Non-symmetric diffusions on a Riemannian manifold

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Abstract.

We consider a non-symmetric diffusion on a Riemannian manifold generated by $\mathfrak{A} = \frac{1}{2}\triangle + b$. We give a sufficient condition for which $\mathfrak{A}$ generates a C_0-semigroup in L^2. To do this, we show that $\mathfrak{A}$ is maximal dissipative. Further we give a characterization of the generator domain.

We also discuss the same issue in L^p $(1 < p < \infty)$ setting and give a sufficient condition for which $\mathfrak{A}$ generates a C_0-semigroup in L^p.

§1. Introduction

We consider diffusion processes on a Riemannian manifold generated by the operator $\frac{1}{2}\triangle + b$. Here $\triangle$ is the Laplace-Beltrami operator and b is a vector field. We assume that coefficients are all C^∞. So we can construct a diffusion process up to the explosion time by solving a stochastic differential equation. Our interest is to construct a L^p semigroup. Symmetry assumption in L^2 setting does not simplify the problem of essentially self-adjointness. So we consider the problem in non-symmetric case.

We will give a sufficient condition to construct a C_0 semigroup, i.e., strongly continuous semigroup, in L^2 or even in L^p. Further, in L^2, we can determine the domain of the generator. To do this, the intertwining property of operators plays an essential role.

The organization of the paper is as follows. In Section 2, we give a sufficient condition for the existence of C_0 semigroup in L^2. We have to show that the operator is maximal dissipative. In Section 3, we determine the domain of the generator. We use the intertwining property and the symmetric part of the associated bilinear form. In Section 4, we construct a C_0 semigroup in L^p and last we give some examples in Section 5.

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§2. Non-symmetric diffusion on a Riemannian manifold

Let (M, g) be a smooth d-dimensional Riemannian manifold. We assume that M is complete but we do not assume that M is compact in general. We consider a diffusion process on M whose generator is

$$\mathfrak{A} = \frac{1}{2}\triangle + b. \tag{2.1}$$

Here $\triangle$ is the Laplace-Beltrami operator, b is a vector field. We assume that b and other vector fields, tensor fields, etc, are all C^∞. We denote the Riemannian volume on M by $m = \mathrm{vol}$ and operators will be considered in $L^2(m)$, or $L^p(m)$ later.

The adjoint operator of $\mathfrak{A}$ is

$$\mathfrak{A}^* = \frac{1}{2}\triangle - b - \operatorname{div} b. \tag{2.2}$$

The symmetrization of $\mathfrak{A}$ is defined by

$$\tilde{\mathfrak{A}} = \frac{1}{2}(\mathfrak{A} + \mathfrak{A}^*) = \frac{1}{2}\triangle - \frac{1}{2}\operatorname{div} b. \tag{2.3}$$

So far, all operators are well-defined on $C_0^\infty(M)$, the set of all smooth functions on M with compact support.

The bilinear form $\mathcal{E}$ associated with $\mathfrak{A}$ is given by

$$\mathcal{E}(u, v) = -(\mathfrak{A}u, v)_2 = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm - \int_M (bu)v\, dm. \tag{2.4}$$

Here $(\ ,\ )_2$ denotes the inner product in $L^2(m)$, $(\cdot, \cdot)$ the Riemannian metric, and ∇ the gradient operator. Further we consider the bilinear form $\tilde{\mathcal{E}}$ associated with $\tilde{\mathfrak{A}}$ as follows:

$$\tilde{\mathcal{E}}(u, v) = \frac{1}{2}\int_M (\nabla u, \nabla v)\, dm + \frac{1}{2}\int_M (\operatorname{div} b)uv\, dm. \tag{2.5}$$

We impose the following condition to ensure that $\mathfrak{A}$ is bounded from above.

(A.1): There exists a constant γ so that $\frac{1}{2}\operatorname{div} b \geq -\gamma$.

To be precise, $\frac{1}{2}(\operatorname{div} b)_x \geq -\gamma$ for all $x \in M$. Under this condition, we can see that $\tilde{\mathcal{E}}$ is bounded from below and so we can take a closure of it. By taking closure, we may assume that $\tilde{\mathcal{E}}$ is closed. Our aim is to deal with semigroups without sector condition.

We denote the metric function on M by d. We fix a reference point $o \in M$ and set $\rho(x) = d(o, x)$. Since ρ is a Lipschitz function, $\nabla\rho$ can be defined as a vector valued bounded function. Using this, we add the following assumption on b:

(A.2): There exists a positive non-increasing function $\kappa\colon [0,\infty) \to [0,1]$ so that $\int_0^\infty \kappa(x)\,dx = \infty$ and $\kappa(\rho)b\rho \geq -1$.

To be precise, $\kappa(\rho(x))b\rho(x) \geq -1$ for all $x \in M$. The function $\kappa(x) = \frac{1}{x}$ is a typical example satisfying $\int_0^\infty \kappa(x)\,dx = \infty$.

To construct a semigroup, it suffices to show that our operator is maximal dissipative. Here, an operator A is called dissipative if $(Au, u) \leq 0$ for any $u \in \mathrm{Dom}(A)$ and if it has no proper dissipative extension, it is called maximal dissipative. For the general theory of semigroup, refer to e.g., Pazy [7, Chapter 1, Section 4] or Goldstein [4, Chapter 1, Section 3].

In symmetric case, i.e., $\mathfrak{A} = \frac{1}{2}\triangle$, this is equivalent to the essentially self-adjointness. This problem of essentially self-adjointness was solved by Gaffney [3] (see also Davies [1]). We have to modify it to handle the vector field b.

Theorem 2.1. Assume **(A.1)** and **(A.2)**. Then the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a Markovian C_0 semigroup in $L^2(m)$. (See, e.g., Ma-Röckner [6] for the Markovian property. To be precise we should say "sub-Markovian" but we use this terminology for simplicity.)

Proof. We first show that $\mathfrak{A} - \gamma$ is dissipative. Here, γ is a constant that appeared in **(A.1)**. From **(A.2)**, we have

$$\begin{aligned}((\mathfrak{A}-\gamma)u, u)_2 &= \int_M (\frac{1}{2}\triangle u + bu - \gamma u)u\,dm \\ &= -\tilde{\mathcal{E}}(u,u) - \gamma(u,u)_2 \\ &= -\frac{1}{2}\int_M (|\nabla u|^2 + u^2 \operatorname{div} b)\,dm - \int_M \gamma u^2\,dm \leq 0,\end{aligned}$$

which shows that $\mathfrak{A} - \gamma$ is dissipative.

To show that the closure of $\mathfrak{A} - \gamma$ generates a contraction semigroup, it suffices to show that the image $(\mathfrak{A} - \gamma - 1)(C_0^\infty(M))$ is dense in $L^2(m)$, which means that $\mathfrak{A} - \gamma$ is maximal-dissipative; in other words, to show that $u = 0$ if

$$\int_M u(\mathfrak{A}-\gamma)\phi\,dm = (u,\phi)_2, \quad \forall \phi \in C_0^\infty(M).$$

Assume this identity. Then, by the hypoellipticity of the elliptic operator, we have $u \in C^\infty(M)$. Using this, we have

$$(u,\phi)_2 = \int_M u(\mathfrak{A}-\gamma)\phi\,dm \tag{2.6}$$

$$= -\frac{1}{2}\int_M \nabla u\cdot\nabla\phi\,dm + \int_M (ub\phi - \gamma u\phi)\,dm, \quad \forall\phi\in C_0^\infty.$$

By an approximation argument, we easily see that the identity holds for $\phi\in H^1_{\mathrm{loc}}(M)$.

