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On equivalence of L^p-norms related to Schrödinger type operators on Riemannian manifolds

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Abstract. In this paper, we study Schrödinger type operator on a Riemannian manifold. Under some assumptions on a potential function, we characterize the domain of the square root of the Schrödinger type operator on L^p space. In the proof, the defective intertwining properties and the Littlewood-Paley inequalities play important roles.

1. Introduction

Let $\triangle$ be the Laplacian on the Euclidean space $\mathbb{R}^d$. Then the following inequalities are well-known: for each $p \in (1, \infty)$, there exist positive constants $c = c_p$ and $C = C_p$ such that for all $f \in C_c^\infty(\mathbb{R}^d)$, it holds that

$$c(\|f\|_p + \|\nabla f\|_p) \leq \|\sqrt{1 - \triangle} f\|_p \leq C(\|f\|_p + \|\nabla f\|_p) \qquad (1.1)$$

where ∇ stands for the gradient. This inequality implies that $\sqrt{1-\triangle}^{-1}\nabla$ is a bounded operator in L^p. It's a variation of Riesz transformation. Inequalities of this kind were discussed by many authors (e.g., P. A. Meyer [8], D. Bakry [2], N. Yoshida [20]).

Meanwhile, we can deduce from these inequalities that the domain of the square root of the Laplacian in L^p is identified with the domain of the gradient, namely

$$\mathrm{Dom}(\sqrt{1 - \triangle})_p = \mathrm{Dom}(\nabla)_p. \qquad (1.2)$$

In this paper, we take this point of view. We mainly deal with a Schrödinger type operator $\triangle - V$ on a Riemannian manifold, where $\triangle$ is the Laplace-Beltrami operator and V is a real valued function called a potential function, and we consider an analogous problem for differential forms. So the aim of this paper is to show that for each $p \in (1, \infty)$, there exist positive constants $c = c_p$, $C = C_p$ and α_p such that for all $f \in C_c^\infty(M)$ and $\alpha \geq \alpha_p$, it holds

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that

$$c(\|f\|_p + \|\nabla f\|_p + \|\sqrt{V} f\|_p) \\ \leq \|\sqrt{\alpha + V - \triangle} f\|_p \\ \leq C(\|f\|_p + \|\nabla f\|_p + \|\sqrt{V} f\|_p) \tag{1.3}$$

where ∇ is the covariant differentiation. This inequality was studied by Z. Shen [12,13] As before, we can show that for sufficiently large α,

$$\mathrm{Dom}(\sqrt{\alpha + V - \triangle})_p = \mathrm{Dom}(\nabla)_p \cap \mathrm{Dom}(\sqrt{V})_p. \tag{1.4}$$

Here all operators are considered in L^p space.

The organization of this paper is as follows. In the section 2, we formulate our problem and give precise results. We also introduce some assumptions for the underlying manifold and a potential function. In the section 3, we introduce semigroups which are needed in proofs of the main theorems. Propositions which play key roles in proofs are the intertwining property and the Littlewood-Paley inequality. In the section 4, we consider the intertwining property, and in the section 5, we consider the Littlewood-Paley inequality. The proofs of the main results are shown in the sections 6 and 7.

2. Formulation and Results

Let (M, g) be an n-dimensional stochastically complete Riemannian manifold, dm be the volume element of M and ∇ be the Levi-Civita connection. We denote the space of smooth functions with compact support by $C_c^\infty = C_c^\infty(M)$ and let $L^p = L^p(M, dm)$ be the set of all p-th integrable functions with the norm $\|f\|_p := (\int_M |f|^p \, dm)^{1/p}$. We also need to consider differential forms. We denoted the k-th exterior product bundle of $T^*(M)$ by Λ_k, and we define $A_k = A_k(M)$ to be the set of all smooth k-forms with compact support, i.e., smooth sections of Λ_k with compact support. We also define $L^p(\Lambda_k) := L^p(M, dm; \Lambda_k)$ to be the set of all p-th integrable k-forms with the norm $\|\omega\|_p := (\int_M |\omega|^p \, dm)^{1/p}$.

As we stated in the section 1, the operator we are mainly concerned in this paper is $\triangle - V$, $\triangle$ being the Hodge-Kodaira Laplacian and V being a real valued function.

Now we introduce assumptions for the manifold M and the potential function V. First we give an assumption for the manifold M. We denote by R the Riemannian curvature $R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]}$. We take an orthonormal basis $\{e_j\}_{j=1}^n$ of $T_x(M)$ for each point x and denote its dual basis by $\{e^k\}_{k=1}^n$. Now our assumption is given as follows:

(A-M) There exists a constant C_M such that for all $\omega \in A_k$ and all $x \in M$, it holds that

$$\left(\sum_{i,\ j=1}^{n} \operatorname{ext}(e^i) \operatorname{int}(e^j) R(e_i, e_j)\omega(x), \omega(x) \right) \geq -C_M |\omega(x)|^2. \tag{2.1}$$

Here ext stands for the exterior product: $\operatorname{ext}(\theta) = \theta \wedge \cdot$, and int stands for the interior product: $(\operatorname{int}(\theta)\omega, \eta) = (\omega, \theta \wedge \eta)$. Clearly, $\operatorname{int}(\theta)$ is the dual operator of $\operatorname{ext}(\theta)$. The left hand side of the inequality above is the 0-th order term that appears in Weitzenböck formula (see e.g., [4, Theorem 3.3.3]).

An assumption for the potential function V is as follows:

(A-V) V is uniformly positive and there exists constants $C_1, C_2 \geq 0$ such that

$$\frac{|\nabla V|}{V} \leq C_1, \quad \frac{|\triangle V|}{V} \leq C_2. \tag{2.2}$$

Under the assumptions (2.1) and (2.2), it is known that $(\triangle - V,\ A_k(M))$ is a self-adjoint operator on $L^2(\Lambda_k)$ (see e.g. Yoshida [20]). We also denote the closure of $(\triangle - V,\ A_k(M))$ by $\triangle - V$. We can define $\sqrt{-\triangle + V}$ by subordination.

The main results of this paper are the following theorems.

Theorem 2.1 *Under Assumptions (A-M) and (A-V), for each $k \in \mathbb{Z}_+$, each $p \in (1, \infty)$, there exist $C = C_{k,p} > 0$, $\alpha = \alpha_{k,p} > 0$ such that it holds that for any $\omega \in A_k$*

$$\begin{aligned} C^{-1}(\|\omega\|_p + &\|d\omega\|_p + \|\delta\omega\|_p + \|\sqrt{V}\omega\|_p) \\ &\leq \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p \\ &\qquad \leq C(\|\omega\|_p + \|d\omega\|_p + \|\delta\omega\|_p + \|\sqrt{V}\omega\|_p). \end{aligned} \tag{2.3}$$

Here d is the exterior differentiation and δ is its dual operator.

Theorem 2.2 *Under Assumptions (A-M) and (A-V), for each $k \in \mathbb{Z}_+$, each $p \in (1, \infty)$, there exist $C = C_{k,p} > 0$, $\alpha = \alpha_{k,p} > 0$ such that it holds that for any $\omega \in A_k$*

$$\begin{aligned} C^{-1}(\|\omega\|_p + &\|d\delta\omega\|_p + \|\delta d\omega\|_p + \|V\omega\|_p) \\ &\leq \|(\triangle - V - \alpha)\omega\|_p \\ &\qquad \leq C(\|\omega\|_p + \|d\delta\omega\|_p + \|\delta d\omega\|_p + \|V\omega\|_p). \end{aligned} \tag{2.4}$$

These inequalities are studied by Z. Shen [12,13] under other assumptions.

Before closing this section, we give examples of potential functions satisfying the assumption (A-V) in case $M = \mathbb{R}$. Let χ be a smooth function such that χ' and χ'' are bounded. (For example, $\chi \in C^\infty$ such that $\chi \geq 0$

and $\chi(x) = |x|$ ($|x| \geq 1$) satisfies these conditions.) Then we can easily see that

$$V(x) = \exp(\chi(x))$$

satisfies (A-V).

3. Preparations

To prove Theorems 2.1 and 2.2, we need to introduce several semigroups. on L^p ($1 < p < \infty$). To do this, we introduce them on L^2, then extend them to L^p.

First, we consider the semigroups associated to $\triangle$ and $\triangle - V$. As we stated in the section 3, $(\triangle,\ A_k(M))$ and $(\triangle - V,\ A_k(M))$ is a essestially self-adjoint operator on L^2. Therefore we can extend them to the self-adjoint operator. We shall denote the closures of them by the same notations.

It is well-known that there exist strongly continuous, contractive semigroups $\{\overrightarrow{T}^{\alpha}_t\}_{t\geq 0}$ and $\{\overrightarrow{T}^{V+\alpha}_t\}_{t\geq 0}$ on L^2 whose generators are $\triangle - \alpha$ and $\triangle - V - \alpha$, respectively.

Let us define the bilinear forms $(\overrightarrow{\mathcal{E}}^{\alpha}, \overrightarrow{\mathcal{F}})$ by

$$\begin{cases} \overrightarrow{\mathcal{E}}^{\alpha} & = \lim_{t\to 0} t^{-1}(\omega - T_t\omega, \omega)_{L^2}, \\ \overrightarrow{\mathcal{F}} & = \{\omega \in L^2 |\ \text{the above limit exists.}\}, \end{cases}$$

$(\overrightarrow{\mathcal{E}}^{V+\alpha}, \overrightarrow{\mathcal{F}}_V)$ is defined in the similar way.

$\overrightarrow{\mathcal{E}}^{\alpha}$ and $\overrightarrow{\mathcal{E}}^{V+\alpha}$ can be written as follows: for any $\omega,\ \eta \in A_k$

$$\overrightarrow{\mathcal{E}}^{\alpha}(\omega,\eta) = \int_M (d\omega, d\eta)\, dm + \int_M (\delta\omega, \delta\eta)\, dm + \alpha \int_M (\omega,\eta)\, dm, \quad (3.1)$$

$$\overrightarrow{\mathcal{E}}^{V+\alpha}(\omega,\eta) = \int_M (d\omega, d\eta)\, dm + \int_M (\delta\omega, \delta\eta)\, dm + \int_M (\omega,\eta)(V+\alpha)\, dm. \quad (3.2)$$

Remark 3.1 *We denote $(\triangle - V - \alpha)_k$, $\overrightarrow{T}^{V+\alpha,k}_t$, $(\overrightarrow{\mathcal{E}}^{V+\alpha,k}, \overrightarrow{\mathcal{F}}_{V,k})$ to specify the base space Λ_k if necessary. Since operators acting on scalar functions take a special place, we delete $\rightarrow$ to distinguish them, e.g., we denote $(\mathcal{E}^{V+\alpha}, \mathcal{F}_V)$ instead of $(\overrightarrow{\mathcal{E}}^{V+\alpha}, \overrightarrow{\mathcal{F}}_V)$ when it is defined on the space of scalar functions $A_0 = C_c^{\infty}(M)$. In the sequel we will use this convention without mention.*

Since $\{\overrightarrow{T}^{\alpha}_t\}$ and $\{\overrightarrow{T}^{V+\alpha}_t\}$ are symmetric with respect to m, they can be extended to the semigroups on L^p. (c.f. [14]) We shall also denote them by the same notations.

Next, we shall consider semigroups whose formal generators are written as

$$\overrightarrow{A - \alpha} = \triangle + \nabla_b - \left(V + \frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2 + \alpha\right),$$

$$\overrightarrow{\hat{A} - \alpha} := \triangle - \nabla_b - \left(V - \frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2 + \alpha\right)$$

and

$$\overrightarrow{\tilde{A}-\alpha} := \triangle + 2\nabla_b - \left(V + \frac{3}{2}\nabla^* b - \frac{5}{4}|b|^2 + \alpha\right)$$

where $b = -\frac{\nabla V}{V}$ and ∇^* is the dual operator of ∇. (For the rigorous definition of the generators above, see (3.3). See also (3.4).)

Here we note that it holds that $\nabla^* b = \frac{\triangle V}{V} + |b|^2$ and hence it follows from Assumption (A-V) that b and $\nabla^* b$ are bounded.

The reason why we shall introduce these semigroups is as follows. $\overrightarrow{A-\alpha}$ and $\overrightarrow{\tilde{A}-\alpha}$ satisfy the following relations: for $\omega \in A_k$

$$\sqrt{V}(\triangle - V - \alpha)\omega = (\overrightarrow{A-\alpha})(\sqrt{V}\omega),$$
$$\sqrt{V}(\overrightarrow{A-\alpha})\omega = \overrightarrow{\tilde{A}-\alpha}(\sqrt{V}\omega).$$

These relations formally imply that for $\omega \in \overrightarrow{\mathcal{F}}_V$

$$\sqrt{V}\overrightarrow{T}_t^{V+\alpha}\omega = \overrightarrow{T}_t^{A-\alpha}(\sqrt{V}\omega),$$
$$\sqrt{V}\overrightarrow{T}_t^{A-\alpha}\omega = \overrightarrow{T}_t^{\tilde{A}-\alpha}(\sqrt{V}\omega).$$

where $\overrightarrow{T}_t^{A-\alpha}$ and $\overrightarrow{T}_t^{\tilde{A}-\alpha}$ are semigroups corresponding to generators $\overrightarrow{A-\alpha}$ and $\overrightarrow{\tilde{A}-\alpha}$, respectively. These relations will play important roles in the proof of Theorem 2.1 and 2.2, especially, in the estimate of $\|\sqrt{V}\omega\|_p$. These relations will be studied in the section 4.

From now on, we shall define $\overrightarrow{A-\alpha}$, $\overrightarrow{\hat{A}-\alpha}$, $\overrightarrow{\tilde{A}-\alpha}$ and $\overrightarrow{T}_t^{A+\alpha}$, $\overrightarrow{T}_t^{\hat{A}+\alpha}$, $\overrightarrow{T}_t^{\tilde{A}+\alpha}$ rigorously. But we will only discuss the semigroup generated by $\overrightarrow{A-\alpha}$ because other semigroups can be discussed similarly.

