A Kähler metric on a based loop group and a covariant differentiation

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1. Introduction

Loop groups have been attracting many authors recently. In this paper, we are discussing a Kähler metric on a loop group. Let G be a d-dimensional compact Lie group and $\mathfrak{g}$ be its Lie algebra ($\equiv$ the space of *left* invariant vector fields). Then, $\mathfrak{g}$ admits an $\mathrm{Ad}(G)$-invariant inner product $(\cdot,\cdot)_{\mathfrak{g}}$ and we fix it through the paper. We denote the G-valued path space on $[0,1]$ by

$$PG := \{\gamma\colon [0,1]\to G\,;\text{continuous and } \gamma(0)=e\} \tag{1.1}$$

e being the unit element of G.

On the other hand, our interest is in the based loop group ΩG over G:

$$\Omega G := \{\gamma\colon [0,1]\to G\,;\text{continuous and } \gamma(0)=\gamma(1)=e\}. \tag{1.2}$$

We develop a differential geometry from an analytic point of view. In particular, we discuss several operators acting on, e.g., tensor fields. Usually, the following Cameron-Martin space H_0 is regarded as a tangent space:

$$H_0 = \left\{ \mathbf{h} : [0,1] \to \mathfrak{g} : \begin{array}{l} \mathbf{h} \text{ is absolutely continuous, } \mathbf{h}(0)=\mathbf{h}(1)=0, \text{ and} \\ \text{the derivative } \dot{\mathbf{h}} \text{ satisfies that } \int_0^1 |\dot{\mathbf{h}}(t)|_{\mathfrak{g}}^2 dt < \infty \end{array} \right\},$$

where $|\cdot|_{\mathfrak{g}} = \sqrt{(\cdot,\cdot)_{\mathfrak{g}}}$. H_0 is a Hilbert space with the inner product

$$(\mathbf{h},\mathbf{k})_{H_0} = \int_0^1 (\dot{\mathbf{h}}(t),\dot{\mathbf{k}}(t))_{\mathfrak{g}}\,dt, \qquad \mathbf{h},\mathbf{k}\in H_0.$$

Using the left translation, this inner product defines a Riemannian metric.

Further, we can introduce an almost complex structure, denoted by J, (see, [11]), but under it, the above metric is not a Kähler metric. The following Kähler form S was introduced by Pressley [11]:

$$S(\mathbf{h},\mathbf{k}) = \int_0^1 (\dot{\mathbf{h}}(t),\mathbf{k}(t))_{\mathfrak{g}}\,dt.$$

The associated Riemannian metric is defined by $B(X,Y) = S(X, JY)$. This metric was discussed by [11, 2] in view of differential geometry. We will discuss it from a probabilistic point of view, in which the pinned Brownian motion measure plays an essential role.

The organization of this paper is as follows. In the section 2, we prepare the fundamental notions of differential geometry. We define vector fields, differential forms, exterior derivatives, etc. We also introduce an almost complex structure. Kähler metric is discussed in the section 3. We will calculate the Levi-Civita covariant derivative and the associated Riemannian curvature. The section 4 is devoted to showing the closability of operators. The Ricci curvature is computed in the section 5.

2. A based loop group and an almost complex structure

In this section, we introduce several notions in differential geometry. The Cameron-Martin space H_0 is a tangent space of ΩG. Defining a bracket by $[\mathbf{h},\mathbf{k}](t) = [\mathbf{h}(t),\mathbf{k}(t)]_{\mathfrak{g}}$, $t \in [0,1]$, H_0 becomes a Lie algebra.

Thinking of H_0 as $T_{\mathbf{e}}(\Omega G)$, the tangent space of ΩG at $\mathbf{e}$, where $\mathbf{e}(s) \equiv e$, one may regard the product space $\Omega G \times H_0$ as the tangent bundle of ΩG. One then defines spaces of tensor fields on ΩG by

$$\begin{aligned}&\mathcal{F}\Gamma_b^\infty(T_q^p(\Omega G))\\ &\quad = \left\{ u : \Omega G \to H_0^{\otimes p} \otimes (H_0^*)^{\otimes q} : \begin{array}{l} u = \sum_j^{\text{finite}} f_j e_j \text{ for some } f_j \in \mathcal{F}C_b^\infty(\Omega G) \\ \text{and } e_j \in H_0^{\otimes p} \otimes (H_0^*)^{\otimes q} \end{array} \right\},\end{aligned}$$

where

$$\begin{aligned}&\mathcal{F}C_b^\infty(\Omega G)\\ &\quad = \left\{ u : \Omega G \to \mathbb{R} : \begin{array}{l} u(\gamma) = f(\gamma(t_1), \dots, \gamma(t_n)) \text{ for some} \\ f \in C_b^\infty(G^n) \text{and } 0 \le t_1 < \dots < t_n \le 1 \end{array} \right\}.\end{aligned}$$