To truncate u, we introduce a bump function χ_n in the following procedure. Take a C^∞ function $\psi:\mathbb{R}\to[0,1]$ such that $\psi(t)=1$ for $t\in[0,1]$ and $\psi(t)=0$ for $t\in[3,\infty)$ and further $-1\le\psi'(t)\le 0$. Define $h(x)$ by

$$h(x) = \int_0^x \kappa(y)\,dy$$

and then set

$$\chi_n(x) = \psi(h(\rho(x))/n). \tag{2.7}$$

Here $\rho(x)=d(o,x)$. Clearly we have

$$b\chi_n = \psi'(h(\rho)/n)\frac{\kappa(\rho)}{n}b\rho.$$

Take $\phi=\chi_n^2 u$ as a test function. Then, from (2.6), we have

$$\begin{aligned}
&\int_M u\chi_n^2 u\,dm\\
&= -\frac{1}{2}\int_M \nabla u\cdot\nabla(\chi_n^2 u)\,dm + \int_M \{ub(\chi_n^2 u) - \gamma u\chi_n^2 u\}\,dm\\
&= -\frac{1}{2}\int_M \nabla u\cdot(\nabla(\chi_n u)\chi_n + \chi_n u\nabla\chi_n)\,dm\\
&\quad + \int_M \{u(b(\chi_n u)\chi_n + u\chi_n u b\chi_n) - \gamma\chi_n^2 u^2\}\,dm\\
&= -\frac{1}{2}\int_M \{\chi_n\nabla u\cdot\nabla(\chi_n u) + \chi_n u\nabla u\cdot\nabla\chi_n\}\,dm\\
&\quad + \int_M \{\chi_n u(b(\chi_n u) + \chi_n u^2 b\chi_n) - \gamma\chi_n^2 u^2\}\,dm\\
&= -\frac{1}{2}\int_M \{(\nabla(\chi_n u) - u\nabla\chi_n)\cdot\nabla(\chi_n u) + \chi_n u\nabla u\cdot\nabla\chi_n\}\,dm\\
&\quad + \int_M \{\frac{1}{2}b(\chi_n^2 u^2) + \chi_n u^2 b\chi_n - \gamma\chi_n^2 u^2\}\,dm\\
&= -\frac{1}{2}\int_M \{|\nabla(\chi_n u)|^2 - u\nabla\chi_n\cdot(\nabla\chi_n u + \chi_n\nabla u) + \chi_n u\nabla u\cdot\nabla\chi_n\}\,dm
\end{aligned}$$

$$
\begin{aligned}
&+ \int_M \{-\frac{1}{2}\chi_n^2 u^2 \operatorname{div} b + \chi_n u^2 \chi_n - \gamma \chi_n^2 u^2\}\, dm \\
&= -\frac{1}{2}\int_M \{|\nabla(\chi_n u)|^2 - u^2 |\nabla \chi_n|^2\}\, dm \\
&+ \int_M \{\chi_n u^2 b\chi_n - \frac{1}{2}\chi_n^2 u^2 \operatorname{div} b - \gamma \chi_n^2 u^2\}\, dm.
\end{aligned}
$$

Hence

$$
\begin{aligned}
(2.8)\quad &\frac{1}{2}\int_M |\nabla(\chi_n u)|^2\, dm + \frac{1}{2}\int_M \chi_n^2 u^2 \operatorname{div} b\, dm + \int_M (\gamma+1)\chi_n^2 u^2\, dm \\
&= \frac{1}{2}\int_M u^2 |\nabla \chi_n|^2\, dm + \int_M \chi_n u^2 b\chi_n\, dm.
\end{aligned}
$$

Note that $b\chi_n = \psi'(h(\rho)/n)\frac{\kappa(\rho)}{n} b\rho$. Using $-1 \le \psi' \le 0$ and the assumption $\kappa(\rho) b\rho \ge -1$, we have

$$
b\chi_n \le \frac{1}{n}
$$

and hence

$$
\int_M \chi_n u^2 b\chi_n\, dm \le \frac{1}{n}\int_M \chi_n u^2\, dm.
$$

Further, since $\nabla \chi_n = \psi'(h(\rho))/n)\frac{\kappa(\rho)}{n}\nabla\rho$ and $|\nabla \rho| \le 1$, we have

$$
|\nabla \chi_n| \le \frac{1}{n}.
$$

Thus, the right hand side of (2.8) is bounded and so $\chi_n u$ has a subsequence which converges weakly in $\tilde{\mathcal{E}}$. We can easily show that the limit is u and, by letting $n \to \infty$, we have

$$
\begin{aligned}
\tilde{\mathcal{E}}(u,u) + (\gamma+1)(u,u)_2 &\le \varliminf_{n\to\infty} \left\{ \frac{1}{2}\int_M u^2 |\nabla \chi_n|^2\, dm + \frac{1}{n}\int_M \chi_n u^2\, dm \right\} \\
&= 0.
\end{aligned}
$$

The positivity of $\tilde{\mathcal{E}}_\gamma = \tilde{\mathcal{E}} + \gamma(\ ,\ )_2$ brings $u = 0$.

Thus we have shown that $(\mathfrak{A} - \gamma - 1)(C_0^\infty(M))$ is dense in $L^2(m)$. This means that the closure of $\mathfrak{A} - \gamma$ with a domain $C_0^\infty(M)$ is maximal dissipertive and so it generates a contraction semigroup. From now on, taking closure, we regard $\mathfrak{A}$ as a closed operaotr. We also note that this means that $C_0^\infty(M)$ is dense in $\operatorname{Dom}(\mathfrak{A})$ with respect to the graph norm.

Last we show the Markovian property. The criterion is the following (see e.g., [6] when $\gamma = 0$ and [12] for general γ):

$$(\mathfrak{A}u, u - u \wedge 1)_2 \le \gamma \|u - u \wedge 1\|_2^2, \quad \forall u \in \mathrm{Dom}(\mathfrak{A}). \tag{2.9}$$

Here, $a \wedge b = \min\{a, b\}$ and $\| \ \|_2$ denotes the L^2 norm. Since we have shown that $C_0^\infty(M)$ is dense in $\mathrm{Dom}(\mathfrak{A})$, it suffices to show (2.9) for $u \in C_0^\infty(M)$. Take $u \in C_0^\infty(M)$. We construct an approximating sequence to the function $t \mapsto t \wedge 1$. Take any $\varepsilon > 0$ and take a C^∞ function φ_ε so that

$$\varphi_\varepsilon(t) = \begin{cases} t, & t \le 1, \\ \in [1, 1+\varepsilon], & 1 \le t \le 1 + 2\varepsilon, \\ 1 + \varepsilon, & t \ge 1 + 2\varepsilon \end{cases}$$

and $0 \le \varphi_\varepsilon' \le 1$. Recall that $\mathcal{E}(u, v) = -(\mathfrak{A}u, v)_2$. We first show

$$\varliminf_{\varepsilon \to 0} \mathcal{E}(\varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) \ge 0. \tag{2.10}$$

To do this, note

$$\begin{aligned} &\mathcal{E}(\varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) \\ &\quad = \int_M \{\frac{1}{2} \nabla \varphi_\varepsilon(u) \cdot \nabla (u - \varphi_\varepsilon(u)) - b\varphi_\varepsilon(u)(u - \varphi_\varepsilon(u))\} \, dm \\ &\quad = \frac{1}{2} \int_M \varphi_\varepsilon'(u)(1 - \varphi_\varepsilon'(u)) |\nabla u|^2 \, dm - \int_M \varphi_\varepsilon'(u) bu \, (u - \varphi_\varepsilon(u)) \, dm. \end{aligned}$$

The first term of the right hand side is non-negative. In the second term, the integrand is not 0 only when $1 \le u \le 1 + 2\varepsilon$ and in this case, $|u - \varphi_\varepsilon(u)| \le \varepsilon$. Hence the second term goes to 0 as $\varepsilon \to 0$ which proves (2.10). In addition, since $\tilde{\mathcal{E}}_\gamma$ is non-negative, we have

$$\tilde{\mathcal{E}}_\gamma(u - \varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) \ge 0.$$

Combining both of them, we get

$$\begin{aligned} 0 &\le \varliminf_{\varepsilon \to 0} \{\mathcal{E}(\varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) + \tilde{\mathcal{E}}_\gamma(u - \varphi_\varepsilon(u), u - \varphi_\varepsilon(u))\} \\ &\le \varliminf_{\varepsilon \to 0} \{\mathcal{E}(\varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) + \mathcal{E}(u - \varphi_\varepsilon(u), u - \varphi_\varepsilon(u)) + \gamma \|u - \varphi_\varepsilon(u)\|_2^2\} \\ &\le \varliminf_{\varepsilon \to 0} \{\mathcal{E}(u, u - \varphi_\varepsilon(u)) + \gamma \|u - \varphi_\varepsilon(u)\|_2^2\} \\ &\le \varliminf_{\varepsilon \to 0} \{-(\mathfrak{A}u, u - \varphi_\varepsilon(u))_2 + \gamma \|u - \varphi_\varepsilon(u)\|_2^2\} \end{aligned}$$

$$\le -(\mathfrak{A}u, u - u \wedge 1)_2 + \gamma \|u - u \wedge 1\|_2^2,$$

which is (2.9) as desired. Q.E.D.