Here we note that, in these cases, we will introduce the semigroups on L^2 first, and will extend them to L^p.

Let us define the following bilinear form on L^2;

$$\overrightarrow{\mathcal{E}}^{A-\alpha}(\omega,\eta) = \int_M \Big\{ (d\omega, d\eta) + (\delta\omega, \delta\eta) - (\nabla_b\omega, \eta) + (\omega,\eta)\Big(V + \frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2 + \alpha\Big)\Big\} dm, \quad \text{for } \omega,\eta \in A_k. \tag{3.3}$$

Owing to Assumptions (A-M) and (A-V), we can easily see that for sufficiently large $\alpha > 0$, $(\overrightarrow{\mathcal{E}}^{A-\alpha}, A_k)$ is a closable coercive bilinear form on $L^2(\Lambda_k)$ and we denote its closure by $(\overrightarrow{\mathcal{E}}^{A-\alpha}, \overrightarrow{\mathcal{F}}_V)$. From the general theory (c.f. Ma-Röckner [7]), there exists the generator $\overrightarrow{A-\alpha}$ of the strongly continuous contraction semigroup on $L^2(\Lambda_k)$ corresponding to the bilinear form $\overrightarrow{\mathcal{E}}^{A-\alpha}$. $\overrightarrow{\mathcal{E}}^{A-\alpha}$ and $\overrightarrow{A-\alpha}$ satisfies the following relation; such that

$$\overrightarrow{\mathcal{E}}^{A-\alpha}(\omega,\eta) = -((\overrightarrow{A-\alpha})\omega, \eta)_2, \qquad \omega,\eta \in A_k. \tag{3.4}$$

Let us denote by $\{\overrightarrow{T}_t^{A-\alpha}\}_{t\geq 0}$ the semigroup generated by $\overrightarrow{A-\alpha}$.

Remark 3.2 $\overrightarrow{\hat{A}-\alpha}$ *satisfies the following relation;*

$$\overrightarrow{\mathcal{E}}^{A-\alpha}(\omega,\eta) = (\omega, -(\overrightarrow{\hat{A}-\alpha})\eta)_2, \tag{3.5}$$

i.e., $\overrightarrow{\hat{A}-\alpha}$ *is the dual operator of* $\overrightarrow{A-\alpha}$.

We prepare the following lemmas.

Lemma 3.3 *For* $\omega \in A_k$ *and* $f \in C^\infty$,

$$\delta(f\omega) = f\delta\omega - \operatorname{int}(df)\omega \tag{3.6}$$

holds.

Proof. For $\omega \in A_k, \eta \in A_{k-1}$ and $f \in C^\infty$, we have

$$\begin{aligned}\int_M (\delta(f\omega),\eta)\,dm &= \int_M (f\omega, d\eta)\,dm \\ &= \int_M (\omega, f d\eta)\,dm \\ &= \int_M (\omega, d(f\eta) - df\wedge\eta)\,dm \\ &= \int_M (f\delta\omega - \operatorname{int}(df)\omega, \eta)\,dm,\end{aligned}$$

and this completes the proof.

Lemma 3.4 *For* $\omega \in A_k$,

$$(d\delta(\sqrt{V}\omega), \sqrt{V}\omega)_2 = (d\delta\omega, V\omega)_2 + (\operatorname{int}(V^{-1}dV)\omega, \operatorname{int}(V^{-1}dV)(V\omega))_2 \tag{3.7}$$

and

$$(\delta d(\sqrt{V}\omega), \sqrt{V}\omega)_2 = (\delta d\omega, V\omega)_2 + (V^{-1}dV\wedge\omega, V^{-1}dV\wedge(V\omega))_2 \tag{3.8}$$

holds.

Proof. For $\omega \in A_k$, we have

$$\begin{aligned}(d\delta(\sqrt{V}\omega), \sqrt{V}\omega)_2 &= (\delta(\sqrt{V}\omega), \delta(\sqrt{V}\omega))_2 \\ &= (\sqrt{V}\delta\omega - \operatorname{int}(d\sqrt{V})\omega, \sqrt{V}\delta\omega - \operatorname{int}(d\sqrt{V})\omega)_2 \\ &= (\sqrt{V}\delta\omega, \sqrt{V}\delta\omega)_2 - 2(\operatorname{int}(d\sqrt{V})\omega, \sqrt{V}\delta\omega)_2 + (\operatorname{int}(d\sqrt{V})\omega, \operatorname{int}(d\sqrt{V})\omega)_2,\end{aligned}$$

where we use (3.6) in the second line. The first term is calculated as follows;

$$\begin{aligned}(\sqrt{V}\delta\omega, \sqrt{V}\delta\omega)_2 &= (d(V\delta\omega), \omega)_2 \\ &= (Vd\delta\omega + dV\wedge(\delta\omega), \omega)_2 \\ &= (d\delta\omega, V\omega)_2 + (\delta\omega, \operatorname{int}(dV)\omega)_2.\end{aligned}$$

The second term is equal to

$$-2(\mathrm{int}(d\sqrt{V})\omega, \sqrt{V}\delta\omega)_2 = -(\mathrm{int}(dV)\omega, \delta\omega)_2,$$

and the third term is

$$(\mathrm{int}(d\sqrt{V})\omega, \mathrm{int}(d\sqrt{V})\omega)_2 = (\mathrm{int}(V^{-1}dV)\omega, \mathrm{int}(V^{-1}dV)V\omega)_2.$$

Thus we have

$$(d\delta(\sqrt{V}\omega), \sqrt{V}\omega)_2 = (d\delta\omega, V\omega)_2 + (\mathrm{int}(V^{-1}dV)\omega, \mathrm{int}(V^{-1}dV)V\omega)_2,$$

and this completes the proof of (3.7). (3.8) is calculated in the similar way.

Lemma 3.5 *Under Assumptions (A-V), there exist* $C > 0$ *such that, for* $\omega \in A_k$*,*

$$\|\triangle\omega\|_2 \le C(\|(\triangle - V)\omega\|_2 + \|\omega\|_2). \tag{3.9}$$

holds.

Proof. It is obvious that the following inequality holds;

$$\begin{aligned}\|\sqrt{V}\omega\|_2^2 &\le \|d\omega\|_2^2 + \|\delta\omega\|_2^2 + \|\sqrt{V}\omega\|_2^2 \\ &= ((-\triangle + V)\omega, \omega)_2.\end{aligned}$$

Using inequality above, we have

$$\begin{aligned}\|V\omega\|_2^2 &\le ((-\triangle + V)(\sqrt{V}\omega), \sqrt{V}\omega)_2 \\ &= (d\delta(\sqrt{V}\omega), \sqrt{V}\omega)_2 + (\delta d(\sqrt{V}\omega), \sqrt{V}\omega)_2 + (V\sqrt{V}\omega, \sqrt{V}\omega,)_2 \\ &= (d\delta\omega, V\omega)_2 + (\mathrm{int}(V^{-1}dV)\omega, \mathrm{int}(V^{-1}dV)(V\omega))_2 \\ &\quad + (\delta d\omega, V\omega)_2 + (V^{-1}dV \wedge \omega, V^{-1}dV \wedge (V\omega))_2 \\ &\qquad + (V\omega, V\omega,)_2 \\ &= ((-\triangle + V)\omega, V\omega)_2 \\ &\quad + (\mathrm{int}(V^{-1}dV)\omega, \mathrm{int}(V^{-1}dV)(V\omega))_2 \\ &\quad + (V^{-1}dV \wedge \omega, V^{-1}dV \wedge (V\omega))_2 \\ &\le \|(\triangle - V)\omega\|_2\|V\omega\|_2 + 2C_1^2\|\omega\|_2\|V\omega\|_2 \\ &\le (1/2)(\|(\triangle - V)\omega\|_2^2 + 2C_1^2\|\omega\|_2^2) + (1/2)\|V\omega\|_2^2,\end{aligned}$$

where we use Assumption (A-V) in the fifth line and C_1 is the constant appearing Assumption (A-V). Thus there exists a constant such that

$$\|V\omega\|_2 \le C(\|(\triangle - V)\omega\|_2 + \|\omega\|_2)$$

holds. Therefore

$$\begin{aligned}\|\triangle\omega\|_2 &= \|(\triangle - V)\omega + V\omega\|_2 \\ &\le \|(\triangle - V)\omega\|_2 + \|V\omega\|_2 \\ &\le (C + 1)\|(\triangle - V)\omega\|_2\end{aligned}$$

holds. This completes the proof.

We will use the following lemma in the section 5.

Lemma 3.6 *A_k is a core of* $\mathrm{Dom}(\overrightarrow{A-\alpha})$ *in* L^2.

Proof. Due to [5, Teorem 1.1. of Chapter IV], it is sufficient to prove that there exist a constant $a > 0$ and $b \in (0, 1)$ such that

$$\|(A-\alpha)\omega\|_2 \leq a\|\omega\|_2 + b\|(\triangle - V - \alpha)\omega\|_2$$

holds.

Since b is bounded, we have

$$\begin{aligned}\|\nabla_b\omega\|_2 &\leq C_1\|\nabla\omega\|_2\\ &= C_1(-\nabla^*\nabla\omega, \omega)_2\\ &\leq \varepsilon\|\nabla^*\nabla\omega\|_2^2 + (4\varepsilon)^{-1}C_1^2\|\omega\|_2^2\\ &\leq \varepsilon\|\triangle\omega\|_2^2 + ((4\varepsilon)^{-1}C_1^2 + C_M\varepsilon)\|\omega\|_2^2\\ &\leq C\varepsilon\|(\triangle - V)\omega\|_2^2 + ((4\varepsilon)^{-1}C_1^2 + C_M\varepsilon)\|\omega\|_2^2,\end{aligned}$$

where we use Assumption (A-V) in the first line, Assumption (A-M) in the fourth line, (3.9) in the last line. The remaining term $\frac{1}{2}\nabla^*b - \frac{1}{4}|b|^2 + \alpha$ is bounded and hence we can get the desired result.

Now, we shall extend $\{\overrightarrow{T}_t^{A-\alpha}\}$ to a C_0-semigroup on $L^p(\Lambda_k)$.

First, we consider the case $k = 0$. In this case, we use the probabilistic representation of $\{T_t^{A-\alpha}\}$. We recall that the generator of $\{T_t^{A-\alpha}\}$ is written as follows:

$$(A-\alpha)u = \triangle u + (b, \nabla u) - \left(V + \frac{1}{2}\nabla^*b - \frac{1}{4}|b|^2 - \alpha\right)u, \text{ for } u \in C_0^\infty(M).$$

When M is the Euclidean space, combining the Girsanov transformation and the Feynman-Kac formula, we obtain a probabilistic representation of $\{T_t^{A-\alpha}\}$ as follows:

$$\begin{aligned}T_t^{A-\alpha}u(x) = \mathbb{E}_x\Big[u(B_t)\exp\Big\{\frac{1}{2}\int_0^t b(B_s)\cdot dB_s - \frac{1}{4}\int_0^t |b(B_s)|^2 ds\\ - \int_0^t (V(B_s) + \frac{1}{2}\nabla^*b(B_s) - \frac{1}{4}|b(B_s)|^2 - \alpha)ds\Big\}\Big]\end{aligned}$$

where $\{B_t = (B_t^1, \cdots, B_t^n)\}_{t\geq 0}$ is a Brownian motion on $\mathbb{R}^n$ with its quadratic variation $\langle B^j\rangle_t = 2t$ $(j = 1, \cdots, n)$. In our case, there is a similar representation of $\{T_t^{A-\alpha}\}$. Since the bilinear form $(\mathcal{E}^0, \mathcal{F})$ on L^2 is a local Dirichlet form, there exists an associated diffusion process $(\{\mathbb{P}_x\}_{x\in M}, \{X_t\}_{t\geq 0})$ on M. Since V is positive and smooth, $\log V \in \mathcal{F}_{loc}$. Therefore, by the Fukushima decomposition, (c.f. [3, Theorem 5.5.1.]) there exist a continuous martingale additive functional $\{M_t^{[\log V]}\}_{t\geq 0}$ and a continuous additive functional of zero energy $\{N_t^{[\log V]}\}_{t\geq 0}$ such that

$$\log V(X_t) - \log V(X_0) = M_t^{[\log V]} + N_t^{[\log V]}.$$

We note that

$$\langle M^{[\log V]}\rangle_t = 2\int_0^t |\nabla \log V(X_s)|^2\, ds = 2\int_0^t |b(X_s)|^2\, ds.$$

Set

$$M_t = \exp\Big\{-\frac{1}{2}M_t^{[\log V]} - \frac{1}{8}\langle M^{[\log V]}\rangle_t\Big\},$$
$$N_t = \exp\Big\{-\int_0^t \Big(V(X_s) + \frac{1}{2}\nabla^* b(X_s) - \frac{1}{4}|b(X_s)|^2 - \alpha\Big)\, ds\Big\},$$

and

$$Z_t = M_t N_t.$$

Then we have:

Proposition 3.7 *For $u \in L^2$, it holds that*

$$T_t^{A-\alpha}u(x) = \mathbb{E}_x[u(X_t)Z_t]. \tag{3.10}$$

This proposition can be proved in the same way as Theorem 3.2 of [6], so we omit the proof.