For $X \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G))$ and $u \in \mathcal{F}C_b^\infty(\Omega G)$, set

$$Xu(\gamma) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\Big(f\big(\gamma e^{\varepsilon X(\gamma)}\big) - f(\gamma) \Big),$$

where for $\mathbf{h} \in H_0$, $\big(e^{\varepsilon \mathbf{h}}\big)(t) = e^{\varepsilon \mathbf{h}(t)}$, the position of the integral curve along $\mathbf{h}(t) \in \mathfrak{g}$ at time ε. As is easily seen, one has that

$$\begin{aligned} Xu(\gamma) &= \sum_{i=1}^n \Big(\big(X(\gamma)\big)(t_i)^{(i)} f \Big)(\gamma(t_1), \dots, \gamma(t_n)) \\ &\qquad \text{for } u(\gamma) = f(\gamma(t_1), \dots, \gamma(t_n)) \in \mathcal{F}C_b^\infty(\Omega G) \end{aligned} \tag{2.1}$$

where, for $\xi \in \mathfrak{g}$,

$$\xi^{(i)} f(g_1, \dots, g_n) = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon}\big(f(g_1, \dots, g_{i-1}, g_i e^{\varepsilon\xi}, g_{i+1}, \dots, g_n) - f(g_1, \dots, g_n)\big).$$

For $X, Y \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G))$, the Lie bracket $[X, Y]$ can be defined as a unique element of $\mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G))$ so that

$$[X, Y]u = XYu - YXu, \quad X, Y \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G)), \; u \in \mathcal{F}C^\infty(\Omega G). \quad (2.2)$$

Defining constant vector field $X^{\mathbf{h}} \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G))$, $\mathbf{h} \in H_0$, by $X^{\mathbf{h}}(\gamma) = \mathbf{h}$, $\gamma \in \Omega G$. Then $X^{\mathbf{h}}$ is a *left* invariant vector field. Due to (2.1), one has that

$$[X^{\mathbf{h}}, X^{\mathbf{k}}] = X^{[\mathbf{h},\mathbf{k}]}, \quad \mathbf{h}, \mathbf{k} \in H_0.$$

By virtue of (2.2), the Jacobi identity can be seen;

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0, \quad X, Y, Z \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G)). \quad (2.3)$$

Let

$$\mathcal{F}\Gamma_b^\infty(\textstyle\bigwedge^p T^*(\Omega G)) = \{u \in \mathcal{F}\Gamma_b^\infty(T_p^0(\Omega G)) : u(\gamma) \text{ is anti-symmetric}\}.$$

The exterior derivative du can be defined, as in the finite dimensional case, for $u \in \mathcal{F}\Gamma_b^\infty(\bigwedge^p T^*(\Omega G))$

$$\begin{aligned} du(X_1, \dots, X_{p+1}) &= \sum_{a=1}^{p+1} (-1)^{a-1} X_a\big(u(X_1, \dots, \hat{X}_a, \dots, X_{p+1})\big) \\ &\quad + \sum_{1 \le a < b \le p+1} (-1)^{a+b} u([X_a, X_b], X_1, \dots, \hat{X}_a, \dots, \hat{X}_b, \dots, X_{p+1}), \end{aligned} \quad (2.4)$$

for $X_1, \dots, X_{p+1} \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G))$, where $\hat{X}_a$ means that X_a is omitted. As in the finite dimensional case, one has that

$$d^2 = 0. \quad (2.5)$$

Indeed, define $\mathcal{L}_X$ and $\iota(X)$ by

$$\begin{aligned} \mathcal{L}_X u(X_1, \dots, X_p) &= X\big(u(X_1, \dots, X_p)\big) \\ &\quad - \sum_{a=1}^{p} u(X_1, \dots, X_{a-1}, [X, X_a], X_{a+1}, \dots, X_p), \\ \iota(X)u(X_1, \dots, X_{p-1}) &= u(X, X_1, \dots, X_{p-1}), \end{aligned}$$

for $X, X_1, \dots, X_p \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G))$, $u \in \mathcal{F}\Gamma_b^\infty(\bigwedge^p T^*(\Omega G))$.

An elementary algebraic computation leads one to the identities

$$\begin{cases} d \circ \iota(X) + \iota(X) \circ d = \mathcal{L}_X, \\ d \circ \mathcal{L}_X = \mathcal{L}_X \circ d, \qquad X \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G)). \end{cases}$$

These yield that $d^2 \circ \iota(X) = \iota(X) \circ d^2$ from which (2.5) follows by induction on p.