From now on, we assume that $\mathfrak{A}$ is closed by taking a closure. The above argument shows that if $u \in \mathrm{Dom}(\mathfrak{A})$, then $u \in \mathrm{Dom}(\tilde{\mathcal{E}})$ and we have

$$\tilde{\mathcal{E}}(u, u) = -(\mathfrak{A}u, u)_2. \tag{2.11}$$

In connection to the Markovian property, we will show the L^1 contraction property. Here, the L^1 contraction property means that semigroup $\{T_t\}$ satisfies the following: for any $u \in L^2 \cap L^1$, we have

$$\|T_t u\|_1 \le \|u\|_1,$$

where $\| \ \|_1$ stands for the L^1-norm.

This is equivalent to the Markovian property of the dual semigroup. But we need an additional assumption to show the Markovian property of the dual semigroup, we give a direct proof of the L^1 contraction property. Then the Markovian property of the dual semigroup follows. We denote the semigroup generated by $\mathfrak{A}$ by $\{T_t\}$.

Proposition 2.2. Assume **(A.1)** and **(A.2)**. Then the semigroup $\{e^{-2t\gamma}T_t\}$ satisfies the L^1 contraction property.

Proof. It is enough to verify that

$$((\mathfrak{A} - 2\gamma)u, u_+ \wedge 1)_2 \le -\gamma \|u_+ \wedge 1\|_2^2, \quad \forall u \in \mathrm{Dom}(\mathfrak{A}), \tag{2.12}$$

(see [12]). To show this, we may assume that $u \in C_0^\infty(M)$.

We divide $\mathfrak{A}$ into tow parts: $\triangle$ and b. For any $\varepsilon > 0$, take φ_ε such that $0 \le \varphi'_\varepsilon \le 1$ and

$$\varphi_\varepsilon(t) = \begin{cases} \varepsilon, & t \le 0, \\ \in [\varepsilon, 2\varepsilon], & 0 \le t \le 2\varepsilon, \\ t, & 2\varepsilon \le t \le 1, \\ \in [1, 1+\varepsilon], & 1 \le t \le 1 + 2\varepsilon, \\ 1 + \varepsilon, & t \ge 1 + 2\varepsilon. \end{cases} \tag{2.13}$$

Then

$$\int_M \triangle u \varphi_\varepsilon(u)\, dm = -\int_M \nabla u \cdot \nabla \varphi_\varepsilon(u)\, dm = -\int_M \varphi'_\varepsilon |\nabla u|^2\, dm \le 0.$$

Now letting $\varepsilon \to 0$, we have

$$\int_M \triangle u(u_+ \wedge 1)\, dm \le 0.$$

Next we consider the bu part. Let Φ be a primitive function of the function $t \to t_+ \wedge 1$. That is

$$\Phi(t) = t(t_+ \wedge 1) - \frac{1}{2}(t_+ \wedge 1)^2 = \begin{cases} 0, & t \le 0, \\ \frac{1}{2}t^2, & 0 \le t \le 1, \\ t - \frac{1}{2}, & t \ge 1. \end{cases} \tag{2.14}$$

Then

$$\begin{aligned}\int_M \{bu(u_+ \wedge 1) - 2\gamma u(u_+ \wedge 1)\}\, dm &= \int_M \{b\Phi(u) - 2\gamma u(u_+ \wedge 1)\}\, dm \\ &= -\int_M (\Phi(u) \operatorname{div} b + 2\gamma u(u_+ \wedge 1))\, dm.\end{aligned}$$

By the assumption **(A.1)** and the definition of Φ, we have

$$-\Phi(u) \operatorname{div} b \le 2\gamma \Phi(u) = 2\gamma u(u_+ \wedge 1) - \gamma(u_+ \wedge 1)^2.$$

Hence

$$\int_M \{bu(u_+ \wedge 1) - 2\gamma u(u_+ \wedge 1)\}\, dm \le -\gamma \|u_+ \wedge 1\|_2^2.$$

Combining both of them, we get the desired result. Q.E.D.

We can deal with $\mathfrak{A}^*$ similarly. This time, the sign of the vector field is opposite and so we assume the following

(A.2)*: There exists a positive non-increasing function $\kappa\colon [0,\infty) \to [0,1]$ such that $\int_0^\infty \kappa(x)\, dx = \infty$ and $\kappa(\rho) b\rho \le 1$.

We now have the following

Theorem 2.3. Assume **(A.1)** and **(A.2)***. Then the closure of $(\mathfrak{A}^*, C_0^\infty(M))$ generates a C_0-semigroup in $L^2(m)$. Further the semigroup satisfies L^1 contraction property. If, in addition, $\operatorname{div} b \ge 0$, then the semigroup satisfies the Markovian property.

Proof. The proof is similar to that of Theorem 2.1. We only see the L^1 contraction property. Define Φ by (2.14). Then $\Phi(t)-t(t_+\wedge 1) = -\frac{1}{2}(t_+\wedge 1)^2$ and therefore

$$\begin{aligned}\int_M &\{-bu(u_+\wedge 1) - u(u_+\wedge 1)\,\mathrm{div}\, b\}\, dm \\ &= \int_M \{\Phi(u)\,\mathrm{div}\, b - u(u_+\wedge 1)\,\mathrm{div}\, b\}\, dm \\ &= -\frac{1}{2}\int_M (u_+\wedge 1)^2\,\mathrm{div}\, b\, dm \\ &\le \gamma \int_M (u_+\wedge 1)^2\, dm.\end{aligned}$$

This shows the L^1 contraction property. Q.E.D.

§3. Domain of the generator

We now proceed to the issue of determining the generator domain. Our main tool is the intertwining property of operators. So we first need to investigate the intertwining property between $\mathfrak{A}$ and ∇. Here ∇ is the covariant differentiation. We always assume that our connection is the Levi-Civita connection. The intertwining property between $\triangle$ and ∇ is well-known as $\nabla\triangle = \Box_1\nabla$, where $\Box_1$ is the Hodge-Kodaira operator $-(dd^*+d^*d)$ acting on 1-forms. In fact, noting that $\nabla = d$ for scalar functions and $d^2=0$, we have $\nabla\triangle = -dd^*d = -(dd^*+d^*d)d = \Box_1\nabla$. Let us recall that $\Box_1 = -\nabla^*\nabla - \mathrm{Ric}$ where Ric denotes the Ricci curvature. We will use this later. What about ∇ and ∇_b? To see this, we note that for any vector field X,

$$\begin{aligned}\nabla_X(bu) &= \nabla_X\langle\nabla u, b\rangle \\ &= (\nabla^2 u)(X,b) + \langle\nabla u, \nabla_X b\rangle \\ &= (\nabla^2 u)(b,X) + \langle\nabla u, \nabla_X b\rangle \quad (\text{symmetry of } \nabla^2 u) \\ &= \langle\nabla_b\nabla u, X\rangle + \langle\nabla u, \nabla_X b\rangle.\end{aligned}$$

Now define an operator $\vec{\mathfrak{A}}$ acting on 1-forms as

$$\vec{\mathfrak{A}}\theta = \frac{1}{2}\Box_1\theta + \nabla_b\theta + \langle\nabla.b, \theta\rangle. \tag{3.1}$$

Here $\langle\nabla.b,\theta\rangle$ is a 1-form defined by $\langle\nabla.b,\theta\rangle(X) = \theta(\nabla_X b)$ for any vector field X. Then

$$\nabla\mathfrak{A} = \vec{\mathfrak{A}}\nabla.$$

To be precise, this relation holds at least on $C_0^\infty(M)$.

Next we will get a symmetric bilinear form $\vec{\mathcal{E}}$ satisfying

$$\vec{\mathcal{E}}(\theta,\theta) = -(\vec{\mathfrak{A}}\theta,\theta)_2. \tag{3.2}$$

To do this, note that

$$\begin{aligned}
-(\vec{\mathfrak{A}}\theta,\theta)_2 &= -\frac{1}{2}(\Box_1\theta,\theta)_2 - \int_M (\nabla_b\theta,\theta)\,dm - \int_M (\langle\nabla.b,\theta\rangle,\theta)\,dm \\
&= \frac{1}{2}(\nabla\theta,\nabla\theta)_2 + \frac{1}{2}\int_M \operatorname{Ric}(\theta,\theta)\,dm - \frac{1}{2}\int_M \nabla_b(\theta,\theta)\,dm \\
&\quad - \int_M (\langle\nabla.b,\theta\rangle,\theta)\,dm \\
&= \frac{1}{2}(\nabla\theta,\nabla\theta)_2 + \frac{1}{2}\int_M \operatorname{Ric}(\theta,\theta)\,dm + \frac{1}{2}\int_M |\theta|^2 \operatorname{div} b\,dm \\
&\quad - \int_M (\langle\nabla.b,\theta\rangle,\theta)\,dm.
\end{aligned}$$

Let B be a symmetrization of ∇b, i.e., $B = \frac{1}{2}(\nabla.b + (\nabla.b)^*)$. Then, $\vec{\mathcal{E}}$ is given by

$$\vec{\mathcal{E}}(\theta,\eta) = \frac{1}{2}(\nabla\theta,\nabla\eta)_2 + \int_M \{\frac{1}{2}\operatorname{Ric}(\theta,\eta) + \frac{1}{2}\operatorname{div} b(\theta,\eta) - (B\theta,\eta)\}\,dm. \tag{3.3}$$

We impose the following assumption to ensure that $\vec{\mathcal{E}}$ is bounded from below.