To prove that $\{T_t^{A-\alpha}\}$ can be extended to C_0-semigroup on L^p, we need the following lemma:

Lemma 3.8 *For $1 < p < \infty$ and sufficiently large $\alpha > 0$,*

$$|T_t^{A-\alpha}u(x)|^p \le T_t^0(|u|^p)(x). \tag{3.11}$$

Proof. We first show that the inequality

$$\mathbb{E}_x[{M_t}^q] \le \exp(\frac{q^2-q}{8}C_1^2 t) \tag{3.12}$$

holds for $1 < q < \infty$. To see this, we note

$$\begin{aligned}
\mathbb{E}_x[M_t^q] &= \mathbb{E}_x\left[\exp(-\frac{q}{2}M_t^{[\log V]} - \frac{q}{8}\langle M^{[\log V]}\rangle_t)\right] \\
&= \mathbb{E}_x\left[\exp(-\frac{q}{2}M_t^{[\log V]} - \frac{q^2}{8}\langle M^{[\log V]}\rangle_t)\exp(\frac{q^2-q}{8}\langle M^{[\log V]}\rangle_t)\right] \\
&\le \mathbb{E}_x\left[\exp(-\frac{q}{2}M_t^{[\log V]} - \frac{q^2}{8}\langle M^{[\log V]}\rangle_t)\right]\exp(\frac{q^2-q}{8}C_1^2 t) \\
&\le \exp(\frac{q^2-q}{8}C_1^2 t).
\end{aligned}$$

In the third line, we used Assumption (A-V) and in the last line, we used the fact that $\exp(-\frac{q}{2}M_t^{[\log V]} - \frac{q^2}{8}\langle M^{[\log V]}\rangle_t)$ is a martingale. Therefore (3.12) follows.

Let q be the conjugate exponent of p: $1/p + 1/q = 1$. Then it follows that

$$\begin{aligned}|T_t^{A-\alpha}u(x)|^p &\le \mathbb{E}_x[|u(X_t)|M_tN_t]^p \\ &\le \mathbb{E}_x[|u(X_t)|M_t]^p \exp(p(C_1^2 + C_2 - \alpha)t) \\ &\le \mathbb{E}_x[|u(X_t)|^p]\mathbb{E}_x[{M_t}^q]^{p/q} \exp(p(C_1^2 + C_2 - \alpha)t) \\ &\le \mathbb{E}_x[|u(X_t)|^p] \exp(\frac{q}{8}C_1^2 t) \exp(p(C_1^2 + C_2 - \alpha)t).\end{aligned}$$

In the second line we used Assumption (A-V), in the third line we used Hölder's inequality and in the last line we used (3.12). Therefore, for sufficiently large $\alpha > 0$, (3.11) holds.

From the contractivity of $\{T_t^0\}$, the following lemma is clear.

Lemma 3.9 *It holds that*

$$\|T_t^{A-\alpha}u\|_p \le \|u\|_p. \tag{3.13}$$

Combining the lemma above with the argument in [14], we can show that $\{T_t^{A-\alpha}\}$ can be extended to the semigroup on L^p.

We now turn to the case where k is general.

Lemma 3.10 *The following semigroup domination holds:*

$$|\overrightarrow{T}_t^{A-(\alpha+C_M)}\omega| \le T_t^{A-\alpha}|\omega|. \tag{3.14}$$

Proof. By Ouhabaz' criterion (see [9,10]), we have to check the following:
(1) If $\omega \in \mathrm{Dom}(\overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)})$, then $|\omega| \in \mathrm{Dom}(\mathcal{E}^{A-\alpha})$ and

$$\mathcal{E}^{A-\alpha}(|\omega|, |\omega|) \le \overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)}(\omega, \omega). \tag{3.15}$$

(2) If $\omega \in \mathrm{Dom}(\overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)})$ and $f \in \mathrm{Dom}(\mathcal{E}^{A-\alpha})$ with $0 \le f \le |\omega|$, then $f \operatorname{sgn} \omega \in \mathrm{Dom}(\overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)})$ and

$$\mathcal{E}^{A-\alpha}(f, |\omega|) \le \overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)}(\omega, f \operatorname{sgn} \omega), \tag{3.16}$$

where $\operatorname{sgn} \omega = \omega/|\omega|$ if $\omega \ne 0$ and $= 0$ if $\omega = 0$ In fact, these are equivalent to (3.14).

To show (1), we recall that $|e^{-t\nabla^*\nabla}\omega| \le e^{t\triangle}|\omega|$. Hence, by Ouhabaz' criterion, we have that if $\omega \in \overrightarrow{\mathcal{F}}_V$, then $|\omega| \in \mathcal{F}_V$ and

$$\mathcal{E}^V(|\omega|, |\omega|) \le \overrightarrow{\mathcal{E}}^{V+C_M}(\omega, \omega) \tag{3.17}$$

and if, in addition, $0 \le f \le |\omega|$,

$$\int_M (\nabla|\omega|, \nabla f) dm \le \int_M (\nabla\omega, \nabla(f \operatorname{sgn} \omega)) dm.$$

Since $\mathrm{Dom}(\overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)}) = \overrightarrow{\mathcal{F}}_V$, we have that if $\omega \in \mathrm{Dom}(\overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)})$, then $|\omega| \in \mathrm{Dom}(\mathcal{E}^{A-\alpha})$ and

$$\begin{aligned}
&\mathcal{E}^{A-\alpha}(|\omega|, |\omega|) \\
&\quad = \mathcal{E}^{V}(|\omega|, |\omega|) \\
&\qquad - \int_M (b,\ \nabla|\omega|)|\omega|\,dm + \int_M \left(\frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2 + \alpha\right)|\omega|^2\,dm \\
&\quad \le \overrightarrow{\mathcal{E}}^{V+C_M}(\omega, \omega) \\
&\qquad - \int_M (b, \nabla|\omega|)|\omega|\,dm + \int_M \left(\frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2 + \alpha\right)|\omega|^2\,dm \\
&\quad = \overrightarrow{\mathcal{E}}^{V+C_M}(\omega, \omega) \\
&\qquad - \frac{1}{2}\int_M (b, \nabla(|\omega|^2))\,dm + \int_M \left(\frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2 + \alpha\right)|\omega|^2\,dm \\
&\quad = \overrightarrow{\mathcal{E}}^{V+C_M}(\omega, \omega) \\
&\qquad - \int_M (\nabla_b\omega, \omega)\,dm + \int_M \left(\frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2 + \alpha\right)|\omega|^2\,dm \\
&\quad = \overrightarrow{\mathcal{E}}^{A-(\alpha+C_M)}(\omega, \omega).
\end{aligned}$$

Here we used (3.17) in the second line, derivation property in the third line and $(b, \nabla(|\omega|^2)) = \nabla_b(|\omega|^2) = 2(\nabla_b\omega, \omega)$ in the fourth line. Thus we have shown (3.15).

As for (2), it is enough to show that $(\nabla_b|\omega|)f = (\nabla_b\omega, f\,\mathrm{sgn}\,\omega)$. To show this, we approximate $|\omega|$ as follows; for any $\varepsilon > 0$, we consider $\sqrt{|\omega|^2+\varepsilon}$. Since $\nabla_b\sqrt{|\omega|^2+\varepsilon} = (\nabla_b\omega, \omega)/\sqrt{|\omega|^2+\varepsilon}$, we have

$$(\nabla_b|\omega|)f = \lim_{\varepsilon\downarrow 0}(\nabla_b\sqrt{|\omega|^2+\varepsilon})f = \lim_{\varepsilon\downarrow 0}\frac{(\nabla_b\omega, \omega)f}{\sqrt{|\omega|^2+\varepsilon}} = (\nabla_b\omega, f\omega/|\omega|).$$

Thus the condition (2) is satisfied.

Now we are in a position to discuss thet the extention of $\{T_t^{A-\alpha}\}$ and $\{T_t^{\hat{A}-\alpha}\}$ to (C_0)-semigroups on $L^p(\Lambda_k)$.

Combining (3.13) and (3.14), we can show that $\{\overrightarrow{T}_t^{A-\alpha}\}$ is contractive in $L^p(\Lambda_k)$ for $1 < p < \infty$. We can also show in the similar manner that $\{\overrightarrow{T}_t^{\hat{A}-\alpha}\}$ is contractive in $L^p(\Lambda_k)$ for $1 < p < \infty$. Therefore we can extend $\{\overrightarrow{T}_t^{A-\alpha}\}$ and $\{\overrightarrow{T}_t^{\hat{A}-\alpha}\}$ to on $L^p(\Lambda_k)$.

Let us turn to the proof of the strong continuity of $\{\overrightarrow{T}_t^{A-\alpha}\}$ and $\{\overrightarrow{T}_t^{\hat{A}-\alpha}\}$ on $L^p(\Lambda_k)$.

First, we discuss the case $2 \ge p < \infty$. Let $r > p$ and $\alpha > 0$ be the number satisfying $p = 2\alpha + (1-\alpha)r$. Then, due to Hölder's inequality, the following inequality holds for $\omega \in L^p(\Lambda_k)$;

$$\|\overrightarrow{T}_t^{A-\alpha}\omega - \omega\|_p \le \|\overrightarrow{T}_t^{A-\alpha}\omega - \omega\|_2^{2\alpha}\|\overrightarrow{T}_t^{A-\alpha}\omega - \omega\|_r^{(1-\alpha)r}.$$

From the inequality above, the strong continuity of $\{\overrightarrow{T}_t^{A-\alpha}\}$ in $L^2(\Lambda_k)$, and the contractivity of $\{\overrightarrow{T}_t^{A-\alpha}\}$ in $L^r(\Lambda_k)$, we can derive the strong continuity $\{\overrightarrow{T}_t^{A-\alpha}\}$ in $L^p(\Lambda_k)$. The strong continuity of $\{\overrightarrow{T}_t^{\hat{A}-\alpha}\}$ in $L^p(\Lambda_k)$ is derived in the similar manner above.

Next, we turn to the case $1 < p < 2$. To prove the strong continuity of $\{\overrightarrow{T}_t^{A-\alpha}\}$ in $L^p(\Lambda_k)$, due to Theorem in page 233 of [19], We only show that, for any $\omega \in L^p(\Lambda_k)$ and $\eta \in L^q(\Lambda_k)$ (where q is the conjugate exponent of p),

$$\lim_{t\to 0}(\overrightarrow{T}^{A-\alpha}\omega - \omega, \eta) = 0.$$

But this is clear from the strong continuity of $\{\overrightarrow{T}_t^{\hat{A}-\alpha}\}$ in $L^q(\Lambda_k)$. Thus we obtain the strong continuity of $\{\overrightarrow{T}_t^{A-\alpha}\}$ in $L^p(\Lambda_k)$.

Before ending this section, we will introduce the Cauchy semigroup. Let $\{\lambda_t\}_{t\geq 0}$ be the family of the measures on $(0,\infty)$ whose Laplace transforms are

$$\int_0^\infty e^{-us}\,\lambda_t(ds) = e^{-\sqrt{u}\,t}.$$

Define

$$\overrightarrow{Q}_t^{V+\alpha}\omega := \int_0^\infty \overrightarrow{T}_s^{V+\alpha}\omega\,\lambda_t(ds).$$

By general theory, $\{\overrightarrow{Q}_t^{V+\alpha}\}_{t\geq 0}$ is a strongly continuous contraction semigroup on $L^p(\Lambda_k)$. We call $\{\overrightarrow{Q}_t^{V+\alpha}\}_{t\geq 0}$ the Cauchy semigroup of $\{\overrightarrow{T}_t^{V+\alpha}\}_{t\geq 0}$. We denote the generator of $\{\overrightarrow{Q}_t^{V+\alpha}\}_{t\geq 0}$ by $-\sqrt{-(\triangle - V - \alpha)}$. The explicit representation of $-\sqrt{-(\triangle - V - \alpha)}$ is as follows (see e.g., [11, Section 32]):

$$\sqrt{-(\triangle - V - \alpha)}\omega = \frac{1}{2\sqrt{\pi}}\int_0^\infty (\omega - \overrightarrow{T}_t^{V+\alpha}\omega)t^{-3/2}\,dt.$$

Cauchy semigroups of $\{\overrightarrow{T}_t^{A-\alpha}\}$, $\{\overrightarrow{T}_t^{\hat{A}-\alpha}\}$ and $\{\overrightarrow{T}_t^{\tilde{A}-\alpha}\}$ can be defined similarly. We denote them by $\{\overrightarrow{Q}_t^{A-\alpha}\}$, $\{\overrightarrow{Q}_t^{\hat{A}-\alpha}\}$ and $\{\overrightarrow{Q}_t^{\tilde{A}-\alpha}\}$, respectively.

4. Defective intertwining property

The first aim of this section is to prove the following proposition.

Proposition 4.1 *The following identities hold: for* $\omega \in \overrightarrow{\mathcal{F}}_V$,

$$\sqrt{V}\,\overrightarrow{T}_t^{V+\alpha}\omega = \overrightarrow{T}_t^{A-\alpha}(\sqrt{V}\omega), \tag{4.1}$$

$$d\,\overrightarrow{T}_t^{V+\alpha}\omega = \overrightarrow{T}_t^{V+\alpha}d\omega - \int_0^t \overrightarrow{T}_{t-s}^{V+\alpha}(\mathrm{ext}(dV)\,\overrightarrow{T}_s^{V+\alpha}\omega)\,ds, \tag{4.2}$$

$$\delta\,\overrightarrow{T}_t^{V+\alpha}\omega = \overrightarrow{T}_t^{V+\alpha}\delta\omega + \int_0^t \overrightarrow{T}_{t-s}^{V+\alpha}(\mathrm{int}(dV)\,\overrightarrow{T}_s^{V+\alpha}\omega)\,ds. \tag{4.3}$$

These identities follows from the defective intertwining property studied in [15]. To state the result in [15], we introduce some notations.

