldsymbol{f}^2]+\boldsymbol{E}[(\boldsymbol{G}\boldsymbol{f})^2]+\boldsymbol{E}[|\boldsymbol{V}|(\boldsymbol{G}\boldsymbol{f})^2]\\
&\qquad +\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_n)^2}-1]\|\boldsymbol{f}\|_2^2\}\\
&\le \frac{1}{\boldsymbol{N}}\{(2+\|\boldsymbol{V}\|_\infty)\|\boldsymbol{f}\|_2^2+\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_n)^2}-1]\|\boldsymbol{f}\|_2^2\}.
\end{aligned}$$

Hence

$$\|(\boldsymbol{V}-\boldsymbol{V}_n)\boldsymbol{G}\|_{\mathrm{op}}^2\le \frac{1}{\boldsymbol{N}}\{2+\|\boldsymbol{V}\|_\infty+\boldsymbol{E}[e^{\boldsymbol{N}(\boldsymbol{V}-\boldsymbol{V}_n)^2}-1]\}.$$

Now letting $\boldsymbol{n} \to \infty$ and then letting $\boldsymbol{N} \to \infty$, we have

$$\lim_{\boldsymbol{n}\to\infty} \|(\boldsymbol{V} - \boldsymbol{V_n})\boldsymbol{G}\|_{\mathbf{op}} = \mathbf{0}.$$

This completes the proof.