(A.3): Ric is bounded from below and there exists a constant δ so that $\frac{1}{2}\operatorname{Ric} + \frac{1}{2}\operatorname{div} b - B \geq -\delta$.

Let us remark that $\vec{\mathcal{E}}$ in (3.3) is defined for C^∞ 1-forms with compact support. Assuming **(A.3)**, we see that $\vec{\mathcal{E}}_\delta = \vec{\mathcal{E}} + \delta(\ ,\ )_2$ becomes non-negative and we can take a closure. So we assume that $\vec{\mathcal{E}}$ is closed from now on. Further, by (3.3), we have

$$\frac{1}{2}\|\nabla\theta\|_2^2 \leq \vec{\mathcal{E}}_\delta(\theta,\theta). \tag{3.4}$$

We are ready to determine the domain of $\mathfrak{A}$.

Theorem 3.1. Assume **(A.1)**, **(A.2)**, **(A.2)*** and **(A.3)**. Then the necessary and sufficient condition for $u \in \operatorname{Dom}(\mathfrak{A})$ is that $u \in \operatorname{Dom}(\triangle)$ and $bu \in L^2(m)$.

Proof. The sufficiency is easily shown by noting that $C_0^\infty(M)$ is dense in $\mathrm{Dom}(\mathfrak{A}^*)$. In fact, by the integration by part, we have

$$(\frac{1}{2}\triangle u + bu, \phi)_2 = (u, \mathfrak{A}^*\phi)_2, \quad \forall \phi \in C_0^\infty(M).$$

It is easy to see that the above identity holds for $\phi \in \mathrm{Dom}(\mathfrak{A}^*)$ by using the denseness of $C_0^\infty(M)$ in $\mathrm{Dom}(\mathfrak{A}^*)$. This implies that $u \in \mathrm{Dom}(\mathfrak{A}^{**}) = \mathrm{Dom}(\mathfrak{A})$.

Next we will show the necessity. Take any $u \in C_0^\infty(M)$. Then

$$\begin{aligned}((\mathfrak{A} - \delta - 1)u, \triangle u)_2 &= -((\mathfrak{A} - \delta - 1)u, \nabla^*\nabla u)_2 \\ &= -(\nabla(\mathfrak{A} - \delta - 1)u, \nabla u)_2 \\ &= -((\vec{\mathfrak{A}} - \delta - 1)\nabla u, \nabla u)_2 \\ &= \vec{\mathfrak{E}}_{\delta+1}(\nabla u, \nabla u).\end{aligned}$$

Hence, by Young's inequality,

$$\vec{\mathfrak{E}}_{\delta+1}(\nabla u, \nabla u) \leq \frac{1}{2\varepsilon}\|(\mathfrak{A} - \delta - 1)u\|_2^2 + \frac{\varepsilon}{2}\|\triangle u\|_2^2.$$

Choose ε to be small so that

$$\frac{\varepsilon}{2}\|\triangle u\|_2^2 \leq \frac{1}{4}(\|\nabla^2 u\|_2 + \|u\|_2^2).$$

Then, by (3.4), we have

$$\begin{aligned}\frac{1}{2\varepsilon}\|(\mathfrak{A} - \delta - 1)u\|_2^2 &\geq \vec{\mathfrak{E}}_{\delta+1}(\nabla u, \nabla u) - \frac{\varepsilon}{2}\|\triangle u\|_2^2 \\ &\geq \vec{\mathfrak{E}}_{\delta+1}(\nabla u, \nabla u) - \frac{1}{4}(\|\nabla^2 u\|_2 + \|u\|_2^2) \\ &\geq \frac{1}{4}\vec{\mathfrak{E}}_{\delta+1}(\nabla u, \nabla u).\end{aligned}$$

Noting that $C_0^\infty(M)$ is dense in $\mathrm{Dom}(\mathfrak{A})$, the above relation implies that $\nabla u \in \mathrm{Dom}(\vec{\mathfrak{E}})$ if $u \in \mathrm{Dom}(\mathfrak{A})$. Therefore, by noting (3.4), we have $\nabla^2 u \in L^2(m)$, i.e., $u \in \mathrm{Dom}(\triangle)$. Since $bu = \mathfrak{A}u - \frac{1}{2}\triangle u$, we have $bu \in L^2(m)$. This completes the proof. Q.E.D.

We can have a similar result for $\mathfrak{A}^*$ but we have to handle the potential term $\operatorname{div} b$ in this case. First we will get the intertwining property between $\mathfrak{A}^*$ and ∇. To do this, it is enough to use that $\nabla(Vu) = u\nabla V + V\nabla u$ for $V = \operatorname{div} b$. So, defining a operator $\vec{\mathfrak{D}}$ acting on 1-forms by

$$\vec{\mathfrak{D}}\theta = \frac{1}{2}\Box_1\theta - \nabla_b\theta - \langle\nabla.b, \theta\rangle - \theta \operatorname{div} b, \tag{3.5}$$

we easily have the following defective intertwining property (see [11]):

$$\nabla \mathfrak{A}^* u = \vec{\mathfrak{D}} \nabla u - u \nabla \operatorname{div} b.$$

Further, denoting the symmetrization of $\nabla . b$ by B and noting that $\square_1 = -\nabla^* \nabla - \operatorname{Ric}$,

$$-(\vec{\mathfrak{D}}\theta, \theta)_2 = \frac{1}{2}(\nabla\theta, \nabla\theta)_2 + \int_M \{\frac{1}{2} \operatorname{Ric}(\theta, \theta) + \frac{1}{2}|\theta|^2 \operatorname{div} b + (B\theta, \theta)\}\, dm.$$

Now we introduce the following assumption.

> **(A.4):** Ric is bounded from below and there exists a constant δ' so that $\operatorname{Ric} + \frac{1}{2} \operatorname{div} b + B \geq -\delta'$. Moreover, $\frac{\nabla \operatorname{div} b}{\operatorname{div} b + 2\gamma + 2}$ is bounded.

Under the above assumptions, we define a bilinear form $\vec{\mathcal{E}}'$ on 1-forms by

(3.6)

$$\vec{\mathcal{E}}'(\theta, \eta) = \frac{1}{2}(\nabla\theta, \nabla\eta)_2 + \int_M \{\frac{1}{2} \operatorname{Ric}(\theta, \eta) + \frac{1}{2}(\theta, \eta) \operatorname{div} b + (B\theta, \eta)\}\, dm.$$

Then $\vec{\mathcal{E}}'$ is bounded from below and so it is closable. Taking a closure, we may assume that $\vec{\mathcal{E}}'$ is closed. We also have the inequality for $\vec{\mathcal{E}}'$ as follows:

$$\frac{1}{2}\|\nabla\theta\|_2^2 \leq \vec{\mathcal{E}}'_{\delta'}(\theta, \theta).$$

Now we can determine the domain of $\mathfrak{A}^*$.

Theorem 3.2. Assume **(A.1)**, **(A.2)**, **(A.2)*** and **(A.4)**. Then the necessary and sufficient condition for $u \in \operatorname{Dom}(\mathfrak{A}^*)$ is that $u \in \operatorname{Dom}(\triangle)$ and $bu + \frac{1}{2} u \operatorname{div} b \in L^2(m)$.

Proof. As in the proof of Theorem 3.1, the sufficiency is shown by using the denseness of $C_0^\infty(M)$ in $\operatorname{Dom}(\mathfrak{A})$.