We now introduce an almost complex structure on ΩG following [11]. To do this, put

$$e_n(t) = \frac{1}{2\pi\sqrt{-1}n}\Big(e^{2\pi\sqrt{-1}nt} - 1\Big), \qquad n \in \mathbb{Z} \setminus \{0\}$$

and we take an orthonormal basis $\{\xi\}_{i=1,\dots,d}$ in $\mathfrak{g}$. We fix it through the paper. We use the following convention. For $\alpha = (n,i)$, $n \in \{1,2,\dots\}$, $i = 1,2,\dots,d$, we define $\bar{\alpha} = (-n,i)$, and

$$\mathbf{e}_\alpha = e_n(t)\xi_i, \quad \mathbf{e}_{\bar{\alpha}} = e_{-n}(t)\xi_i. \tag{2.6}$$

Every $\mathbf{h} \in H_0$ can be expanded as

$$\mathbf{h} = \sum_{i=1}^d \sum_{n \neq 0} (\mathbf{h}, \mathbf{e}_{-n,i})_{H_0} \mathbf{e}_{n,i} \quad \text{in } H_0,$$

where $(\mathbf{h}, \mathbf{k}_1 + \sqrt{-1}\mathbf{k}_2)_{H_0} = (\mathbf{h}, \mathbf{k}_1)_{H_0} + \sqrt{-1}(\mathbf{h}, \mathbf{k}_2)_{H_0}$, $\mathbf{h}, \mathbf{k}_1, \mathbf{k}_2 \in H_0$. An almost complex structure $J : H_0 \to H_0$ on ΩG is defined by

$$J\mathbf{h} = \sqrt{-1}\sum_{i=1}^d \sum_{n>0} (\mathbf{h}, \mathbf{e}_{-n,i})_{H_0} \mathbf{e}_{n,i} - \sqrt{-1}\sum_{i=1}^d \sum_{n>0} (\mathbf{h}, \mathbf{e}_{n,i})_{H_0} \mathbf{e}_{-n,i}.$$

See [11]. As is easily seen, it holds that

$$J^2\mathbf{h} = -\mathbf{h} \quad \text{and} \quad (J\mathbf{h}, J\mathbf{k})_{H_0} = (\mathbf{h}, \mathbf{k})_{H_0}, \quad \mathbf{h}, \mathbf{k} \in H_0.$$

Put $H_0^{\mathbb{C}} = H_0 \oplus \sqrt{-1}H_0$ and

$$H_0^{(1,0)} = \{\boldsymbol{\eta} \in H_0^{\mathbb{C}} : J\boldsymbol{\eta} = \sqrt{-1}\boldsymbol{\eta}\}, \quad H_0^{(0,1)} = \{\boldsymbol{\eta} \in H_0^{\mathbb{C}} : J\boldsymbol{\eta} = -\sqrt{-1}\boldsymbol{\eta}\}.$$

Here we extend J to $H_0^{\mathbb{C}}$ by complex linearity. Obviously $H_0^{\mathbb{C}} = H_0^{(1,0)} \oplus H_0^{(0,1)}$. $J^*\colon H_0^* \to H_0^*$ satisfies the same property and therefore $H_0^{*(1,0)}$, $H_0^{*(0,1)}$ can be defined similarly. For $\mathbf{h} \in H_0$, define

$$\pi_+\mathbf{h} = \mathbf{h}^{(1,0)} = \tfrac{1}{2}(\mathbf{h} - \sqrt{-1}J\mathbf{h}), \tag{2.7}$$

$$\pi_-\mathbf{h} = \mathbf{h}^{(0,1)} = \tfrac{1}{2}(\mathbf{h} + \sqrt{-1}J\mathbf{h}). \tag{2.8}$$

Then $\mathbf{h}^{(1,0)} \in H_0^{(1,0)}$, $\mathbf{h}^{(0,1)} \in H_0^{(0,1)}$, and $\mathbf{h} = \mathbf{h}^{(1,0)} + \mathbf{h}^{(0,1)}$. By a straightforward computation one sees that

$$\mathbf{h}^{(1,0)} = \sum_{i=1}^{d} \sum_{n>0} (\mathbf{h}, \mathbf{e}_{-n,i})_{H_0} \mathbf{e}_{n,i}, \quad \mathbf{h}^{(0,1)} = \sum_{i=1}^{d} \sum_{n>0} (\mathbf{h}, \mathbf{e}_{n,i})_{H_0} \mathbf{e}_{-n,i}. \quad (2.9)$$

Since

$$[\mathbf{e}_{n,i}, \mathbf{e}_{m,j}](t) = \frac{1}{2\pi\sqrt{-1}mn} \big\{(m+n)e_{m+n}(t) - ne_n(t) - me_m(t)\big\}[\xi_i, \xi_j],$$

one can conclude from (2.9) that

$$[\mathbf{h}^{(1,0)}, \mathbf{k}^{(1,0)}] \in H_0^{(1,0)} \quad \text{and} \quad [\mathbf{h}^{(0,1)}, \mathbf{k}^{(0,1)}] \in H_0^{(0,1)} \quad \mathbf{h}, \mathbf{k} \in H_0. \quad (2.10)$$