Let $\{T_t\}$ and $\{\hat{T}_t\}$ be a (C_0)-semigroups on (real) Hilbert spaces H and $\hat{H}$. The generators of $\{T_t\}$ and $\{\hat{T}_t\}$ are denoted by A and $\hat{A}$, respectively. We assume that they are bounded from below in the following sense: there exists $\beta \geq 0$ such that

$$(-Ax, x)_H \geq -\beta |x|_H^2,$$
$$(-\hat{A}\theta, \theta)_{\hat{H}} \geq -\beta |\theta|_{\hat{H}}^2$$

hold. Hence $A - \beta$ and $\hat{A} - \beta$ generate contraction semigroups. We further assume that they satisfy the weak sector condition.(For the weak sector condition, see [7]) We denoted the associated quadratic form by $\mathcal{E}$ and $\hat{\mathcal{E}}$, namely,

$$\mathcal{E}(x, y) = (-Ax, y)_H, \quad \text{for } x \in \mathrm{Dom}(A), y \in \mathrm{Dom}(\mathcal{E}),$$
$$\hat{\mathcal{E}}(\xi, \eta) = (-\hat{A}\xi, \eta)_{\hat{H}}, \quad \text{for } \xi \in \mathrm{Dom}(\hat{A}), \eta \in \mathrm{Dom}(\hat{\mathcal{E}}).$$

We fix $\gamma > \beta$ and set

$$\mathcal{E}_\gamma(x, y) = \mathcal{E}(x, y) + \gamma(x, y)_H.$$

Then $\mathcal{F} = \mathrm{Dom}(\mathcal{E})$ is a Hilbert space with the inner product

$$(x, y)_{\mathcal{F}} = \frac{1}{2} \left\{ \mathcal{E}_\gamma(x, y) + \mathcal{E}_\gamma(y, x) \right\}.$$

By the weak sector condition, $\mathcal{E}$ is a bounded bilinear form on $\mathcal{F} \times \mathcal{F}$, i.e., there exists a constant $C > 0$ such that

$$|\mathcal{E}(x, y) \leq C(x, x)_{\mathcal{F}}^{1/2} (y, y)_{\mathcal{F}}^{1/2}.$$

Similarly, we define

$$\hat{\mathcal{E}}_\gamma(\xi, \eta) = \hat{\mathcal{E}}(\xi, \eta) + \gamma(\xi, \eta)_{\hat{H}}$$

and a Hilbert space $\hat{\mathcal{F}} = \mathrm{Dom}(\hat{\mathcal{E}})$ with the inner product

$$(\xi, \eta)_{\hat{\mathcal{F}}} = \frac{1}{2} \left\{ \hat{\mathcal{E}}_\gamma(\xi, \eta) + \hat{\mathcal{E}}_\gamma(\eta, \xi) \right\}.$$

We shall denote by H^* and $\mathcal{F}^*$ the set of all continuous linear functional on H and $\mathcal{F}$, respectively. Clearly $H^* \subset \mathcal{F}^*$ and, by the Riesz theorem, we can identify H^* with H. Hence we have a triplet $\mathcal{F} \subset H \subset \mathcal{F}^*$. Moreover A can be extended to a bounded linear operator from $\mathcal{F}$ to $\mathcal{F}^*$ and a (C_0)-semigroup $\{\tilde{T}_t\}$ on $\mathcal{F}^*$ (see [18, §]). We denoted the generator of $\{\tilde{T}_t\}$ by $\tilde{A}$.

Similarly, we can define a triplet $\hat{\mathcal{F}} \subset \hat{H} \subset \hat{\mathcal{F}}^*$ and $\tilde{\hat{A}} : \hat{\mathcal{F}} \to \hat{\mathcal{F}}^*$ which is an extension of $\hat{A}$. The semigroup generated by $\tilde{\hat{A}}$ is denoted by $\{\tilde{\hat{T}}_t\}$. Associated resolvent is denoted by $\tilde{\hat{G}}_\lambda$.

Suppose also that we are given a closed operator D from H into $\hat{H}$ satisfying $\mathrm{Dom}(A) \subset \mathrm{Dom}(D)$. Now we consider the following defective intertwining property

$$DA = \hat{A}D + R,$$

where R is a linear operator from H into $\hat{H}$. We shall assume that
(B.1) R is a bounded linear operator from $\mathrm{Dom}(A)$ into $\hat{\mathcal{F}}^*$.
Then we have the following theorem.

Theorem 4.2 (Theorem 3.1 and Theorem 3.3 of [15]) *We assume that* $\mathrm{Dom}(A) \subset \mathrm{Dom}(D)$ *and (B.1). Furthermore, we assume* $\hat{\mathcal{F}} \subset \mathrm{Dom}(D^*)$. *Then the following three conditions are equivalent.*

(1) There exist a dense subspace $\mathcal{D} \subset \mathrm{Dom}(\hat{\mathcal{E}})$ *and a dense subspace* $\hat{\mathcal{D}} \subset \mathrm{Dom}(\hat{\mathcal{E}})$ *such that* $D\mathcal{D} \subset \mathrm{Dom}(\hat{\mathcal{E}})$ *and*

$$(Ax, D^*\theta)_H = -\hat{\mathcal{E}}(Dx, \theta) + {}_{\hat{\mathcal{F}}^*}(Rx, \theta)_{\hat{\mathcal{F}}}, \quad \forall x \in \mathcal{D}, \forall \theta \in \hat{D} \tag{4.4}$$

(2) For sufficiently large λ,

$$DG_\lambda x = \hat{G}_\lambda Dx + \hat{G}_\lambda \tilde{R} G_\lambda x, \quad \forall x \in \mathrm{Dom}(D). \tag{4.5}$$

(3) $\{T_t\}$ *is a* (C_0)*-semigroup on* $\mathrm{Dom}(D)$ *and the following holds:*

$$DT_t x = \hat{T}_t Dx + \int_0^t \hat{T}_{t-s} \tilde{R} T_s x \, ds, \quad \forall\, \mathrm{Dom}(D), \tag{4.6}$$

where the integral is the limit of Riemann sum in $\hat{\mathcal{F}}^*$.

Set

$$\begin{aligned}
\mathrm{Dom}(D) &:= \overrightarrow{\mathcal{F}}_{V,k} \\
D\omega &:= (d\omega, \delta\omega, \sqrt{V}\omega) \\
\hat{\mathscr{F}} &:= \overrightarrow{\mathcal{F}}_{V,k+1} \times \overrightarrow{\mathcal{F}}_{V,k-1} \times \overrightarrow{\mathcal{F}}_{V,k} \\
\hat{\mathscr{E}}(\omega, \eta) &:= \overrightarrow{\mathcal{E}}^{V+\alpha,k+1}(\omega_1, \eta_1) + \overrightarrow{\mathcal{E}}^{V+\alpha,k-1}(\omega_2, \eta_2) + \overrightarrow{\mathcal{E}}^{A-\alpha,k}(\omega_3, \eta_3) \\
R\omega &:= (-\operatorname{ext}(dV)\omega, \operatorname{int}(dV)\omega, 0).
\end{aligned}$$

Due to Theorem 4.2, it is sufficient to prove the following lemma

Lemma 4.3 *Under the notations above, we have*

(1) R *is a bounded linear operator from* $\mathrm{Dom}(D)$ *into* $\hat{\mathscr{F}}^*$, *where* $\hat{\mathscr{F}}^*$ *is a dual space of* $\hat{\mathscr{F}}$;
(2) A_k *is a dense subspace of* $\mathrm{Dom}((\triangle - V - \alpha)_k)$ *and* $A_{k+1} \times A_{k-1} \times A_k$ *is a dense subspace of* $\hat{\mathscr{F}}$ *such that* $DA_k \subset \hat{\mathscr{F}}$;
(3) For $\omega \in A_k$ *and* $\theta \in A_{k+1} \times A_{k-1} \times A_k$,

$$((\triangle - V - \alpha)_k \omega, D^*\theta)_{L^2(\Lambda_k)} = -\hat{\mathscr{E}}(D\omega, \theta) + {}_{\hat{\mathscr{F}}^*}(R\omega, \theta)_{\hat{\mathscr{F}}}. \tag{4.7}$$

Proof. (1) can be shown similarly as in [15, Section 4]. (2) is trivial by definition. We only need to prove (3).

We first note the following identities:

$$\begin{aligned} d(\triangle - V - \alpha)\omega &= (\triangle - V - \alpha)d\omega - \mathrm{ext}(dV)\omega, \\ (\triangle - V - \alpha)\sqrt{V}\omega &= \sqrt{V}(\overrightarrow{\hat{A} - \alpha})\omega. \end{aligned} \tag{4.8}$$

Then, for $\omega \in A_k$ and $\theta = (\theta_1, \theta_2, \theta_3) \in A_{k+1} \times A_{k-1} \times A_k$, we have

$$\begin{aligned} &(\text{LHS of } (4.7)) \\ &\quad = (d(\triangle - V - \alpha)\omega, \theta_1) \\ &\qquad + (\omega, (\triangle - V - \alpha)d\theta_2) + (\omega, (\triangle - V - \alpha)\sqrt{V}\theta_3) \\ &\quad = ((\triangle - V - \alpha)d\omega, \theta_1) - (\mathrm{ext}(dV)\omega, \theta_1) \\ &\qquad + (\omega, d(\triangle - V - \alpha)\theta_2) + (\omega, \mathrm{ext}(dV)\theta_2) + (\omega, \sqrt{V}(\overrightarrow{\hat{A} - \alpha})\theta_3) \\ &\quad = ((\triangle - V - \alpha)d\omega, \theta_1) - (\mathrm{ext}(dV)\omega, \theta_1) \\ &\qquad + ((\triangle - V - \alpha)\delta\omega, \theta_2) + (\mathrm{int}(dV)\omega, \theta_2) + ((\overrightarrow{A - \alpha})\sqrt{V}\omega, \theta_3) \\ &\quad = (\text{RHS of } (4.7)). \end{aligned}$$

Here we used (4.8) in the second line.

We proceed to the Cauchy operators. Set

$$\begin{aligned} \Phi^d_t\omega &:= \int_0^t \overrightarrow{T}^{V+\alpha}_{t-s}(\mathrm{ext}(dV)\overrightarrow{T}^{V+\alpha}_s\omega)\, ds, \\ \Phi^\delta_t\omega &:= \int_0^t \overrightarrow{T}^{V+\alpha}_{t-s}(\mathrm{int}(dV)\overrightarrow{T}^{V+\alpha}_s\omega)\, ds, \\ \Psi^d_t\omega &:= \int_0^\infty \Phi^d_s\omega\, d\lambda_t(ds), \\ \Psi^\delta_t\omega &:= \int_0^\infty \Phi^\delta_s\omega\, d\lambda_t(ds), \end{aligned}$$

and

$$\begin{aligned} \Theta^d\omega &:= \frac{1}{2\sqrt{\pi}}\int_0^\infty \Phi^d_t\omega t^{-3/2}\, dt, \\ \Theta^\delta\omega &:= \frac{1}{2\sqrt{\pi}}\int_0^\infty \Phi^\delta_t\omega t^{-3/2}\, dt. \end{aligned}$$

By the definition of Cauchy semigroup and Proposition 4.1, we can obtain the following corollaries.

Corollary 4.4 *The following identities hold: for* $\omega \in \overrightarrow{\mathcal{F}}_V$,

$$\sqrt{V}\overrightarrow{Q}_t^{V+\alpha}\omega = \overrightarrow{Q}_t^{A-\alpha}(\sqrt{V}\omega), \tag{4.9}$$
$$d\overrightarrow{Q}_t^{V+\alpha}\omega = \overrightarrow{Q}_t^{V+\alpha}d\omega - \Psi_t^d\omega, \tag{4.10}$$
$$\delta\overrightarrow{Q}_t^{V+\alpha}\omega = \overrightarrow{Q}_t^{V+\alpha}\delta\omega + \Psi_t^\delta\omega. \tag{4.11}$$

Corollary 4.5 *The following identities hold: for* $\omega \in \mathrm{Dom}(\triangle - V - \alpha)$ *in* L^2,

$$\sqrt{V}\sqrt{-(\triangle - V - \alpha)}\omega = \sqrt{-(\overrightarrow{A - \alpha})}(\sqrt{V}\omega), \tag{4.12}$$
$$d\sqrt{-(\triangle - V - \alpha)}\omega = \sqrt{-(\triangle - V - \alpha)}d\omega - \Theta^d\omega, \tag{4.13}$$
$$\delta\sqrt{-(\triangle - V - \alpha)}\omega = \sqrt{-(\triangle - V - \alpha)}\delta\omega + \Theta^\delta\omega. \tag{4.14}$$

In the same way, we can prove the following intertwining property.

Proposition 4.6 *The following identities hold: for* $\omega \in \overrightarrow{\mathcal{F}}_V$,

$$\sqrt{V}\overrightarrow{T}_t^{A-\alpha}\omega = \overrightarrow{T}_t^{\tilde{A}-\alpha}(\sqrt{V}\omega), \tag{4.15}$$
$$\sqrt{V}\overrightarrow{Q}_t^{A-\alpha}\omega = \overrightarrow{Q}_t^{\tilde{A}-\alpha}(\sqrt{V}\omega). \tag{4.16}$$

Now we give another type of the intertwining property. We will use this later in the estimate of Φ_t^d and Φ_t^δ.