We will show the necessity. We set $V = \operatorname{div} b$. From the assumption, we take a constant M so that $\frac{|\nabla V|}{V + 2\gamma + 2} \leq M$. Take any $u \in C_0^\infty(M)$. Then

$$\begin{aligned}
&((\mathfrak{A}^* - \delta' - 1)u, \triangle u)_2 \\
&\quad = -((\mathfrak{A}^* - \delta' - 1)u, \nabla^* \nabla u)_2 \\
&\quad = -(\nabla(\mathfrak{A}^* - \delta' - 1)u, \nabla u)_2 \\
&\quad = -((\vec{\mathfrak{D}} - \delta' - 1)\nabla u, \nabla u)_2 + (u \nabla V, \nabla u)_2
\end{aligned}$$

$$
\begin{aligned}
&= \vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) + (\frac{\nabla V}{V+2\gamma+2}\sqrt{V+2\gamma+2}u, \sqrt{V+2\gamma+2}\nabla u)_2 \\
&\geq \vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) \\
&\quad - M\Big\{\frac{1}{2\varepsilon}\int_M \{(V+2\gamma+2)u^2\,dm + \frac{\varepsilon}{2}\int_M (V+2\gamma+2)|\nabla u|^2\}\,dm \\
&\geq \vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) - \frac{M}{\varepsilon}\tilde{\mathcal{E}}_{\gamma+1}(u,u) - M\varepsilon\vec{\mathcal{E}}'_{\gamma+1}(\nabla u, \nabla u).
\end{aligned}
$$

Choose $\varepsilon > 0$ to be small so that $\frac{\varepsilon}{2}\|\triangle u\|_2 \leq \frac{1}{2}\vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) + \|u\|_2^2$. Then

$$
\begin{aligned}
&(1-M\varepsilon)\vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) \\
&\quad \leq ((\mathfrak{A}^* - \delta' - 1)u, \triangle u)_2 + \frac{M}{\varepsilon}\tilde{\mathcal{E}}_{\gamma+1}(u,u) \\
&\quad = ((\mathfrak{A}^* - \delta' - 1)u, \triangle u)_2 - \frac{M}{\varepsilon}((\mathfrak{A}^* - \gamma - 1)u, u)_2 \\
&\quad \leq \frac{1}{2\varepsilon}\|(\mathfrak{A}^* - \delta' - 1)u\|_2 + \frac{\varepsilon}{2}\|\triangle u\|_2 + \frac{M}{2\varepsilon}\|(\mathfrak{A}^* - \gamma - 1)u\|_2^2 + \frac{M}{2\varepsilon}\|u\|_2^2 \\
&\quad \leq \frac{1}{2\varepsilon}\|(\mathfrak{A}^* - \delta' - 1)u\|_2 + \frac{1}{2}\vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) \\
&\qquad + \|u\|_2^2 + \frac{M}{2\varepsilon}\|(\mathfrak{A}^* - \gamma - 1)u\|_2^2 + \frac{M}{2\varepsilon}\|u\|_2^2.
\end{aligned}
$$

Eventually we have

$$
\begin{aligned}
&(\frac{1}{2} - M\varepsilon)\vec{\mathcal{E}}'_{\delta'+1}(\nabla u, \nabla u) \\
&\qquad \leq \frac{1}{2\varepsilon}\|(\mathfrak{A}^* - \delta' - 1)u\|_2 + \frac{M}{2\varepsilon}\|(\mathfrak{A}^* - \gamma - 1)u\|_2 + (\frac{M}{2\varepsilon} + 1)\|u\|_2^2.
\end{aligned}
$$

Here we take again ε to be small so that the coefficient of the left hand side becomes positive. This inequality holds for $u \in C_0^\infty(M)$ but, using the denseness, we can see that the above inequality holds for $u \in \mathrm{Dom}(\mathfrak{A}^*)$. It brings $\nabla u \in \mathrm{Dom}(\vec{\mathcal{E}}')$. Therefore we have $\nabla^2 u \in L^2(m)$ and hence $u \in \mathrm{Dom}(\triangle)$. Now $bu + \frac{1}{2}u\,\mathrm{div}\,b \in L^2(m)$ follows easily. The proof is completed. Q.E.D.

§4. Construction of L^p semigroup

So far we considered the L^2 case. In this section, we will construct a semigroup in L^p setting where $1 < p < \infty$. We can show it along the same line as before but the discussion becomes complicated. We consider two cases separately: $p \leq 2$ and $p \geq 2$. In the case $p \leq 2$, we have the following.

Theorem 4.1. Assume conditions **(A.1)** and **(A.2)**. Then, in $L^p(m)$ $(1 < p \leq 2)$, the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a C_0 semigroup.

Proof. Set $\gamma_p = \frac{p}{2}\gamma$. Then, from **(A.1)**, we have

$$\frac{1}{p}\operatorname{div} b \geq -\gamma_p. \tag{4.1}$$

We first show that $\mathfrak{A} - \gamma_p$ is dissipative. To do this, we take any $u \in C_0^\infty(M)$ and show that

$$\int_M \triangle u \operatorname{sgn}(u)|u|^{p-1}\, dm \leq 0. \tag{4.2}$$

For $\varepsilon > 0$, define φ_ε by

$$\varphi_\varepsilon(t) = t(t^2+\varepsilon)^{(p/2)-1}. \tag{4.3}$$

Then

$$\begin{aligned}\varphi_\varepsilon'(t) &= (t^2+\varepsilon)^{(p/2)-1} + t(\frac{p}{2}-1)(t^2+\varepsilon)^{(p/2)-2}2t \\ &= (t^2+\varepsilon)^{(p/2)-2}(t^2+\varepsilon+(p-2)t^2) \\ &= (t^2+\varepsilon)^{(p/2)-2}((p-1)t^2+\varepsilon) \geq 0.\end{aligned}$$

Therefore, we have

$$\int_M \triangle u\varphi_\varepsilon(u)\, dm = -\int_M \nabla u\varphi_\varepsilon'(u)\nabla u\, dm = -\int_M \varphi_\varepsilon'(u)|\nabla u|^2\, dm \leq 0.$$

Letting $\varepsilon \to 0$, we can get (4.2).

Let us deal with bu. This time, we set $\varphi(t) = |t|^p$. Then, φ is a C^1 function and $\varphi'(t) = p\operatorname{sgn}(t)|t|^{p-1}$. Hence

$$b(|u|^p) = b\varphi(u) = p\operatorname{sgn}(u)|u|^{p-1}bu$$

and

$$\int_M bu\operatorname{sgn}(u)|u|^{p-1}\, dm = \frac{1}{p}\int_M b(|u|^p)\, dm = -\frac{1}{p}\int_M |u|^p \operatorname{div} b\, dm.$$

Combining them, we have

$$\begin{aligned}&\int_M (\mathfrak{A}-\gamma_p)u\operatorname{sgn}(u)|u|^{p-1}\, dm \\ &\quad = \int_M \triangle u\operatorname{sgn}(u)|u|^{p-1} - \int_M (\frac{1}{p}\operatorname{div} b + \gamma_p)|u|^p\, dm \leq 0,\end{aligned}$$

which shows that $\mathfrak{A}-\gamma_p$ is dissipative.

Next we show that its closure generates a C_0 semigroup. To do this, it suffices to show that the image of $C_0^\infty(M)$ by $\mathfrak{A}-\alpha$ is dense in $L^p(m)$ for sufficiently large α. So let q be the conjugate exponent of p and assume that $u\in L^q(m)$ satisfies

$$\int_M u(\mathfrak{A}-\alpha)\phi\, dm=0,\quad \forall\phi\in C_0^\infty.$$

Our aim is to deduce $u=0$ from this condition. By using the hypoellipticity of the elliptic operator, we have $u\in C^\infty(M)$. Therefore, the above identity can be rewritten as

$$-\frac{1}{2}\int_M \nabla u\nabla\phi\, dm+\int_M ub\phi=\alpha\int_M \phi u\, dm,\quad \forall\phi\in C_0^\infty.$$

It is easy to see that the above identity holds for any $\phi\in H^1(M)$ with compact support. We now set

$$\psi(t)=\operatorname{sgn}(t)|t|^{q-1}.$$

ψ is a C^1-function and $\psi'(t)=(q-1)|t|^{q-2}$. We take $\phi=\chi_n^q\psi(u)$ where χ_n is a function defined by (2.7) in Section 2. Then we have

$$\begin{aligned}\alpha\int_M \chi_n^q\psi(u)u\, dm&=-\frac{1}{2}\int_M \nabla(\chi_n^q\psi(u))\cdot\nabla u\, dm+\int_M ub(\chi_n^q\psi(u))\, dm\\&=: I_1+I_2.\end{aligned}$$

We compute I_1, I_2 respectively. As for I_1,

$$\begin{aligned}I_1&=-\frac{1}{2}\int_M \nabla(\chi_n^q\psi(u))\cdot\nabla u\, dm\\&=-\frac{1}{2}\int_M \{q\chi_n^{q-1}\operatorname{sgn}(u)|u|^{q-1}\nabla\chi_n\cdot\nabla u+(q-1)\chi_n^q|u|^{q-2}|\nabla u|^2\}\, dm\\&=-\frac{q}{2}\int_M q\chi_n^{q-1}\operatorname{sgn}(u)|u|^{q-1}\nabla\chi_n\cdot\nabla u\, dm\\&\quad+(q-1)\int_M \chi_n^q|u|^{q-2}|\nabla u|^2\, dm.\end{aligned}$$