Let us define the Newlander-Nirenberg tensor N as follows:

$$N(X,Y) = J[X,Y] - [JX,Y] - [X,JY] - J[JX,JY].$$

By (2.10), we can easily see $N = 0$ and in this sense, J is "integrable." Put

$$\textstyle\bigwedge^{p,q} = \underbrace{H_0^{*(1,0)} \wedge \cdots \wedge H_0^{*(1,0)}}_{p\text{-times}} \wedge \underbrace{H_0^{*(0,1)} \wedge \cdots \wedge H_0^{*(0,1)}}_{q\text{-times}}$$

and define

$$\mathcal{F}\Gamma_b^\infty(\textstyle\bigwedge^{p,q} T^*(\Omega G)) = \left\{ u : \Omega G \to \textstyle\bigwedge^{p,q} : \begin{array}{l} u = \sum_i^{\text{finite}} (f_i + \sqrt{-1} g_i) e_i \\ f_i, g_i \in \mathcal{F}C_b^\infty(\Omega G), e_i \in \bigwedge^{p,q} \end{array} \right\}.$$

Observing that

$$\mathcal{F}\Gamma_b^\infty(\textstyle\bigwedge^{p,q} T^*(\Omega G)) \subset \mathcal{F}\Gamma_b^\infty(T_{p+q}^0(\Omega G)) \oplus \sqrt{-1}\mathcal{F}\Gamma_b^\infty(T_{p+q}^0(\Omega G)),$$

we can extend the exterior derivative d in (2.1) to $\mathcal{F}\Gamma_b^\infty(\bigwedge^{p,q}(T^*(\Omega G)))$, and due to (2.4), obtain that

$$du(\gamma) \in \textstyle\bigwedge^{p+1,q} \oplus \bigwedge^{p,q+1}, \ \gamma \in \Omega G, \quad \text{for } u \in \mathcal{F}\Gamma_b^\infty(\bigwedge^{p,q} T^*(\Omega G)).$$

Thus, operators

$$\partial : \mathcal{F}\Gamma_b^\infty(\textstyle\bigwedge^{p,q} T^*(\Omega G)) \to \mathcal{F}\Gamma_b^\infty(\bigwedge^{p+1,q} T^*(\Omega G))$$
$$\overline{\partial} : \mathcal{F}\Gamma_b^\infty(\textstyle\bigwedge^{p,q} T^*(\Omega G)) \to \mathcal{F}\Gamma_b^\infty(\bigwedge^{p,q+1} T^*(\Omega G))$$

can be defined so that for $u \in \mathcal{F}\Gamma_b^\infty(\bigwedge^{p,q} T^*(\Omega G))$, $\partial u(\gamma)$ (resp. $\overline{\partial} u(\gamma)$) is the projection of $du(\gamma)$ onto $\bigwedge^{p+1,q}$ (resp. $\bigwedge^{p,q+1}$). Obviously $d = \partial + \overline{\partial}$. Then, by virtue of (2.2), one obtains that

$$\partial^2 = 0, \quad \overline{\partial}^2 = 0, \quad \text{and} \quad \partial\overline{\partial} + \overline{\partial}\partial = 0 \quad (2.11)$$

on $\mathcal{F}\Gamma_b^\infty(\bigwedge^{p,q} T^*(\Omega G))$ for $p, q \geq 0$.

3. A Kähler metric

We now introduce a Kähler metric on ΩG following [11]. Define $S \in \mathcal{F}\Gamma_b^\infty(\bigwedge^2 T^*(\Omega G))$ by

$$S(X^{\mathbf{h}}, X^{\mathbf{k}}) = \int_0^1 (\dot{\mathbf{h}}(t), \mathbf{k}(t))_{\mathfrak{g}} dt, = -\int_0^1 (\mathbf{h}(t), \dot{\mathbf{k}}(t))_{\mathfrak{g}} dt, \quad \mathbf{h}, \mathbf{k} \in H_0.$$

Note that

$$|S(X^{\mathbf{h}}, X^{\mathbf{k}})| \le |\mathbf{h}|_{H_0} |\mathbf{k}|_{H_0},$$

and hence that S is well-defined. By the integration by parts and the $\mathrm{Ad}(G)$-invariance of $(\cdot,\cdot)_{\mathfrak{g}}$, we obtain the following cyclic formula:

$$S([X^{\mathbf{h}}, X^{\mathbf{k}}], X^{\mathbf{l}}) + S([X^{\mathbf{k}}, X^{\mathbf{l}}], X^{\mathbf{h}}) + S([X^{\mathbf{l}}, X^{\mathbf{h}}], X^{\mathbf{k}}) = 0.$$