Lemma 4.7 *For* $\alpha > 0$ *and* $f \in \mathcal{F}_V$ *with* $f \geq 0$, *it holds that*

$$T_t^\alpha f - T_t^{V+\alpha} f = \int_0^t T_{t-s}^\alpha (V T_s^{V+\alpha} f)\, ds. \tag{4.17}$$

Proof. We give the probabilistic proof. Let $(\{X_t\}_{t\geq 0}, \{\mathbb{P}_x\}_{x\in M})$ be the diffusion process associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$ and set $\mathcal{F}_t := \sigma\{X_s;\ 0 \leq s \leq t\}$. Then it is well-known that $\{T_t^\alpha\}$ and $\{T_t^{V+\alpha}\}$ have the following probabilistic representation:

$$T_t^\alpha f(x) = e^{-\alpha t}\mathbb{E}_x[f(X_t)],$$
$$T_t^{V+\alpha} f(x) = e^{-\alpha t}\mathbb{E}_x[f(X_t)\exp(-\int_0^t V(X_s)ds)].$$

Therefore we have

$$\begin{aligned}
&T_t^{\alpha} f(x) - T_t^{V+\alpha} f(x) \\
&\quad = e^{-\alpha t}\mathbb{E}_x[f(X_t)(1-\exp(-\int_0^t V(X_s)\,ds))] \\
&\quad = e^{-\alpha t}\mathbb{E}_x[f(X_t)\int_0^t V(X_{t-s})\exp(-\int_{t-s}^t V(X_r)\,dr)\,ds] \\
&\quad = e^{-\alpha t}\int_0^t \mathbb{E}_x[f(X_t)V(X_{t-s})\exp(-\int_{t-s}^t V(X_r)\,dr)]\,ds \\
&\quad = e^{-\alpha t}\int_0^t \mathbb{E}_x[V(X_{t-s})\mathbb{E}_{X_{t-s}}[f(X_s)\exp(-\int_0^s V(X_r)\,dr)]]\,ds \\
&\quad = \int_0^t P_{t-s}^{\alpha}(VP_s^{V+\alpha} f)\,ds.
\end{aligned}$$

Here we used the Markov property in the fourth line. This completes the proof

To prove Theorem 2.1, we have to estimates Φ_t^d and Φ_t^δ. The next lemma plays an important role in a proof of Theorem 2.1.

Lemma 4.8 *For sufficiently large $\alpha > 0$, it holds that*

$$|\Phi_t^d \omega| \le T_t^{\alpha - C_M}|\omega| - T_t^{V+\alpha-C_M}|\omega|, \tag{4.18}$$

$$|\Phi_t^\delta \omega| \le T_t^{\alpha - C_M}|\omega| - T_t^{V+\alpha-C_M}|\omega|. \tag{4.19}$$

Proof. We only show the first inequality. The second one can be proved similarly.

$$\begin{aligned}
|\Phi_t^d \omega| &\le \int_0^t T_{t-s}^{V+\alpha-C_M}|\operatorname{ext}(dV)\overrightarrow{T}_s^{V+\alpha}\omega|\,ds \\
&\le C_1\int_0^t T_{t-s}^{V+\alpha-C_M} V|\overrightarrow{T}_s^{V+\alpha}\omega|\,ds \\
&\le C_1\int_0^t T_{t-s}^{V+\alpha-C_M} V T_s^{V+\alpha-C_M}|\omega|\,ds \\
&\le C_1\int_0^t T_{t-s}^{\alpha-C_M} V T_s^{V+\alpha-C_M}|\omega|\,ds \\
&= C_1(T_t^{\alpha-C_M}|\omega| - T_t^{V+\alpha-C_M}|\omega|).
\end{aligned}$$

Here we used Assumption (A-V) in the second line and lemma 4.7 in the last line. This completes the proof.

5. Littlewood-Paley inequality

In this section we will prove the so-called "Littlewood-Paley inequality." To state the results, we introduce some notations. Set

$$g_{V+\alpha}^{\rightarrow}\omega(x,t) := |\partial_t \overrightarrow{Q}_t^{V+\alpha}\omega(x)|, \tag{5.1}$$
$$g_{V+\alpha}^{d}\omega(x,t) := |d\overrightarrow{Q}_t^{V+\alpha}\omega(x)|, \tag{5.2}$$
$$g_{V+\alpha}^{\delta}\omega(x,t) := |\delta\overrightarrow{Q}_t^{V+\alpha}\omega(x)|, \tag{5.3}$$
$$g_{V+\alpha}^{\nabla}\omega(x,t) := |\nabla\overrightarrow{Q}_t^{V+\alpha}\omega(x)|, \tag{5.4}$$
$$g_{V+\alpha}^{V}\omega(x,t) := |\sqrt{V}\,\overrightarrow{Q}_t^{V+\alpha}\omega(x)|, \tag{5.5}$$
$$\tag{5.6}$$

and set

$$G_{V+\alpha}^{\rightarrow}\omega(x) := \left\{\int_0^\infty t\, g_{V+\alpha}^{\rightarrow}\omega(x,t)^2\, dt\right\}^{1/2}. \tag{5.7}$$

$G_{V+\alpha}^{\delta}\omega(x), G_{V+\alpha}^{\nabla}\omega(x), G_{V+\alpha}^{V}\omega(x)$ are defined similarly. In this setting, the Littlewood-Paley inequality is stated as follows.

Proposition 5.1 *For $1 < p < \infty$ and $\alpha > C_M$, it holds that for any $\omega \in A_k$*

$$\|G_{V+\alpha}^{\rightarrow}\omega\|_p \lesssim \|\omega\|_p, \tag{5.8}$$
$$\|G_{V+\alpha}^{d}\omega\|_p \lesssim \|\omega\|_p, \tag{5.9}$$
$$\|G_{V+\alpha}^{\delta}\omega\|_p \lesssim \|\omega\|_p, \tag{5.10}$$
$$\|G_{V+\alpha}^{V}\omega\|_p \lesssim \|\omega\|_p, \tag{5.11}$$

and

$$\|\omega\|_p \lesssim \|G_{V+\alpha}^{\rightarrow}\omega\|_p.. \tag{5.12}$$

Here, for two norms $\|\omega\|$ and $\|\omega\|'$, $\|\omega\| \lesssim \|\omega\|'$ means that there exists a constant C independent of ω such that $\|\omega\| \le C\|\omega\|'$ holds. In the sequel, we use this convention without mention.

In this section, we prove another type of Littlewood-Paley inequality. Before stating the proposition, we need further notations. Set

$$g_{A-\alpha}^{\rightarrow}\omega(x,t) := |\partial_t \overrightarrow{Q}_t^{A-\alpha}\omega(x)|, \tag{5.13}$$
$$g_{A-\alpha}^{d}\omega(x,t) := |d\overrightarrow{Q}_t^{A-\alpha}\omega(x)|, \tag{5.14}$$
$$g_{A-\alpha}^{\delta}\omega(x,t) := |\delta\overrightarrow{Q}_t^{A-\alpha}\omega(x)|, \tag{5.15}$$
$$g_{A-\alpha}^{\nabla}\omega(x,t) := |\nabla\overrightarrow{Q}_t^{A-\alpha}\omega(x)|, \tag{5.16}$$
$$g_{A-\alpha}^{V}\omega(x,t) := |\sqrt{V}\,\overrightarrow{Q}_t^{A-\alpha}\omega(x)|, \tag{5.17}$$

and set

$$G^{\rightarrow}_{A-\alpha}\omega(x) := \left\{ \int_0^\infty t\, g^{\rightarrow}_{A-\alpha}\omega(x,t)^2\, dt \right\}^{1/2}. \tag{5.18}$$

$G^d_{A-\alpha}\omega(x), G^\delta_{A-\alpha}\omega(x), G^\nabla_{A-\alpha}\omega(x), G^V_{A-\alpha}\omega(x)$ are defined similarly. $g^{\rightarrow}_{\hat{A}-c}\omega$, $G^{\rightarrow}_{\hat{A}-c}\omega, \dots$ are defined in the same manner.

We will also prove the following proposition.

Proposition 5.2 *For $1 < p < \infty$ and sufficiently large $\alpha > 0$, it holds that*

$$\|G^{\rightarrow}_{A-\alpha}\omega\|_p \lesssim \|\omega\|_p, \qquad \|G^{\rightarrow}_{\hat{A}-\alpha}\omega\|_p \lesssim \|\omega\|_p, \tag{5.19}$$

$$\|G^V_{A-\alpha}\omega\|_p \lesssim \|\omega\|_p, \qquad \|G^V_{\hat{A}-\alpha}\omega\|_p \lesssim \|\omega\|_p, \tag{5.20}$$

$$\|\omega\|_p \lesssim \|G^{\rightarrow}_{A-\alpha}\omega\|_p, \qquad \|\omega\|_p \lesssim \|G^{\rightarrow}_{\hat{A}-\alpha}\omega\|_p. \tag{5.21}$$

In the rest of this section, we prove Propositions 5.1 and 5.2. To do this, we need the following facts proved by P. A. Meyer [8]. See also [16], Section 3.2 (h).

Let $(\{\mathbb{P}_x\}_{x\in M}, \{X_t\}_{t\geq 0})$ be the diffusion process on M associated with the Dirichlet form $(\mathcal{E}^0,\ \mathcal{F})$ and $(\{\mathbb{P}_a\}_{a\in\mathbb{R}},\ \{B_t\}_{t\geq 0})$ be 1-dimensional Brownian motion with $\langle B\rangle_t = 2t$. We define $\tau := \inf\{t > 0 :\ B_t = 0\}$. Let h be a non-negative measurable function on $[0,\infty)$. Then the following identity hold:

$$\mathbb{E}_{\delta_a}\left[\int_0^\tau h(B_s)ds\right] = \int_0^\infty (a\wedge s)h(s)ds. \tag{5.22}$$

Let j be a non-negative measurable function on $M\times[0,\infty)$. Then the following identities hold:

$$\mathbb{E}_{m\otimes\delta_a}\left[\int_0^\tau j(X_s,B_s)ds\right] = \int_B dm(x)\int_0^\infty (a\wedge s)j(x,s)ds, \tag{5.23}$$

$$\mathbb{E}_{m\otimes\delta_a}\left[\int_0^\tau j(X_s,B_s)ds\middle| X_\tau\right] = \int_0^\infty (a\wedge s)Q_s j(X_\tau,s)ds \tag{5.24}$$

Let $\{Z_t\}_{t\in[0,\infty]}$ be a submartingale that is decomposed as $Z_t = M_t + A_t$, where $\{M_t\}$ is a right continuous martingale with left limits and $\{A_t\}$ is a continuous increasing process with $A_0 = 0$. Then, for $p \geq 1$, the following inequality hold:

$$\mathbb{E}\left[A^p_\infty\right] \leq (2p)^p\mathbb{E}\left[\left(\sup_{0\leq t\leq\infty}|Z_t|\right)^p\right]. \tag{5.25}$$

We are now ready to give proofs. We divide the proofs into several steps.

First, we will prove (5.8), (5.9), (5.10), and (5.11) in the case of $p = 2$. In this case, we use the spectral decomposition. Let $\{E_s\}_{s\geq 0}$ be the spectral decomposition of $-(\triangle - V)$. Then

$$\begin{aligned}
\|G^{\to}_{V+\alpha}\omega\|_2^2 &= \int_M\int_0^\infty t\,|\sqrt{\triangle - V - \alpha}\,\overrightarrow{Q}_t^{V+\alpha}\omega(x)|^2\,dt\,dm(x)\\
&= \int_0^\infty\int_0^\infty t\,(s+\alpha)e^{-2\sqrt{s+\alpha}\,t}\,d(E_s\omega,\omega)\,dt\\
&= \frac{1}{4}\int_0^\infty d(E_s\omega,\omega)\\
&= \frac{1}{4}\|\omega\|_2^2,
\end{aligned}$$

where we used $\int_0^\infty t\,(s+\alpha)e^{-2\sqrt{s+\alpha}\,t}\,dt = 1/4$ in the third line. Therefore we obtain

$$\|G^{\to}_{V+\alpha}\omega\|_2 = \frac{1}{2}\|\omega\|_2. \tag{5.26}$$

We note that

$$\|G^{\to}_{V+\alpha}\omega\|_2^2 = \|G^d_{V+\alpha}\omega\|_2^2 + \|G^\delta_{V+\alpha}\omega\|_2^2 + \|G^V_{V+\alpha}\omega\|_2^2.$$

Then, from (5.26), we obtain

$$\|G^d_{V+\alpha}\omega\|_2 \leq \frac{1}{2}\|\omega\|_2, \tag{5.27}$$

$$\|G^\delta_{V+\alpha}\omega\|_2 \leq \frac{1}{2}\|\omega\|_2, \tag{5.28}$$

and

$$\|G^V_{V+\alpha}\omega\|_2 \leq \frac{1}{2}\|\omega\|_2. \tag{5.29}$$

Now completed the proof of (5.8), (5.9), (5.10), and (5.11) in the case of $p = 2$.

Next, we will prove (5.8), (5.9), (5.10), and (5.11) in the case of $p < 2$. Here, we recall that $(\{\mathbb{P}_x\}_{x\in M}, \{X_t\}_{t\geq 0})$ is the diffusion process on M associated with the Dirichlet form $(\mathcal{E}^0,\ \mathcal{F})$ and $(\{\mathbb{P}_a\}_{a\in\mathbb{R}},\ \{B_t\}_{t\geq 0})$ is 1-dimensional Brownian motion with $\langle B\rangle_t = 2t$. We put $\overrightarrow{R}\omega := \sum_{i,j=1}^n \mathrm{ext}(e^i)\,\mathrm{int}(e^j)R_{e_i,e_j}\omega$.