The first term of the left hand side is estimated as follows.

$$\begin{aligned}&\frac{q}{2}\left|\int_M \chi_n^{q-1}\operatorname{sgn}(u)|u|^{q-1}\nabla\chi_n\cdot\nabla u\, dm\right|\\&\qquad\qquad\le\frac{q}{2}\int_M \chi_n^{q-1}|u|^{q-1}|\nabla\chi_n|\,|\nabla u|\, dm\end{aligned}$$

$$\begin{aligned}
&\le \frac{q}{2n}\int_M \chi_n^{\frac{q}{2}-1}|u|^{\frac{q}{2}}|u|^{\frac{q-2}{2}}|\nabla u|\chi_n^{\frac{q}{2}}\,dm \quad (\because |\nabla \chi_n| \le \frac{1}{n})\\
&\le \frac{q}{4n}\int_M \{\chi_n^{q-2}|u|^q n^q |u|^{q-2}|\nabla u|^2\}\,dm\\
&\le \frac{q}{4n}\int_M \chi_n^{q-2}|u|^q\,dm + \frac{q}{4n}\int_M \chi_n^q|u|^{q-2}|\nabla u|^2\,dm.
\end{aligned}$$

Thus we have

$$I_1 \le \frac{q}{4n}\int_M \chi_n^{q-2}|u|^q\,dm - \{(q-1)-\frac{q}{4n}\}\int_M \chi_n^q|u|^{q-2}|\nabla u|^2\,dm.$$

If we take n to be large so that $(q-1)-\frac{q}{4n} > 0$, then we can get

$$I_1 \le \frac{q}{4n}\int_M \chi_n^{q-2}|u|^q\,dm.$$

As for I_2,

$$\begin{aligned}
I_2 &= \int_M \{ub\chi_n\chi_n^{q-1}\psi(u) + u\chi_n b(\chi_n^{q-1}\psi(u))\}\,dm\\
&= \int_M \{(b\chi_n)\chi_n^{q-1}|u|^q - b(u\chi_n)\chi_n^{q-1}\psi(u) - (\operatorname{div} b)u\chi_n\chi_n^{q-1}\psi(u)\}\,dm\\
&= \int_M \{(b\chi_n)\chi_n^{q-1}|u|^q - \frac{1}{q}b(\chi_n^q|u|^q) - (\operatorname{div} b)\chi_n^q|u|^q\}\,dm\\
&= \int_M \{(b\chi_n)\chi_n^{q-1}|u|^q + \frac{1}{q}\operatorname{div} b(\chi_n^q|u|^q) - \operatorname{div} b\chi_n^q|u|^q\}\,dm\\
&= \int_M \{(b\chi_n)\chi_n^{q-1}|u|^q - (1-\frac{1}{q})(\operatorname{div} b)\chi_n^q|u|^q\}\,dm.
\end{aligned}$$

From the assumption, $b\chi_n \le \frac{1}{n}$ and hence

$$\int_M (b\chi_n)\chi_n^{q-1}|u|^q\,dm \le \frac{1}{n}\int_M \chi_n^{q-1}|u|^q\,dm$$

and therefore

$$I_2 \le \frac{1}{n}\int_M \chi_n^{q-1}|u|^q\,dm - (1-\frac{1}{q})\int_M (\operatorname{div} b)\chi_n^q|u|^q\,dm.$$

Summing up both of them, we have

$$\begin{aligned}
\alpha\int_M \chi_n^q|u|^q\,dm \le{}& \frac{q}{4n}\int_M \chi_n^{q-2}|u|^q\,dm + \frac{1}{n}\int_M \chi_n^{q-1}|u|^q\,dm\\
&- (1-\frac{1}{q})\int_M (\operatorname{div} b)\chi_n^q|u|^q\,dm.
\end{aligned}$$

Now we take α large enough so that $\alpha - (1 - \frac{1}{q}) \operatorname{div} b \geq 1$. Then

$$\int_M \chi_n^q |u|^q \, dm \leq \frac{q}{4n} \int_M \chi_n^{q-2} |u|^q \, dm + \frac{1}{n} \int_M \chi_n^{q-1} |u|^q \, dm.$$

Since $u \in L^q(m)$, by letting $n \to \infty$, we get

$$\int_M |u|^q \, dm \leq 0,$$

which implies $u = 0$ and the proof is completed. Q.E.D.

We can treat the case $p \geq 2$ similarly but we have to adopt a different approximation method.

Theorem 4.2. Assume **(A.1)** and **(A.2)**. Then, the closure of $(\mathfrak{A}, C_0^\infty(M))$ generates a C_0 semigroup in $L^p(m)$ $(p \geq 2)$.

Proof. Setting $\gamma_p = \frac{p}{2}\gamma$, we have

$$\frac{1}{p} \operatorname{div} b \geq -\gamma_p. \tag{4.4}$$

Let us first show that $\mathfrak{A} - \gamma_p$ is dissipative. We note that for $u \in C_0^\infty(M)$,

$$\begin{aligned}
\int_M \triangle u \operatorname{sgn}(u) |u|^{p-1} \, dm &= -\int_M \nabla u \cdot \nabla(\operatorname{sgn}(u)|u|^{p-1}) \, dm \\
&= -\int_M \nabla u \cdot ((p-1)|u|^{p-2} \nabla u) \, dm \\
&= -\int_M (p-1)|u|^{p-2} |\nabla u|^2 \, dm \leq 0.
\end{aligned}$$

To deal with bu, we note

$$b(|u|^p) = b\varphi(u) = p \operatorname{sgn}(u)|u|^{p-1} bu,$$

and hence we have

$$\int_M \nabla u \operatorname{sgn}(u)|u|^{p-1} \, dm = \frac{1}{p} \int_M b(|u|^p) \, dm = -\frac{1}{p} \int_M (\operatorname{div} b)|u|^p \, dm.$$

Therefore

$$\begin{aligned}
&\int_M (\mathfrak{A} - \gamma_p) u \operatorname{sgn}(u)|u|^{p-1} \, dm \\
&\qquad = \int_M \triangle u \operatorname{sgn}(u)|u|^{p-1} \, dm - \int_M (\frac{1}{p} \operatorname{div} b + \gamma_p)|u|^p \, dm \leq 0,
\end{aligned}$$

which shows that $\mathfrak{A} - \gamma_p$ is dissipative.

Next we show that its closure generates a C_0 semigroup. To do this, we need to show that it is maximal dissipative, i.e., for large enough α, the image of $C_0^\infty(M)$ by $\mathfrak{A}-\alpha$ is dense in $L^p(m)$. Let q be the conjugate exponent of p and suppose that $u \in L^q(m)$ satisfies

$$\int_M u(\mathfrak{A} - \alpha)\phi \, dm = 0, \quad \forall \phi \in C_0^\infty.$$

We need to show that $u = 0$. We note that, by the hypoellipticity of the elliptic operator, $u \in C^\infty(M)$. So the above identity can be rewritten as

$$-\frac{1}{2}\int_M \nabla u \nabla \phi \, dm + \int_M ub\phi = \alpha \int_M \phi u \, dm, \quad \forall \phi \in C_0^\infty.$$

Further the above identity holds for $\phi \in H^1(M)$ with a compact support. For $\varepsilon > 0$, we set, as in (4.3), $\varphi_\varepsilon(t) = t(t^2+\varepsilon)^{\frac{q}{2}-1}$. Then φ_ε is a C^1 function and satisfies $\varphi_\varepsilon'(t) = (t^2+\varepsilon)^{\frac{q}{2}-2}((q-1)t^2+\varepsilon)$. We again use χ_n defined by (2.7). Taking $\phi = \chi_n^p \varphi_\varepsilon(u)$,

$$\begin{aligned}\alpha \int_M \chi^p \varphi_\varepsilon(u)\, dm &= -\frac{1}{2}\int_M \nabla(\chi_n^p\varphi_\varepsilon(u)) \cdot \nabla u \, dm + \int_M ub(\chi_n^p \varphi_\varepsilon(u))\, dm \\ &=: I_1 + I_2.\end{aligned}$$