Combine this with (2.1), we can show that $dS = 0$. Define

$$B(X, Y) = S(X, JY), \quad X, Y \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G)).$$

We note that $S(X, Y) = B(JX, Y)$. Let $T : H_0^{\mathbb{C}} \to H_0^{\mathbb{C}}$ be a continuous linear operator so that $T\mathbf{e}_{n,i} = \frac{1}{2|n|\pi}\mathbf{e}_{n,i}$. Observe then that

$$S(X^{\mathbf{e}_{n,i}}, X^{\mathbf{e}_{m,j}}) = \frac{1}{2\pi m\sqrt{-1}}\delta_{n,-m}\delta_{i,j},$$

and hence that

$$B(X^{\mathbf{h}}, X^{\mathbf{k}}) = (T\mathbf{h}, \mathbf{k})_{H_0}. \tag{3.1}$$

In particular, for $X, Y \in H_0$,

$$B(X, X) \ge 0 \quad \text{and "} = 0\text{" if and only if } X = 0 \tag{3.2}$$

$$B(X, Y) = B(Y, X) \in \mathbb{R}. \tag{3.3}$$

Thus one have obtained the Kähler metric B on ΩG. We denote the completion of H_0 with respect to B by H_1. From now on, we regard H_0 as a tangent space. Moreover, by noting $JT = TJ$, J also defines an almost complex structure in H_1.

We now turn to the Levi-Civita covariant derivative. As in the finite dimensional case, the Levi-Civita covariant derivative is characterized by the following identity:

$$\begin{aligned} 2B(\nabla_X Y, Z) &= XB(Y, Z) + YB(X, Z) - ZB(X, Y) \\ &\quad + B([X, Y], Z) + B([Z, X], Y) + B(X, [Z, Y]). \end{aligned}$$

In particular, taking left invariant vector fields, we have

$$2B(\nabla_{X^{\mathbf{h}}} X^{\mathbf{k}}, X^{\mathbf{l}}) = B([X^{\mathbf{h}}, X^{\mathbf{k}}], X^{\mathbf{l}}) + B([X^{\mathbf{l}}, X^{\mathbf{h}}], X^{\mathbf{k}}) + B(X^{\mathbf{h}}, [X^{\mathbf{l}}, X^{\mathbf{k}}]).$$

Furthermore, due to the identity (see, e.g., [8, Proposition IX.4.2])

$$4B((\nabla_X J)Y,Z) = 6dS((X,JY,JY) - 6dS(X,Y,Z) + B(N(Y,Z),JX),$$

we have $\nabla J = 0$, i.e., the almost complex structure is parallel.

We easily see that

$$B(\nabla_X Y,Z) = 0 \quad \text{if } Y,Z \in H^{(1,0)} \text{ or } Y,Z \in H^{(0,1)}. \tag{3.4}$$

To see this, we note that the almost complex structure J is parallel. For example, if $Y,Z \in H^{(1,0)}$, then

$$\begin{aligned} B(\nabla_X Y,Z) &= -\sqrt{-1}B(\nabla_X JY,Z) \\ &= -\sqrt{-1}B(J\nabla_X Y,Z) \\ &= \sqrt{-1}B(\nabla_X Y,JZ) \\ &= -B(\nabla_X Y,Z). \end{aligned}$$

Thus we have (3.4).