Setting $u(x,\ a) := \overrightarrow{Q}_a^{V+\alpha}\omega(x)$, we have

$$\partial_a^2 u = -(\triangle - V - \alpha)u. \tag{5.30}$$

By Ito's formula,

$$
\begin{aligned}
&d(|u(X_t,B_t)|^2)\\
&\quad = dM_t^u + \{\triangle(|u|^2)(X_t,B_t) + \partial_a^2(|u|^2)(X_t,B_t)\}\,dt\\
&\quad = dM_t^u + 2\{(\triangle u(X_t,B_t),u(X_t,B_t)) + (\overrightarrow{R}u(X_t,B_t),u(X_t,B_t))\\
&\qquad + |\nabla u(X_t,B_t)|^2 + (\partial_a u(X_t,B_t),u(X_t,B_t)) + |\partial_a u(X_t,B_t)|^2\}\,dt\\
&\quad = dM_t^u + 2\{(V(X_t)+\alpha)|u(X_t,B_t)|^2 + (\overrightarrow{R}u(X_t,B_t),u(X_t,B_t))\\
&\qquad + |\nabla u(X_t,B_t)|^2 + |\partial_a u(X_t,B_t)|^2\}\,dt
\end{aligned}
$$

where M^u is a square integrable local martingale satisfying

$$
d\langle M^u\rangle_t = 2\{|\nabla(|u|^2)(X_t,B_t)|^2 + |\partial_a(|u|^2)(X_t,B_t)|^2\}\,dt. \tag{5.31}
$$

We used Weitzenböck's formula in the second line and (5.30) in the last line. Hence the following semimartingale decomposition hold:

$$
\begin{aligned}
&d(|u(X_t,B_t)|^2)\\
&\quad = dM_t^u + 2\{(V(X_t)+\alpha)|u(X_t,B_t)|^2 + (\overrightarrow{R}u(X_t,B_t),u(X_t,B_t))\\
&\qquad + |\nabla u(X_t,B_t)|^2 + |\partial_a u(X_t,B_t)|^2\}\,dt.
\end{aligned} \tag{5.32}
$$

Applying Ito's formula to $(|u(X_t,B_t)|^2+\varepsilon)^{p/2}$, we have

$$
\begin{aligned}
&d(|u(X_t,B_t)|^2+\varepsilon)^{p/2}\\
&\quad = \frac{p}{2}(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}\{dM_t^u + 2\{(V(X_t)+\alpha)|u(X_t,B_t)|^2\\
&\qquad + (\overrightarrow{R}u(X_t,B_t),u(X_t,B_t)) + |\nabla u(X_t,B_t)|^2 + |\partial_a u(X_t,B_t)|^2\}\,dt\}\\
&\qquad + \frac{1}{2}\frac{p}{2}\left(\frac{p}{2}-1\right)(|u(X_t,B_t)|^2+\varepsilon)^{p/2-2}d\langle M^u\rangle_t\\
&\quad \geq \frac{p}{2}(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}dM_t^u\\
&\qquad + p(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}(V(X_t)+\alpha-C_M)|u(X_t,B_t)|^2\,dt\\
&\qquad + p(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}(|\nabla u(X_t,B_t)|^2+|\partial_a u(X_t,B_t)|^2)\,dt\\
&\qquad + p(p-2)(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}(|\nabla u(X_t,B_t)|^2+|\partial_a u(X_t,B_t)|^2)\,dt\\
&\quad \geq \frac{p}{2}(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}dM_t^u\\
&\qquad + p(p-1)(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}\{(g_{V+\alpha}^{\rightarrow}\omega(X_t,B_t))^2\\
&\qquad\quad + (g_{V+\alpha}^{\nabla}\omega(X_t,B_t))^2 + (g_{V+\alpha}^{V}\omega(X_t,B_t))^2\}\,dt,
\end{aligned}
$$

where we used Assumption (A-V) and (5.31) in the second line. The last inequality holds if we take $\alpha > C_M$ because $0 < p-1 < 1$.

Put $(g_{V+\alpha}\omega(x,t))^2 := (g^{\rightarrow}_{V+\alpha}\omega(x,t))^2 + (g^{\nabla}_{V+\alpha}\omega(x,t))^2 + (g^{V}_{V+\alpha}\omega(x,t))^2$. Taking expectation with respect to $\mathbb{P}_{(m,a)} := \mathbb{P}_m \otimes \mathbb{P}_a$,

$$
\begin{aligned}
\mathbb{E}_{(m,a)}&\left[\int_0^\tau p(p-1)(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}(g_{V+\alpha}\omega(X_t,B_t))^2\,dt\right] \\
&\le \mathbb{E}_{(m,a)}[(|u(X_\tau,B_\tau)|^2+\varepsilon)^{p/2} - (|u(X_0,B_0)|^2+\varepsilon)^{p/2}] \\
&\le \mathbb{E}_{(m,a)}[|u(X_\tau,B_\tau)|^p + \varepsilon^{p/2}] \\
&\le \int_M |u(x,0)|^p\,dm(x) + \varepsilon^{p/2} \\
&\le \int_M |\omega(x)|^p\,dm(x) + \varepsilon^{p/2},
\end{aligned}
$$

where we use $(a+b)^{p/2} \le a^{p/2} + b^{p/2}$ for $a,b>0$, and $p<2$ in the second line and $u(x,0)=\omega(x)$ in the last line.

On the other hand,

$$
\begin{aligned}
\mathbb{E}_{(m,a)}&\left[\int_0^\tau p(p-1)(|u(X_t,B_t)|^2+\varepsilon)^{p/2-1}(g_{V+\alpha}\omega(X_t,B_t))^2\,dt\right] \\
&= p(p-1)\int_M\int_0^\infty (t\wedge a)(|u(x,t)|^2+\varepsilon)^{p/2-1}(g_{V+\alpha}\omega(x,t))^2\,dt\,dm(x) \\
&\ge p(p-1)\int_M\int_0^\infty (t\wedge a)(|u^*(x)|^2+\varepsilon)^{p/2-1}(g_{V+\alpha}\omega(x,t))^2\,dt\,dm(x)
\end{aligned}
$$

where we used (5.22) in the first line and we put $u^*(x) := \sup_{t\ge 0}|u(x,t)|$ in the last line. Therefore

$$
p(p-1)\int_M\int_0^\infty (t\wedge a)(|u^*|^2+\varepsilon)^{p/2-1}(g_{V+\alpha}\omega)^2\,dt\,dm \le \int_M |\omega|^p\,dm + \varepsilon^{p/2}
$$

and letting $a\to\infty$ and $\varepsilon\to 0$, we obtain

$$
p(p-1)\int_M\int_0^\infty t|u^*|^{p-2}(g_{V+\alpha}\omega)^2\,dt\,dm \le \int_M |\omega|^p\,dm. \tag{5.33}
$$

From this inequality, we can prove Lemma 5.1. For example, we show (5.8).

$$
\begin{aligned}
&\|G^{\rightarrow}_{V+\alpha}\omega\|_p^p \\
&= \int_M \left\{ \int_0^\infty t|g^{\rightarrow}_{V+\alpha}\omega|^2\,dt \right\}^{p/2} dm \\
&\le \left\{ \int_M |u^*|^p\,dm \right\}^{1-p/2} \left\{ \int_M \int_0^\infty t|u^*|^{p-2}|g^{\rightarrow}_{V+\alpha}\omega|^2\,dt\,dm \right\}^{p/2} \\
&\le \left\{ \int_M |u^*|^p\,dm \right\}^{1-p/2} \left\{ \int_M \int_0^\infty t|u^*(x)|^{p-2}(g_{V+\alpha}\omega)^2\,dt\,dm \right\}^{p/2} \\
&\lesssim \left\{ \int_M |u^*|^p\,dm \right\}^{1-p/2} \left\{ \int_M |\omega|^p\,dm \right\}^{p/2} \\
&\lesssim \left\{ \int_M |\omega|^p\,dm \right\}^{1-p/2} \left\{ \int_M |\omega|^p\,dm \right\}^{p/2} \\
&\le \int_M |\omega|^p\,dm,
\end{aligned}
$$

where we used Hölder's inequality for $2/p$ and $2/(2-p)$, $(g^{\rightarrow}_{V+\alpha}\omega)^2 \le (g_{V+\alpha}\omega)^2$ in the third line, (5.33) in the fourth line, and the maximum ergodic inequality in the fifth line.

Noting that d is antisymmetrization of ∇ and δ is contraction of ∇, it holds that $g^d_{V+\alpha}\omega \lesssim g^\nabla_{V+\alpha}\omega$ and $g^\delta_{V+\alpha}\omega \lesssim g^\nabla_{V+\alpha}\omega$. Thus we can show (5.9), (5.10) and (5.11) similarly.

Now we turn to the proof of (5.8), (5.9), (5.10) and (5.11) in the case of $p > 2$.

We define H-functions as follows:

$$
\begin{aligned}
H^{\rightarrow}_{V+\alpha}\omega(x) &:= \left\{ \int_0^\infty t\,Q^0_t(g^{\rightarrow}_{V+\alpha}\omega^2)(t,x)\,dt \right\}^{1/2}, \\
H^{d}_{V+\alpha}\omega(x) &:= \left\{ \int_0^\infty t\,Q^0_t(g^{d}_{V+\alpha}\omega^2)(t,x)\,dt \right\}^{1/2}, \\
H^{\delta}_{V+\alpha}\omega(x) &:= \left\{ \int_0^\infty t\,Q^0_t(g^{\delta}_{V+\alpha}\omega^2)(t,x)\,dt \right\}^{1/2}, \\
H^{\nabla}_{V+\alpha}\omega(x) &:= \left\{ \int_0^\infty t\,Q^0_t(g^{\nabla}_{V+\alpha}\omega^2)(t,x)\,dt \right\}^{1/2},
\end{aligned}
$$

and

$$
H^{V}_{V+\alpha}\omega(x) := \left\{ \int_0^\infty t\,Q^0_t(g^{V}_{V+\alpha}\omega^2)(t,x)\,dt \right\}^{1/2}.
$$

We first show that G-functions are dominated by H-functions.

Lemma 5.3 *It holds that*

$$G_{V+\alpha}^{\rightarrow}\omega(x) \le 2H_{V+\alpha}^{\rightarrow}\omega(x).$$

Proof. From the definition of $G_{V+\alpha}^{\rightarrow}$, we have

$$\begin{aligned}
G_{V+\alpha}^{\rightarrow}\omega(x,t)^2 &= \int_0^\infty t|\sqrt{\triangle - V - \alpha}\,\overrightarrow{Q}_t^{V+\alpha}\omega(x)|^2\,dt \\
&= 4\int_0^\infty t|\sqrt{\triangle - V - \alpha}\,\overrightarrow{Q}_{2t}^{V+\alpha}\omega(x)|^2\,dt \\
&= 4\int_0^\infty t|\overrightarrow{Q}_t^{V+\alpha}\sqrt{\triangle - V - \alpha}\,\overrightarrow{Q}_t^{V+\alpha}\omega(x)|^2\,dt \\
&\le 4\int_0^\infty tQ_t^0(|\sqrt{\triangle - V - \alpha}\,\overrightarrow{Q}_t^{V+\alpha}\omega|^2)(x)\,dt \\
&= 4H_{V+\alpha}^{\rightarrow}\omega(x)^2
\end{aligned}$$

where we used $|\overrightarrow{Q}_t^{V+\alpha}\omega| \le Q_t^0|\omega|$ in the fourth line. This completes the proof

Lemma 5.4 *It holds that*

$$\begin{aligned}
G_{V+\alpha}^{d}\omega(x) &\lesssim H_{V+\alpha}^{d}\omega(x) + \sup_{s\ge 0}\{T_s^0(|\omega|^2)\}(x), \\
G_{V+\alpha}^{\delta}\omega(x) &\lesssim H_{V+\alpha}^{\delta}\omega(x) + \sup_{s\ge 0}\{T_s^0(|\omega|^2)\}(x).
\end{aligned} \tag{5.34}$$

Proof. We only prove the first inequality. The second one can be proved similarly.

$$\begin{aligned}
G_{V+\alpha}^{d}\omega(x) &= \left\{\int_0^\infty t|d\overrightarrow{Q}_t^{V+\alpha}\omega(x)|^2\,dt\right\}^{1/2} \\
&= 2\left\{\int_0^\infty t|d\overrightarrow{Q}_{2t}^{V+\alpha}\omega(x)|^2\,dt\right\}^{1/2} \\
&= 2\left\{\int_0^\infty t|\overrightarrow{Q}_t^{V+\alpha}d\overrightarrow{Q}_t^{V+\alpha}\omega(x) + \Psi_t^d\overrightarrow{Q}_t^{V+\alpha}\omega(x)|^2\,dt\right\}^{1/2} \\
&\le 2\left\{\int_0^\infty t|\overrightarrow{Q}_t^{V+\alpha}d\overrightarrow{Q}_t^{V+\alpha}\omega(x)|^2\,dt\right\}^{1/2} \\
&\quad + 2\left\{\int_0^\infty t|\Psi_t^d\overrightarrow{Q}_t^{V+\alpha}\omega(x)|^2\,dt\right\}^{1/2} \\
&=: I_1 + I_2
\end{aligned}$$

where we used Corollary 4.4 in the third line.

It is easy to see that $I_1 \le 2H_{V+\alpha}^d\omega$. We estimate the second term I_2. By Lemma 4.8, we have

$$|\Psi_t^d\theta| \le \int_0^\infty |\Phi_s^d\theta|\,\lambda_t(ds) \le \int_0^\infty T_s^{\alpha'}|\theta|\,\lambda_t(ds) = Q_t^{\alpha'}|\theta|$$

where α' is any positive constant less than α. Letting $\theta = \overrightarrow{Q}_t^{V+\alpha}\omega$ in the inequality above,

$$\begin{aligned}
|\Psi_t^d \overrightarrow{Q}_t^{V+\alpha}\omega| &\le Q_s^{\alpha'} |\overrightarrow{Q}_t^{V+\alpha}\omega| \\
&\le Q_{2s}^{\alpha'}|\omega| \\
&= \int_0^\infty T_{2s}^{\alpha'}|\omega| \, \lambda_t(ds) \\
&\le \int_0^\infty e^{-2\alpha' s} \sup_{s\ge 0}(T_s^0|\omega|) \, \lambda_t(ds) \\
&= e^{-\sqrt{2\alpha'}t} \sup_{s\ge 0}(T_s^0|\omega|)
\end{aligned}$$

where we used the Laplace transform of $\lambda_t(ds)$. Thus

$$\begin{aligned}
I_2 &\le \left\{ \int_0^\infty t e^{-2\sqrt{2\alpha'}t} \sup_{s\ge 0}(T_s^0|\omega|)^2 \, dt \right\}^{1/2} \\
&= \left\{ \int_0^\infty t e^{-2\sqrt{2\alpha'}t} \, dt \right\}^{1/2} \sup_{s\ge 0}(T_s^0|\omega|)
\end{aligned}$$

which completes the proof since $\{\int_0^\infty t e^{-2\sqrt{2\alpha'}t} \, dt\}^{1/2} < \infty$.