We estimate I_1 and I_2 respectively. As for I_1,

$$\begin{aligned}I_1 &= -\frac{1}{2}\int_M \nabla(\chi_n^p \varphi_\varepsilon(u)) \cdot \nabla u \, dm \\ &= -\frac{1}{2}\int_M \{p\chi_n^{p-1}\varphi_\varepsilon(u)\nabla\chi_n \cdot \nabla u + \chi_n^p \varphi_\varepsilon'(u) \nabla u \cdot \nabla u\}\, dm \\ &= -\frac{p}{2}\int_M \chi_n^{p-1}\varphi_\varepsilon(u)\nabla\chi_n \cdot \nabla u \, dm \\ &\quad - \frac{1}{2}\int_M \chi_n^p (u^2+\varepsilon)^{\frac{q}{2}-2}((q-1)u^2+\varepsilon)|\nabla u|^2 \, dm.\end{aligned}$$

The first term is estimated as

$$\begin{aligned}&\left|\frac{p}{2}\int_M \chi_n^{p-1}\varphi_\varepsilon(u)\nabla\chi_n \cdot \nabla u \, dm\right| \\ &\qquad \le \frac{p}{2}\int_M \chi_n^{p-1}|\varphi_\varepsilon(u)|\,|\nabla\chi_n|\,|\nabla u|\, dm \\ &\qquad \le \frac{Mp}{2n}\int_M \chi_n^{p-1}|\varphi_\varepsilon(u)|\,|\nabla u|\, dm\end{aligned}$$

$$\begin{aligned}
&= \frac{Mp}{2n} \int_M \chi_n^{p-1} |u| (u^2+\varepsilon)^{\frac{q}{2}-1} |\nabla u| \, dm \\
&= \frac{Mp}{2n} \int_M \chi_n^{\frac{p}{2}-1} |u| (u^2+\varepsilon)^{\frac{q}{4}} ((q-1)u^2+\varepsilon)^{-\frac{1}{2}} \\
&\qquad\qquad \times \chi_n^{\frac{p}{2}} (u^2+\varepsilon)^{\frac{q}{4}-1} ((q-1)u^2+\varepsilon)^{\frac{1}{2}} |\nabla u| \, dm \\
&\le \frac{Mp}{4n} \int_M \{ \chi_n^{p-2} |u|^2 (u^2+\varepsilon)^{\frac{q}{2}} ((q-1)u^2+\varepsilon)^{-1} \\
&\qquad\qquad + \chi_n^p (u^2+\varepsilon)^{\frac{q}{2}-2} ((q-1)u^2+\varepsilon) |\nabla u|^2 \} \, dm.
\end{aligned}$$

In the first term of the above integrand,

$$\begin{aligned}
\frac{|u|^2 (u^2+\varepsilon)^{\frac{q}{2}}}{(q-1)u^2+\varepsilon} &= \frac{|u|^2 (u^2+\varepsilon)^{\frac{q}{2}}}{((q-1)u^2+\varepsilon)^{\frac{q}{2}} ((q-1)u^2+\varepsilon)^{1-\frac{q}{2}}} \\
&\le \frac{|u|^2 (u^2+\varepsilon)^{\frac{q}{2}}}{((q-1)u^2+(q-1)\varepsilon)^{\frac{q}{2}} ((q-1)u^2)^{1-\frac{q}{2}}} \\
&= \frac{|u|^2 (u^2+\varepsilon)^{\frac{q}{2}}}{(q-1)(u^2+\varepsilon)^{\frac{q}{2}} |u|^{2-q}} \\
&= \frac{|u|^q}{q-1}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
I_1 &\le \frac{Mp}{4n(q-1)} \int_M \chi_n^{p-1} |u|^q \, dm \\
&\quad - \left(\frac{1}{2} - \frac{Mp}{4n} \right) \int_M \chi_n^p (u^2+\varepsilon)^{\frac{q}{2}-2} ((q-1)u^2+\varepsilon) |\nabla u|^2 \, dm.
\end{aligned}$$

Taking n to be large enough so that $\frac{1}{2} - \frac{Mp}{4n} > 0$,

$$I_1 \le \frac{Mp}{4n(q-1)} \int_M \chi_n^{p-2} |u|^q \, dm.$$

As for I_2,

$$\begin{aligned}
I_2 &= \int_M ub(\chi_n^p \varphi_\varepsilon(u)) \, dm \\
&= \int_M ub(\chi_n^{\frac{p}{q}} \chi_n^{p-\frac{p}{q}} \varphi_\varepsilon(u)) \, dm \\
&= \int_M \{ u\frac{p}{q} \chi_n^{\frac{p}{q}-1} b\chi_n \, \chi_n^{p-\frac{p}{q}} \varphi_\varepsilon(u) + u\chi_n^{\frac{p}{q}} b(\chi_n^{p-\frac{p}{q}} \varphi_\varepsilon(u)) \} \, dm
\end{aligned}$$

$$= \frac{p}{q}\int_M \chi_n^{p-1} b\chi_n u^2(u^2+\varepsilon)^{\frac{q}{2}-1}\,dm - \int_M b(u\chi_n^{\frac{p}{q}})\chi_n^{\frac{p}{q}(q-1)}\varphi_\varepsilon(u)\,dm$$
$$- \int_M \chi_n^p u\varphi_\varepsilon(u)\operatorname{div} b\,dm.$$

In the third line, we have used the Leibniz rule which requires the differentiability of $\chi_n^{p-\frac{p}{q}}$. But this is clear since $p - \frac{p}{q} = 1$. Now,

$$\frac{p}{q}\int_M \chi_n^{p-1} b\chi_n u^2(u^2+\varepsilon)^{\frac{q}{2}-1}\,dm \le \frac{p}{nq}\int_M \chi_n^{p-1}u^2(u^2+\varepsilon)^{\frac{q}{2}-1}\,dm$$
$$\le \frac{p}{nq}\int_M \chi_n^{p-1}|u|^q\,dm$$

and hence,

$$I_2 \le \frac{p}{nq}\int_M \chi_n^{p-1}|u|^q\,dm - \int_M b(u\chi_n^{\frac{p}{q}})\chi_n^{\frac{p}{q}(q-1)}\varphi_\varepsilon(u)\,dm$$
$$- \int_M \chi_n^p u\varphi_\varepsilon(u)\operatorname{div} b\,dm.$$

Combining both estimates of I_1 and I_2, we have

$$\alpha\int_M \chi_n^p u\,\varphi_\varepsilon(u)\,dm$$
$$\le \frac{Mp}{4n(q-1)}\int_M \chi_n^{p-1}|u|^q\,dm + \frac{p}{nq}\int_M \chi_n^{p-1}|u|^q\,dm$$
$$- \int_M b(u\chi_n^{\frac{p}{q}})\chi_n^{\frac{p}{q}(q-1)}\varphi_\varepsilon(u)\,dm - \int_M \chi_n^p u\,\varphi_\varepsilon(u)\operatorname{div} b\,dm.$$

Letting $\varepsilon \to 0$,

$$\alpha\int_M \chi_n^p|u|^q\,dm$$
$$\le \frac{Mp}{4n(q-1)}\int_M \chi_n^{p-2}|u|^q\,dm + \frac{p}{nq}\int_M \chi_n^{p-1}|u|^q\,dm$$
$$- \int_M b(u\chi_n^{\frac{p}{q}})\chi_n^{\frac{p}{q}(q-1)}\operatorname{sgn}(u)|u|^{q-1}\,dm - \int_M \chi_n^p|u|^q\operatorname{div} b\,dm.$$

Recalling

$$b(|u|^q\chi_n^p) = b(|u|\chi_n^{\frac{p}{q}})^q = q\operatorname{sgn}(u)|u|^{q-1}\chi_n^{\frac{p}{q}(q-1)}b(u\chi_n^{\frac{p}{q}}),$$

we have

$$- \int_M b(|u|\chi_n^{\frac{p}{q}})\chi_n^{\frac{p}{q}(q-1)}\operatorname{sgn}(u)|u|^{q-1}\,dm = -\frac{1}{q}\int_M b(|u|^q\chi_n^p)\,dm$$

$$= \frac{1}{q}\int_M (\operatorname{div} b)|u|^q \chi_n^p \, dm.$$

Thus, eventually we have

$$\alpha \int_M \chi_n^p |u|^q \, dm \le \frac{Mp}{4n(q-1)} \int_M \chi_n^{p-2}|u|^q \, dm + \frac{p}{nq}\int_M \chi_n^{p-1}|u|^q \, dm - (1-\frac{1}{q})\int_M (\operatorname{div} b)\chi_n^p |u|^q \, dm.$$

Taking α to be large enough so that $\alpha + (1-\frac{1}{q})\operatorname{div} b \ge 1$ and letting $n \to \infty$, we get

$$\int_M |u|^q \, dm \le 0,$$

which deduces $u = 0$ as desired. Q.E.D.