Let us calculate the covariant derivative. From the definition,

$$\begin{aligned} &B(\nabla_{X^{\mathbf{h}}} X^{\mathbf{k}}, X^{\mathbf{l}}) \\ &\quad = B([X^{\mathbf{h}},X^{\mathbf{k}}],X^{\mathbf{l}}) + B([X^{\mathbf{l}},X^{\mathbf{h}}],X^{\mathbf{k}}) + B(X^{\mathbf{h}},[X^{\mathbf{l}},X^{\mathbf{k}}]) \\ &\quad = S(X^{\mathbf{l}},J[X^{\mathbf{h}},X^{\mathbf{k}}]) + S([X^{\mathbf{l}},X^{\mathbf{h}}],JX^{\mathbf{k}}) + S([X^{\mathbf{l}},X^{\mathbf{k}}],JX^{\mathbf{h}},) \\ &\quad = \int_0^1 (\dot{\mathbf{l}}(t),J[\mathbf{h},\mathbf{k}](t))_{\mathfrak{g}}dt + \int_0^1 (\frac{d}{dt}[\mathbf{l}(t),\mathbf{h}(t)],J\mathbf{k}(t))_{\mathfrak{g}}dt \\ &\qquad + \int_0^1 (\frac{d}{dt}[\mathbf{l}(t),\mathbf{k}(t)],J\mathbf{h}(t))_{\mathfrak{g}}dt \\ &\quad = \int_0^1 (\dot{\mathbf{l}}(t),J[\mathbf{h},\mathbf{k}](t))_{\mathfrak{g}}dt + \int_0^1 ([\dot{\mathbf{l}}(t),\mathbf{h}(t)],J\mathbf{k}(t))_{\mathfrak{g}}dt \\ &\qquad + \int_0^1 ([\mathbf{l}(t),\dot{\mathbf{h}}(t)],J\mathbf{k}(t))_{\mathfrak{g}}dt + \int_0^1 ([\dot{\mathbf{l}}(t),\mathbf{k}(t)],J\mathbf{h}(t))_{\mathfrak{g}}dt \\ &\qquad + \int_0^1 ([\mathbf{l}(t),\dot{\mathbf{k}}(t)],J\mathbf{h}(t))_{\mathfrak{g}}dt \\ &\quad = \int_0^1 (\dot{\mathbf{l}}(t),J[\mathbf{h},\mathbf{k}](t))_{\mathfrak{g}}dt + \int_0^1 (\dot{\mathbf{l}}(t),[\mathbf{h}(t),J\mathbf{k}(t)])_{\mathfrak{g}}dt \\ &\qquad + \int_0^1 (\mathbf{l}(t),[\dot{\mathbf{h}}(t),J\mathbf{k}(t)])_{\mathfrak{g}}dt + \int_0^1 (\dot{\mathbf{l}}(t),[\mathbf{k}(t),J\mathbf{h}(t)])_{\mathfrak{g}}dt \\ &\qquad + \int_0^1 (\mathbf{l}(t),[\dot{\mathbf{k}}(t),J\mathbf{h}(t)])_{\mathfrak{g}}dt \\ &\quad = \int_0^1 (\dot{\mathbf{l}}(t),J[\mathbf{h},\mathbf{k}](t) + [\mathbf{h}(t),J\mathbf{k}(t)] + [\mathbf{k}(t),J\mathbf{h}(t)])_{\mathfrak{g}}dt \\ &\qquad + \int_0^1 (\mathbf{l}(t),[\dot{\mathbf{h}}(t),J\mathbf{k}(t)] + [\dot{\mathbf{k}}(t),J\mathbf{h}(t)])_{\mathfrak{g}}dt. \end{aligned}$$

First we consider the case $X^{\mathbf{h}}, X^{\mathbf{k}} \in H^{(1,0)}$ or $X^{\mathbf{h}}, X^{\mathbf{k}} \in H^{(0,1)}$. We set $JX^{\mathbf{k}} = \varepsilon\sqrt{-1}X^{\mathbf{k}}$.

$$
\begin{aligned}
&2B(\nabla_{X^{\mathbf{h}}} X^{\mathbf{k}}, X^{\mathbf{l}}) \\
&\quad = \int_0^1 (\dot{\mathbf{l}}(t), \varepsilon\sqrt{-1}[\mathbf{h}, \mathbf{k}](t) + \varepsilon\sqrt{-1}[\mathbf{h}(t), \mathbf{k}(t)] + \varepsilon\sqrt{-1}[\mathbf{k}(t), \mathbf{h}(t)])_{\mathfrak{g}} dt \\
&\qquad + \int_0^1 (\mathbf{l}(t), \varepsilon\sqrt{-1}[\dot{\mathbf{h}}(t), \mathbf{k}(t)] + \varepsilon\sqrt{-1}[\dot{\mathbf{k}}(t), \mathbf{h}(t)])_{\mathfrak{g}} dt \\
&\quad = \int_0^1 (\dot{\mathbf{l}}(t), \varepsilon\sqrt{-1}[\mathbf{h}, \mathbf{k}](t))_{\mathfrak{g}} dt \\
&\qquad + \int_0^1 (\mathbf{l}(t), \varepsilon\sqrt{-1}\frac{d}{dt}[\mathbf{h}(t), \mathbf{k}(t)] - 2\varepsilon\sqrt{-1}[\mathbf{h}(t), \dot{\mathbf{k}}(t)])_{\mathfrak{g}} dt \\
&\quad = 2\int_0^1 (\dot{\mathbf{l}}(t), \varepsilon\sqrt{-1}\int_0^t [\mathbf{h}(s), \dot{\mathbf{k}}(s)] ds)_{\mathfrak{g}} dt \\
&\quad = 2S(\mathbf{l}, \varepsilon\sqrt{-1}\int_0^{\cdot} [\mathbf{h}(s), \dot{\mathbf{k}}(s)] ds)_{\mathfrak{g}} \\
&\quad = -2B(\mathbf{l}, \varepsilon\sqrt{-1}J\int_0^{\cdot} [\mathbf{h}(s), \dot{\mathbf{k}}(s)] ds) \\
&\quad = 2B(\mathbf{l}, \int_0^{\cdot} [\mathbf{h}(s), \dot{\mathbf{k}}(s)] ds).
\end{aligned}
$$