Lemma 5.5 *It holds that*

$$G_{V+\alpha}^V \omega(x) \lesssim H_{V+\alpha}^V \omega(x). \tag{5.35}$$

Proof. By Corollary 4.4, we have

$$\begin{aligned}
G_{V+\alpha}^V \omega(x)^2 &= \int_0^\infty t|\sqrt{V} Q_t^{V+\alpha}\omega(x)|^2 \, dt \\
&= 4\int_0^\infty t|\sqrt{V} Q_{2t}^{V+\alpha}\omega(x)|^2 \, dt \\
&= 4\int_0^\infty t|Q_t^{A-\alpha}\sqrt{V} Q_t^{V+\alpha}\omega(x)|^2 \, dt \\
&\le 4\int_0^\infty t Q_t^0(|\sqrt{V} Q_t^{V+\alpha}\omega|^2)(x) \, dt \\
&= 4H_{V+\alpha}^V \omega(x)^2.
\end{aligned}$$

This completes the proof.

Next, we show the following lemma.

Lemma 5.6 *For* $2 < p < \infty$ *and* $\alpha > C_M$,

$$\|H_{V+\alpha}\omega\|_p \lesssim \|\omega\|_p \tag{5.36}$$

where $H_{V+\alpha}\omega(x) := \left\{\int_0^\infty t \, Q_t^0(g_{V+\alpha}^{\rightarrow}\omega^2 + g_{V+\alpha}^{\nabla}\omega^2 + g_{V+\alpha}^{V}\omega^2)(t,x) \, dt\right\}^{1/2}$.

Proof. We recall the following semimartingale decomposition (c.f. (5.32)):

$$d(|u(X_t,\ B_t)|^2) = dM_t^u + dA_t^u, \tag{5.37}$$

where

$$\begin{aligned} dA_t^u := 2\{&(V(X_t)+\alpha)|u(X_t,B_t)|^2 + (\overrightarrow{R}u(X_t,B_t), u(X_t,B_t)) \\ &+ |\nabla u(X_t,B_t)|^2 + |\partial_a u(X_t,B_t)|^2\}\, dt \end{aligned} \tag{5.38}$$

If we take $\alpha > C_M$, then, due to Assumption (A-M), the bounded variation part is increasing. Therefore, by (5.25) and Doob's inequality, we get

$$\begin{aligned} \mathbb{E}_{(m,a)}\left[(A_\tau^u)^{p/2}\right] &\lesssim \mathbb{E}_{(m,a)}[(|u(X_\tau,\ B_\tau)|^2 - |u(X_0,\ B_0)|^2)^{p/2}] \\ &\le \mathbb{E}_{(m,a)}[|u(X_\tau,\ B_\tau)|^p] \\ &= \int_M |\omega|^p\, dm \end{aligned} \tag{5.39}$$

Using this inequality,

$$\begin{aligned} &\|H_{V+\alpha}\omega\|_p^p \\ &\quad= \lim_{a\to\infty}\int_M \left\{\int_0^\infty (t\wedge a)\, Q_t^0(g_{V+\alpha}\omega^2)(t,x)\, dt\right\}^{p/2} dm(x) \\ &\quad= \lim_{a\to\infty}\int_M \mathbb{E}_{(m,a)}\left[\int_0^\tau g_{V+\alpha}\omega^2(X_s,B_s)ds \;\middle|\; X_\tau = x\right]^{p/2} m(dx) \\ &\quad\le \liminf_{a\to\infty}\int_M \mathbb{E}_{(m,a)}\left[\left\{\int_0^\tau g_{V+\alpha}\omega^2(X_s,B_s)ds\right\}^{p/2} \;\middle|\; X_\tau = x\right] m(dx) \\ &\quad= \liminf_{a\to\infty}\mathbb{E}_{(m,a)}\left[\left\{\int_0^\tau (g_{V+\alpha}\omega^2(X_s,B_s))\, ds\right\}^{p/2}\right] \\ &\quad\le \liminf_{a\to\infty}\mathbb{E}_{(m,a)}\left[(A_\tau^u)^{p/2}\right] \\ &\quad\lesssim \|\omega\|_p^p, \end{aligned}$$

where we used (5.24) in the second line, Jensen's inequality in the third line, the fact that dm is the invariant measure of $\{X_t\}$, in the fourth line, Assumption (A-M) in the fifth line, and (5.39) in the last line. This completes the proof.

Combining the lemmas above, we can obtain Proposition 5.1 easily.

Now let us turn to a proof of (5.12). We note that, due to the polarization of (5.26), the following equality holds:

$$\int_M (\omega,\eta)\, dm = \frac{1}{4}\int_M\int_0^\infty t(\partial_t \overrightarrow{Q}_t^{V+\alpha}\omega, \partial_t \overrightarrow{Q}_t^{V+\alpha}\eta)\, dt\, dm. \tag{5.40}$$

Hence we can obtain (5.12) as follows:

$$\begin{aligned}
\left|\int_M (\omega,\eta)\,dm\right| &\le \frac{1}{4}\int_M\int_0^\infty t|\partial_t \overrightarrow{Q}_t^{V+\alpha}\omega|\,|\,\partial_t \overrightarrow{Q}_t^{V+\alpha}\eta|\,dt\,dm \\
&\le \frac{1}{4}\left\{\int_M \left\{\int_0^\infty t g_{\overrightarrow{V+\alpha}}\omega(t)^2\,dt\right\}^{p/2} dm(x)\right\}^{1/p} \\
&\quad\times \left\{\int_M \left\{\int_0^\infty t g_{\overrightarrow{V+\alpha}}\eta(t)^2\,dt\right\}^{q/2} dm\right\}^{1/q} \\
&= \frac{1}{4}\|G_{\overrightarrow{V+\alpha}}\omega\|_p \|G_{\overrightarrow{V+\alpha}}\eta\|_q \\
&\le \frac{1}{4}\|G_{\overrightarrow{V+\alpha}}\omega\|_p \|\eta\|_q
\end{aligned}$$

where we used Schwarz' inequality and Hölder's inequality in the second line and (5.8) in the last line.

Before ending this section, we will prove proposition 5.2. Our approach is similar to that of the proof of Proposition 5.1. To clarify difference between the former and the latter, we will deal with the case $p = 2$.

Define $U := \frac{1}{2}\nabla^* b - \frac{1}{4}|b|^2$ and set $v(x,a) := \overrightarrow{Q}_a^{A-\alpha}\omega(x)$. Then v satisfies

$$\partial_a^2 v = -\triangle v - \nabla_b v + (V + U + \alpha)v. \tag{5.41}$$

Main difference comes from the second term. By Ito's formula,

$$\begin{aligned}
&d(|v|^2(X_t,B_t)) \\
&\quad = dM_t^v \\
&\qquad + \{-2(\nabla_b v(X_t,B_t), v(X_t,B_t)) + 2(V(X_t) + U(X_t) + \alpha)|v(X_t,B_t)|^2 \\
&\qquad + (2(\overrightarrow{R} v(X_t,B_t), v(X_t,B_t)) + 2|\nabla v(X_t,B_t)|^2 + 2|\partial_a v(X_t,B_t)|^2\}\,dt,
\end{aligned} \tag{5.42}$$

where $\{M_t^v\}$ is a continuous local martingale satisfying

$$d\langle M^v\rangle_t = 2\{|\nabla|v|^2(X_t,B_t)|^2 + |\partial_a|v|^2(X_t,B_t)|\}\,dt.$$

Lemma 5.7 *For sufficiently large $\alpha > 0$, it holds that*

$$-(\nabla_b\eta,\ \eta) + |\nabla\eta|^2 + (\overrightarrow{R}\eta,\eta) + (U+\alpha)|\eta|^2 \ge 0. \tag{5.43}$$

Proof. By Assumption (A-V), we have

$$\begin{aligned}
-(\nabla_b\eta,\ \eta) &\ge -|b||\nabla\eta||\eta| \\
&\ge -C_1|\nabla\eta||\eta| \\
&\ge -1/2|\nabla\eta|^2 - C_1^2/2|\eta|^2.
\end{aligned} \tag{5.44}$$

Here we used $xy \le (1/2)x^2 + (1/2)y^2$ for $x, y \ge 0$. By using this, we have

$$\begin{aligned}
-(\nabla_b \eta,\ \eta) + &|\nabla \eta|^2 + (\overrightarrow{R}\eta, \eta) + (U+\alpha)|\eta|^2 \\
&\ge -1/2|\nabla \eta|^2 + |\nabla \eta|^2 + (\overrightarrow{R}\eta, \eta) + (U + \alpha - 1/2C_1^2)|\eta|^2 \\
&\ge 0.
\end{aligned}$$

The last inequality holds if we take sufficiently large α since U and $\overrightarrow{R}$ are bounded from below by Assumptions (A-V) and (A-M) respectively. This completes the proof.

We now turn to (5.19) and (5.20). By Lemma 5.7,

$$\begin{aligned}
&\|G^{\rightarrow}_{A-\alpha}\omega\|_2^2 + \|G^V_{A-\alpha}\omega\|_2^2 \\
&= \int_M \int_0^\infty t\{|\partial_t \overrightarrow{Q}_t^{A-\alpha}\omega(x)|^2 + V(x)|\overrightarrow{Q}_t^{A-\alpha}\omega(x)|^2\}\, dt\, dm(x) \\
&\le \liminf_{a\to\infty} \int_M \int_0^\infty (t \wedge a)\{|\partial_t v(x,t)|^2 + |\nabla v(x,t)|^2 \\
&\qquad - (\nabla_b v(x,t), v(x,t)) + (\overrightarrow{R} v(x,t), v(x,t)) \\
&\qquad + (V(x) + U(x) + \alpha)|v(x,t)|^2\}\, dt\, dm(x) \\
&= \liminf_{a\to\infty} \mathbb{E}_{(m,a)}\Big[\int_0^\tau \{|\partial_t v(X_t, B_t)|^2 + |\nabla v(X_t, B_t)|^2 \\
&\qquad - (\nabla_b v(X_t, B_t), v(X_t, B_t)) \\
&\qquad + (\overrightarrow{R} v(X_t, B_t),\ v(X_t, B_t)) \\
&\qquad + (V(X_t) + U(X_t) + \alpha)|v(X_t, B_t)|^2\}\, dt\Big] \\
&= \liminf_{a\to\infty} \mathbb{E}_{(m,a)}[|v(X_\tau, B_\tau)|^2 - |v(X_0, B_0)|^2] \\
&\le \|\omega\|_2^2,
\end{aligned}$$

where we used Proposition 5.22 in the third line, and (5.42) in the fourth line. This shows the first equality of (5.19) and the first equality of (5.20) in the case $p = 2$. The second ones can be proved in the same way.

For a proof of (5.21), we need the following lemma.

Lemma 5.8 *It holds that*

$$\int_M \int_0^\infty t(\partial_t \overrightarrow{Q}_t^{A-\alpha}\omega, \partial_t \overrightarrow{Q}_t^{\hat{A}-\alpha}\eta)\, dt\, dm = \frac{1}{4}\int_M (\omega, \eta)\, dm. \tag{5.45}$$

Proof. Since $\overrightarrow{\hat{A}-\alpha}$ is the dual operator of $\overrightarrow{A-\alpha}$, we have

$$\begin{aligned}
(\text{LHS of } (5.45)) &= \frac{1}{4}\int_M \int_0^\infty t(\partial_t^2 \overrightarrow{Q}_{2t}^{A-\alpha}\omega, \eta)\, dt\, dm \\
&= -\frac{1}{4}\int_M \int_0^\infty (\partial_t \overrightarrow{Q}_{2t}^{A-\alpha}\omega, \eta)\, dt\, dm \\
&= \int_M (\omega,\ \eta)\, dm.
\end{aligned}$$

Here we used the integration by parts with respect to t twice. This completes the proof.

Now we can show the first inequality of (5.21) in the case $p = 2$ as follows:

$$
\begin{aligned}
|(\omega, \theta)_{L^2(\Lambda_k)}| &= \left| \left(\int_0^\infty t(\partial_t \overrightarrow{Q}_t^{A-\alpha}\omega, \partial_t \overrightarrow{Q}_t^{\hat{A}-\alpha}\eta)\, dt \right)_{L^2(\Lambda_k)} \right| \\
&= |(G_{A-\alpha}^{\rightarrow}\omega, G_{\hat{A}-\alpha}^{\rightarrow}\theta)_{L^2(\Lambda_k)}| \\
&\le \|G_{A-\alpha}^{\rightarrow}\omega\|_2 \|G_{\hat{A}-\alpha}^{\rightarrow}\theta\|_2 \\
&\lesssim \|G_{A-\alpha}^{\rightarrow}\omega\|_2 \|\theta\|_2 .
\end{aligned}
$$

Here we used (5.45) in the first line and the second inequality of (5.19) in the last line.

The second inequality of (5.21) can be proved in the same way.

6. Proof of Theorem 2.1

We divide a proof into three steps. The first step is to prove

$$\|\sqrt{V}\omega\|_p \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p. \tag{6.1}$$

The second step is to prove

$$\|d\omega\|_p \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p, \tag{6.2}$$

and

$$\|\delta\omega\|_p \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p. \tag{6.3}$$

The third step is to prove

$$\|\sqrt{-(\triangle - V - \alpha)}\omega\|_p \lesssim \|\omega\|_p + \|d\omega\|_p + \|\delta\omega\|_p + \|\sqrt{V}\omega\|_p. \tag{6.4}$$

First step: $\|\sqrt{V}\omega\|_p \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p$.