We can also show the Markovian property and the L^1 contraction property of the semigroup in L^p setting as follows.

Theorem 4.3. Assume **(A.1)** and **(A.2)**. Then the semigroups $\{T_t\}$ obtained in Theorem 4.1 and Theorem 4.2 satisfy the Markovian property. Moreover $\{e^{-2\gamma t}T_t\}$ satisfies the L^1 contraction property.

Proof. Let us prove the Markovian property, which follows if we show the following (see Jacob [5, Lemma 4.6.6] when $\gamma = 0$ and [12] for general γ):

$$\int_M \mathfrak{A}u\,(u-1)_+^{p-1} \, dm \le \frac{2\gamma}{p}\|(u-1)_+\|_p^p. \tag{4.5}$$

We show this inequality for $u \in C_0^\infty$. We treat $\triangle$ and b separately. Take any $\varepsilon > 0$ and define $\varphi_\varepsilon \in C^\infty(\mathbb{R})$ as follows. $\varphi_\varepsilon = \varepsilon$ for $t \le 1$ and $\varphi(t) = t - 1$ for $t \ge 1 + 2\varepsilon$. Then

$$\begin{aligned}\int_M \triangle u\,\varphi_\varepsilon(u)^{p-1} \, dm &= -\int_M \nabla u \cdot \nabla(\varphi_\varepsilon(u)^{p-1}) \, dm \\ &= -\int_M \nabla u \cdot ((p-1)\varphi_\varepsilon(u)^{p-2}\varphi_\varepsilon'(u)\nabla u) \, dm \\ &= -(p-1)\int_M \varphi_\varepsilon(u)^{p-2}\varphi_\varepsilon'(u)|\nabla u|^2 \, dm \le 0.\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\int_M \triangle u\,(u-1)_+^{p-1} \, dm \le 0.$$

As for b part, set $\Phi(t) = \frac{1}{p}(t-1)_+^p$. Then $\Phi'(t) = (t-1)_+^{p-1}$ and hence

$$\begin{aligned}\int_M bu\,\varphi_\varepsilon(u)^{p-1}\,dm &= \int_M bu\,\Phi'(u)\,dm \\ &= \int_M b\Phi(u)\,dm \\ &= -\int_M (\operatorname{div} b)\Phi(u)\,dm \\ &= -\frac{1}{p}\int_M \operatorname{div} b(u-1)_+^p\,dm \\ &\le \frac{2\gamma}{p}\int_M (u-1)_+^p\,dm \\ &= \frac{2\gamma}{p}\|(u-1)_+\|_p^p.\end{aligned}$$

Combining both of them, we can see that (4.5) holds for $u \in C_0^\infty$. The rest is easy if we notice that C_0^∞ is dense in $\mathrm{Dom}(\mathfrak{A})$.

Next we show the L^1 contraction property. It suffices to show the following (see [12]):

$$\int_M (\mathfrak{A} - 2\gamma)u\,(u_+ \wedge 1)^{p-1}\,dm \le 2\gamma(\frac{1}{p} - 1)\|u_+ \wedge 1\|_p^p. \tag{4.6}$$

To show this, for any $\varepsilon > 0$, define φ_ε by (2.13) in Section 2. Then

$$\begin{aligned}\int_M \triangle u\,\varphi_\varepsilon(u)^{p-1}\,dm &= -\int_M \nabla u \cdot \nabla(\varphi_\varepsilon(u)^{p-1})\,dm \\ &= -\int_M \nabla u \cdot ((p-1)\varphi_\varepsilon(u)^{p-2}\varphi_\varepsilon'(u)\nabla u)\,dm \\ &= -(p-1)\int_M \varphi_\varepsilon(u)^{p-2}\varphi_\varepsilon'(u)|\nabla u|^2\,dm \le 0.\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$\int_M \triangle u\,(u_+ \wedge 1)^{p-1}\,dm \le 0.$$

This shows the $\triangle$ part.

Let us treat b part. Define Φ by

$$\Phi(t) = \begin{cases} \displaystyle\int_0^t (v \wedge 1)^{p-1}\,dv, & t \ge 0 \\ 0, & t \le 0. \end{cases}$$

This means that $\Phi(t) = \frac{1}{p}t^p$ for $0 \le t \le 1$ and $\Phi(t) = t - 1 + \frac{1}{p}$ for $t \ge 1$. Then, $\Phi'(t) = (t_+ \wedge 1)^{p-1}$ and hence

$$\begin{aligned}\int_M \{bu(u_+ \wedge 1)^{p-1} - 2\gamma u(u_+ \wedge 1)^{p-1}\}\, dm \\ &= \int_M \{bu\Phi'(u) - 2\gamma u(u_+ \wedge 1)^{p-1}\}\, dm \\ &= \int_M \{b\Phi(u) - 2\gamma u(u_+ \wedge 1)^{p-1}\}\, dm \\ &= -\int_M \{(\operatorname{div} b)\Phi(u) + 2\gamma u(u_+ \wedge 1)^{p-1}\}\, dm.\end{aligned}$$

From the definition of Φ, we have

$$-(\operatorname{div} b)\Phi(u) \le 2\gamma\Phi(u) = 2\gamma u(u_+ \wedge 1)^{p-1} + 2\gamma(\frac{1}{p} - 1)(u_+ \wedge 1)^p$$

and therefore

$$\begin{aligned}\int_M \{bu(u_+ \wedge 1)^{p-1} - 2\gamma u(u_+ \wedge 1)^{p-1}\}\, dm &\le 2\gamma(\frac{1}{p} - 1)\int_M (u_+ \wedge 1)^p\, dm \\ &= 2\gamma(\frac{1}{p} - 1)\|u_+ \wedge 1\|_p^p.\end{aligned}$$

Combining both of them, (4.6) holds for $u \in C_0^\infty(M)$. Now, by the fact that $C_0^\infty(M)$ is dense in $\mathrm{Dom}(\mathfrak{A})$, (4.6) holds for $u \in \mathrm{Dom}(\mathfrak{A})$ and the proof is completed. Q.E.D.

To determine the domain of the generator in L^p setting is also an interesting problem. But it seems that we need a technique different from the L^2 case. It is left as a future problem.

§5. Examples of non-symmetric diffusions

We give some examples. Suppose that $M = \mathbb{R}^2$ eqipped with the Euclidean metric. We denote the coordinates in $\mathbb{R}^2$ by (x^1, x^2). Define a vector field $b = b^1\frac{\partial}{\partial x^1} + b^2\frac{\partial}{\partial x^2}$ as

$$\begin{aligned}b^1 &= c^1 + x^1 b_1^1 + x^2 b_2^1 \\ b^2 &= c^2 + x^1 b_1^2 + x^2 b_2^2\end{aligned}$$

and consider the operator $\mathfrak{A} = \frac{1}{2}\triangle + b$. Then ∇b may be represented with respect to the canonical coordinate as

$$\nabla b = \begin{pmatrix} b_1^1 & b_2^1 \\ b_1^2 & b_2^2 \end{pmatrix}$$

and therefore $\operatorname{div} b = b_1^1 + b_2^2$ and the symmetrization of ∇b is

$$B = \begin{pmatrix} b_1^1 & \frac{1}{2}(b_2^1 + b_1^2) \\ \frac{1}{2}(b_2^1 + b_1^2) & b_2^2 \end{pmatrix}.$$

We easily see that all conditions **(A.1)**, **(A.2)**, **(A.2)*** **(A.3)**, **(A.4)** in Section 2 and Section 3 are satisfied. Hence the operator $\mathfrak{A}$ generates a Markovian semigroup in $L^p(\mathbb{R}^2, dx^1\, dx^2)$. To be precise, the closure of it with the domain $C_0^\infty(\mathbb{R}^2)$ generates a semigroup. In L^2, we have moreover that the domain is the set of all u so that $u \in \operatorname{Dom}(\triangle)$ and $bu \in L^2$. The corresponding SDE is linear so that it can be solved explicitly, but, to his knowledge, the author chould not find the literature which gives the characterization of the generator domain.

We can also treat the perturbation of the Ornstein-Uhlenbeck operator. Here the Ornstein-Uhlenbeck operator L is defined by

$$L = \triangle - x^1 \frac{\partial}{\partial x^1} - x^2 \frac{\partial}{\partial x^2}.$$

In this case, we need to change the measure from the Lebesgue measure to the Gaussian measure $\mu = \frac{1}{2\pi} \exp\{-((x^1)^2 + (x^2)^2)/2\} dx^1 dx^2$. If we take a vector field b as above, we can show that $L + b$ generates a Markovian semigroup. This is not exactly within the framework of the previous sections, but we can show it with a minor change.

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