Here we used $J\int_0^{\cdot}[\mathbf{h}(s), \dot{\mathbf{k}}(s)]ds = \varepsilon\sqrt{-1}\int_0^{\cdot}[\mathbf{h}(s), \dot{\mathbf{k}}(s)]ds$ due to the expression of $\mathbf{e}_\alpha$. Setting

$$
A(\mathbf{h}, \mathbf{k})(t) = \int_0^t [\mathbf{h}(s), \dot{\mathbf{k}}(s)] ds, \tag{3.5}
$$

we have

$$
\nabla_{X^{\mathbf{h}}} X^{\mathbf{k}} = A(\mathbf{h}, \mathbf{k}).
$$

In the case that $X^{\mathbf{h}} \in H^{(1,0)}$ and $X^{\mathbf{k}} \in H^{(0,1)}$, or $X^{\mathbf{h}} \in H^{(0,1)}$ and $X^{\mathbf{k}} \in H^{(1,0)}$, we set $JX^{\mathbf{k}} = \varepsilon\sqrt{-1}X^{\mathbf{k}}$. Then

$$
\begin{aligned}
&2B(\nabla_{X^{\mathbf{h}}} X^{\mathbf{k}}, X^{\mathbf{l}}) \\
&\quad = \int_0^1 (\dot{\mathbf{l}}(t), J[\mathbf{h}, \mathbf{k}](t) + \varepsilon\sqrt{-1}[\mathbf{h}(t), \mathbf{k}(t)] - \varepsilon\sqrt{-1}[\mathbf{k}(t), \mathbf{h}(t)])_{\mathfrak{g}} dt \\
&\qquad + \int_0^1 (\mathbf{l}(t), \varepsilon\sqrt{-1}[\dot{\mathbf{h}}(t), \mathbf{k}(t)] - \varepsilon\sqrt{-1}[\dot{\mathbf{k}}(t), \mathbf{h}(t)])_{\mathfrak{g}} dt \\
&\quad = \int_0^1 (\dot{\mathbf{l}}(t), J[\mathbf{h}, \mathbf{k}](t) + \varepsilon\sqrt{-1}[\mathbf{h}(t), \mathbf{k}(t)] - \varepsilon\sqrt{-1}[\mathbf{k}(t), \mathbf{h}(t)])_{\mathfrak{g}} dt \\
&\qquad - \int_0^1 (\dot{\mathbf{l}}(t), \varepsilon\sqrt{-1}[\mathbf{h}(t), \mathbf{k}(t)])_{\mathfrak{g}} dt \\
&\quad = S(\mathbf{l}, J[\mathbf{h}, \mathbf{k}](t) + \varepsilon\sqrt{-1}[\mathbf{h}, \mathbf{k}])
\end{aligned}
$$

$$
\begin{aligned}
&= -B(\mathbf{l}, J^2[\mathbf{h}, \mathbf{k}](t) + \varepsilon\sqrt{-1}J[\mathbf{h}, \mathbf{k}]) \\
&= B(\mathbf{l}, [\mathbf{h}, \mathbf{k}] - \varepsilon\sqrt{-1}J[\mathbf{h}, \mathbf{k}]).
\end{aligned}
$$

Thus we have (recall the definition of $\pi_\pm$ in (2.7), (2.8))

$$\nabla_{X^{\mathbf{h}}} X^{\mathbf{k}} = \frac{1}{2}\{[\mathbf{h}, \mathbf{k}] - \varepsilon\sqrt{-1}J[\mathbf{h}, \mathbf{k}]\} = \pi_\varepsilon[\mathbf{h}, \mathbf{k}].$$

We sum up into a theorem.

Theorem 3.1. *The covariant derivatives are given as follows:*

$$\nabla_{X^{\mathbf{h}}} X^{\mathbf{k}} = \begin{cases} A(\mathbf{h}, \mathbf{k}), & \text{if } \mathbf{h}, \mathbf{k} \in H^{(1,0)} \text{ or } \mathbf{h}, \mathbf{k} \in H^{(0,1)}, \\ \pi_+[\mathbf{h}, \mathbf{k}], & \text{if } \mathbf{h} \in H^{(0,1)} \text{ and } \mathbf{k} \in H^{(1,0)}, \\ \pi_-[\mathbf{h}, \mathbf{k}], & \text{if } \mathbf{h} \in H^{(1,0)} \text{ and } \mathbf{k} \in H^{(0,1)}. \end{cases}$$