To show this,

$$
\begin{aligned}
\|\sqrt{V}\omega\|_p &\lesssim \|G_{A-\alpha}^{\rightarrow}(\sqrt{V}\omega)\|_p \\
&= \left\| \left\{ \int_0^\infty t |\sqrt{-(\overrightarrow{A - V - \alpha})}\, \overrightarrow{Q}_t^{A-\alpha}(\sqrt{V}\omega)|^2\, dt \right\}^{1/2} \right\|_p \\
&= \left\| \left\{ \int_0^\infty t |\sqrt{V}\, \overrightarrow{Q}_t^{V+\alpha} \sqrt{-(\triangle - V - \alpha)}\omega|^2\, dt \right\}^{1/2} \right\|_p \\
&= \|G_{V+\alpha}^{V}(\sqrt{-(\triangle - V - \alpha)}\omega)\|_p \\
&\lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p
\end{aligned}
$$

where we used Proposition 5.2 in the first line, Corollaries 4.4 and 4.5 in the third line, and Proposition 5.1 in the last line. This is the desired result.

Second step:
$\|d\omega\|_p \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p$ and $\|\delta\omega\|_p \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p$.

To show this, we note

$$
\begin{aligned}
\|d\omega\|_p &\lesssim \|G^{\rightarrow}_{V+\alpha}(d\omega)\|_p \\
&= \left\| \left\{ \int_0^\infty t|\sqrt{-(\triangle - V - \alpha)}\,\overrightarrow{Q}_t^{V+\alpha}(d\omega)|^2\,dt \right\}^{1/2} \right\|_p \\
&= \left\| \left\{ \int_0^\infty t|\overrightarrow{Q}_t^{V+\alpha}(d\sqrt{-(\triangle - V - \alpha)}\omega) + \overrightarrow{Q}_t^{V+\alpha}\Theta^d\omega|^2\,dt \right\}^{1/2} \right\|_p \\
&= \left\| \left\{ \int_0^\infty t|d\overrightarrow{Q}_t^{V+\alpha}\sqrt{-(\triangle - V - \alpha)}\omega \right.\right. \\
&\qquad\qquad \left.\left. + \Psi_t^d(\sqrt{-(\triangle - V - \alpha)}\omega) + \overrightarrow{Q}_t^{V+\alpha}\Theta^d\omega|^2\,dt \right\}^{1/2} \right\|_p \\
&\le \left\| \left\{ \int_0^\infty t|d\overrightarrow{Q}_t^{V+\alpha}\sqrt{-(\triangle - V - \alpha)}\omega|^2\,dt \right\}^{1/2} \right\|_p \\
&\quad + \left\| \left\{ \int_0^\infty t|\Psi_t^d(\sqrt{-(\triangle - V - \alpha)}\omega)|^2\,dt \right\}^{1/2} \right\|_p \\
&\quad + \left\| \left\{ \int_0^\infty t|\overrightarrow{Q}_t^{V+\alpha}\Theta^d\omega|^2\,dt \right\}^{1/2} \right\|_p \\
&=: I_1 + I_2 + I_3,
\end{aligned}
$$

where we used Proposition 5.1 in the first line, Corollary 4.5 in the third line, and Corollary 4.4 in the fourth line.

First, we estimate I_1. By Proposition 5.1,

$$I_1 = \|G^d_{V+\alpha}(\sqrt{-(\triangle - V - \alpha)}\omega)\|_p \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p.$$

Second, we estimate I_2. Noting that

$$|\Psi_t^d\theta| \le e^{-\alpha' t} \sup_{s\ge 0} T_s^0|\theta|,$$

that has already been proved in the proof of Lemma 5.4, we have

$$
\begin{aligned}
I_2 &= \left\| \left\{ \int_0^\infty t |\Psi_t^d(\sqrt{-(\triangle - V - \alpha)}\omega)|^2 \, dt \right\}^{1/2} \right\|_p \\
&\le \left\| \left\{ \int_0^\infty t e^{-2\alpha' t} (\sup_{s \ge 0} \{T_s^0 |\sqrt{-(\triangle - V - \alpha)}\omega|\})^2 \, dt \right\}^{1/2} \right\|_p \\
&\lesssim \| \sup_{s\ge 0} \{T_s^0 |\sqrt{-(\triangle - V - \alpha)}\omega|\} \|_p \\
&\lesssim \| \sqrt{-(\triangle - V - \alpha)}\omega \|_p,
\end{aligned}
$$

where we used the maximum ergodic inequality (c.f. [16, Theorem 3.3.]) in the last line.

For the estimate of I_3, we need to prepare some inequalities.

First, we estimate $|\Theta^d \omega|$.

$$
\begin{aligned}
|\Phi_t^d \omega| &\le \int_0^t |\overrightarrow{T}_{t-s}^{V+\alpha}(\mathrm{ext}(dV)\overrightarrow{T}_s^{V+\alpha}\omega)| \, ds \\
&\le \int_0^t T_{t-s}^{V+\alpha-C_M}(|dV| \; |\overrightarrow{T}_s^{V+\alpha}\omega)|) \, ds \\
&\lesssim \int_0^t T_{t-s}^{V+\alpha-C_M}(V \; |\overrightarrow{T}_s^{V+\alpha}\omega)|) \, ds \\
&= \int_0^t \sqrt{V} T_{t-s}^{\hat{A}-\alpha+C_M}(\sqrt{V} \; |\overrightarrow{T}_s^{V+\alpha}\omega)|) \, ds,
\end{aligned}
$$

where we use $|\overrightarrow{T}_t^{V+\alpha}\omega| \le T_t^{V+\alpha-C_M}|\omega|$ in the second line, Assumption (A-V) in the third line, Proposition 4.1 in the last line. Therefore we get

$$
|\Theta^d \omega| \le \frac{1}{2\sqrt{\pi}} \sqrt{V} \int_0^\infty \left\{ \int_0^t T_{t-s}^{\hat{A}-\alpha+C_M} |\sqrt{V}\overrightarrow{T}_s^{V+\alpha}\omega| \, ds \right\} t^{-3/2} \, dt = \sqrt{V} J, \tag{6.5}
$$

where we put

$$
J := \frac{1}{2\sqrt{\pi}} \int_0^\infty \left\{ \int_0^t T_{t-s}^{\hat{A}-\alpha+C_M} |\sqrt{V}\overrightarrow{T}_s^{V+\alpha}\omega| \, ds \right\} t^{-3/2} \, dt.
$$

Before estimating I_3, we will estimate $\|J\|_p$. Let ε be a (small) positive constant. Then

$$
\begin{aligned}
\|J\|_p &= \frac{1}{2\sqrt{\pi}} \left\| \int_0^\infty \left\{ \int_0^t T_{t-s}^{\hat{A}-\alpha+C_M} |\overrightarrow{T}_s^{A-\alpha}(\sqrt{V}\omega)| \, ds \right\} t^{-3/2} \, dt \right\|_p \\
&\le \frac{1}{2\sqrt{\pi}} \int_0^\infty \left\{ \int_0^t \|T_{t-s}^{\hat{A}-\alpha+C_M} |\overrightarrow{T}_s^{A-\alpha}(\sqrt{V}\omega)|\|_p \, ds \right\} t^{-3/2} \, dt \\
&\le \frac{1}{2\sqrt{\pi}} \int_0^\infty \left\{ \int_0^t e^{-\varepsilon t} \|\sqrt{V}\omega\|_p \, ds \right\} t^{-3/2} \, dt \\
&\lesssim \|\sqrt{V}\omega\|_p,
\end{aligned} \tag{6.6}
$$

where we use Proposition 4.1 in the first line.

Now we are ready to estimate I_3. Using (6.5), we have

$$\begin{aligned}
I_3 &= \left\| \left\{ \int_0^\infty t |\overrightarrow{Q}_t^{V+\alpha} \Theta^d \omega|^2 \, dt \right\}^{1/2} \right\|_p \\
&\leq \left\| \left\{ \int_0^\infty t (Q_t^{V+\alpha-C_M} |\Theta^d \omega|)^2 \, dt \right\}^{1/2} \right\|_p \\
&\leq \left\| \left\{ \int_0^\infty t (Q_t^{V+\alpha-C_M} \sqrt{V} J)^2 \, dt \right\}^{1/2} \right\|_p \\
&= \left\| \left\{ \int_0^\infty t (\sqrt{V} Q_t^{\hat{A}-\alpha+C_M} J)^2 \, dt \right\}^{1/2} \right\|_p .
\end{aligned}$$

Applying Proposition 5.2 to the inequality above, we have

$$I_3 \leq \|J\|_p.$$

Therefore, by (6.6), we have

$$I_3 \lesssim \|\sqrt{V}\omega\|_p.$$

Thus, due to the first step, we obtain

$$I_3 \lesssim \|\sqrt{-(\triangle - V - \alpha)}\omega\|_p.$$

This completes the proof.

<u>*Third step*</u>: $\|\sqrt{\triangle - V - \alpha}\omega\|_p \lesssim \|\omega\|_p + \|d\omega\|_p + \|\delta\omega\|_p + \|\sqrt{V}\omega\|_p$.

Note that

$$\begin{aligned}
&(d\omega, d\eta) + (\delta\omega, \delta\eta) + (\sqrt{V}\omega, \sqrt{V}\eta) + \alpha(\omega, \eta) \\
&\qquad = (\sqrt{-(\triangle - V - \alpha)}\omega, \sqrt{-(\triangle - V - \alpha)}\eta).
\end{aligned}$$

Hence, if we show the upper estimate, then the lower estimate follows by the duality argument as follows. Set $\theta := \sqrt{-(\triangle - V - \alpha)}^{-1}\eta$. Let $p \in (1, \infty)$ and q be the conjugate exponent of p. Then we have

$$\begin{aligned}
&|(\sqrt{-(\triangle - V - \alpha)}\omega, \eta)| \\
&\quad = |(\sqrt{-(\triangle - V - \alpha)}\omega, \sqrt{-(\triangle - V - \alpha)}\theta)| \\
&\quad = |(d\omega, d\theta) + (\delta\omega, \delta\theta) + (\sqrt{V}\omega, \sqrt{V}\theta) + \alpha(\omega, \theta)| \\
&\quad \leq \|d\omega\|_p \, \|d\theta\|_q + \|\delta\omega\|_p \, \|\delta\theta\|_q + \|\sqrt{V}\omega\|_p \, \|\sqrt{V}\theta\|_q + \alpha\|\omega\|_p \, \|\theta\|_q \\
&\quad \lesssim (\|d\omega\|_p + \|\delta\omega\|_p + \|\sqrt{V}\omega\|_p)\|\sqrt{-(\triangle - V - \alpha)}\theta\|_q + \alpha\|\omega\|_p\|\theta\|_q \\
&\quad \leq (\|d\omega\|_p + \|\delta\omega\|_p + \|\sqrt{V}\omega\|_p)\|\eta\|_q + \alpha\|\omega\|_p\|\eta\|_q.
\end{aligned}$$

Thus we have obtained

$$\|\sqrt{-(\triangle - V - \alpha)}\omega\|_p \lesssim \|d\omega\|_p + \|\delta\omega\|_p + \|\sqrt{V}\omega\|_p + \alpha\|\omega\|_p.$$

7. Proof Theorem 2.2

The second inequality in (2.4) is nothing but a triangular inequality because $-\triangle + V = d\delta + \delta d + V$.

To prove the first inequality, we need several types of equivalence of L^p norms. The first one is the special case of Theorem 2.1. If we put $V = 0$ in Theorem 2.1, then we obtain

Lemma 7.1 *For $p > 1$, $\alpha > 0$, it holds that*

$$\|d\omega\|_p + \|\delta\omega\|_p \lesssim \|\sqrt{(\alpha - \triangle)}\omega\|_p \lesssim \|\omega\|_p + \|d\omega\|_p + \|\delta\omega\|_p \qquad (7.1)$$

for all $\omega \in A_k$.

From this inequalities, we can derive the following corollary in the usual way;

Corollary 7.2 *For $p > 1$, $\alpha > 0$ it holds that and*

$$\|d\delta\omega\|_p + \|\delta d\omega\|_p \lesssim \|(\alpha - \triangle)\omega\|_p \lesssim \|\omega\|_p + \|d\delta\omega\|_p + \|\delta d\omega\|_p \qquad (7.2)$$

for all $\omega \in A_k$.

The second one is similar to the first inequality in Theorem 2.1.

Lemma 7.3 *For $p > 1$, there exists sufficiently large $\alpha > 0$ such that*

$$\|\sqrt{V}\omega\|_p \lesssim \|\sqrt{-(A - V - \alpha)}\omega\|_p \qquad (7.3)$$

for all $\omega \in A_k$.

This lemma can be proved in the same way as the first step of the proof of Theorem 2.1, so we omit the proof.

We are now in a position to prove the first inequality of (2.4). First, we prove

$$\|V\omega\|_p \lesssim \|(\triangle - V - \alpha)\omega\|_p. \qquad (7.4)$$

To show this, we note

$$\begin{aligned} \|V\omega\|_p &\lesssim \|\sqrt{-(A - V - \alpha)}\sqrt{V}\omega\|_p \\ &\lesssim \|\sqrt{V}\sqrt{-(\triangle - V - \alpha)}\omega\|_p \\ &\lesssim \|\sqrt{-(\triangle - V - \alpha)}^2\omega\|_p \\ &= \|(\triangle - V - \alpha)\omega\|_p \end{aligned}$$

where we used (7.3) in the first line, Corollary 4.5 in the second line, and Theorem 2.1 in the third line. Thus we have obtained (7.4).

Next, we prove

$$\|d\delta\omega\|_p + \|\delta d\omega\|_p \lesssim \|(\triangle - V - \alpha)\omega\|_p. \qquad (7.5)$$

To show this,

$$
\begin{aligned}
\|d\delta\omega\|_p + \|\delta d\omega\|_p &\lesssim \|(\triangle - \alpha)\omega\|_p \\
&\le \|(\triangle - V - \alpha)\omega\|_p + \|V\omega\|_p \\
&\lesssim \|(\triangle - V - \alpha)\omega\|_p,
\end{aligned}
$$

where we used (7.2) in the first line and (7.4) in the third line. Thus we have obtained (7.5), and then (7.4) and (7.5) conclude the proof of the first inequality in (2.4).

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