Now we are ready to compute the Riemannian curvature. The Riemannian curvature is defined by

$$R(X, Y) := [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}.$$

From the definition, it is easy to see that R satisfies

$$B(R(X, Y)Z, W) = -B(R(Y, X)Z, W) = B(R(Z, W)X, Y).$$

Accordingly, the non-trivial term is $B(R(X_\alpha, X_{\bar\beta})X_\gamma, X_{\bar\delta})$ where $X_\alpha = X^{\mathbf{e}_\alpha}$, $X_{\bar\beta} = X^{\mathbf{e}_{\bar\beta}}$. The Riemannian curvature is given as follows:

Theorem 3.2. *It holds that*

$$
\begin{aligned}
&B(R(X_\alpha, X_{\bar\beta})X_\gamma, X_{\bar\delta}) \\
&\quad= -\sqrt{-1}\int_0^1 (\pi_+[\mathbf{e}_{\bar\beta}, \mathbf{e}_\gamma], [\dot{\mathbf{e}}_\alpha, \mathbf{e}_{\bar\delta}])dt - \sqrt{-1}\int_0^1 (\pi_+[\mathbf{e}_{\bar\beta}, \mathbf{e}_\gamma], [\mathbf{e}_\alpha, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&\qquad + \sqrt{-1}\int_0^1 (A(\mathbf{e}_\alpha, \mathbf{e}_\gamma), [\mathbf{e}_{\bar\beta}, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&\qquad + \sqrt{-1}\int_0^1 (\pi_+[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], [\dot{\mathbf{e}}_\gamma, \mathbf{e}_{\bar\delta}])dt - \sqrt{-1}\int_0^1 (\pi_-[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], [\mathbf{e}_\gamma, \dot{\mathbf{e}}_{\bar\delta}])dt.
\end{aligned}
$$

Proof. By the definition,

$$
\begin{aligned}
&B(R(X_\alpha, X_{\bar\beta})X_\gamma, X_{\bar\delta}) \\
&\quad= B(\nabla_{X_\alpha}\nabla_{X_{\bar\beta}}X_\gamma - \nabla_{X_{\bar\beta}}\nabla_{X_\alpha}X_\gamma - \nabla_{[X_\alpha, X_{\bar\beta}]}X_\gamma, X_{\bar\delta}) \\
&\quad= B(\nabla_{X_\alpha}\pi_+[X_{\bar\beta}, X_\gamma] - \nabla_{X_{\bar\beta}}A(X_\alpha, X_\gamma) - \nabla_{\pi_+[X_\alpha, X_{\bar\beta}]}X_\gamma \\
&\qquad - \nabla_{\pi_-[X_\alpha, X_{\bar\beta}]}X_\gamma, X_{\bar\delta}) \\
&\quad= B(A(X_\alpha, \pi_+[X_{\bar\beta}, X_\gamma]) - \pi_+[X_{\bar\beta}, A(X_\alpha, X_\gamma)] - A(\pi_+[X_\alpha, X_{\bar\beta}], X_\gamma) \\
&\qquad - \pi_+[\pi_-[X_\alpha, X_{\bar\beta}], X_\gamma], X_{\bar\delta}) \\
&\quad= -\sqrt{-1}S(A(X_\alpha, \pi_+[X_{\bar\beta}, X_\gamma]) - [X_{\bar\beta}, A(X_\alpha, X_\gamma)]
\end{aligned}
$$

$$
\begin{aligned}
&\quad - A(\pi_+[X_\alpha, X_{\bar\beta}], X_\gamma) - [\pi_-[X_\alpha, X_{\bar\beta}], X_\gamma], X_{\bar\delta}) \\
&= \sqrt{-1}\int_0^1 (A(\mathbf{e}_\alpha, \pi_+[\mathbf{e}_{\bar\beta}, \mathbf{e}_\gamma]), \dot{\mathbf{e}}_{\bar\delta})dt - \sqrt{-1}\int_0^1 ([\mathbf{e}_{\bar\beta}, A(\mathbf{e}_\alpha, \mathbf{e}_\gamma)], \dot{\mathbf{e}}_{\bar\delta})dt \\
&\quad -\sqrt{-1}\int_0^1 (A(\pi_+[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], \mathbf{e}_\gamma), \dot{\mathbf{e}}_{\bar\delta})dt - \sqrt{-1}\int_0^1 ([\pi_-[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], \mathbf{e}_\gamma], \dot{\mathbf{e}}_{\bar\delta})dt \\
&= -\sqrt{-1}\int_0^1 ([\mathbf{e}_\alpha, \frac{d}{dt}\pi_+[\mathbf{e}_{\bar\beta}, \mathbf{e}_\gamma]], \mathbf{e}_{\bar\delta})dt + \sqrt{-1}\int_0^1 (A(\mathbf{e}_\alpha, \mathbf{e}_\gamma), [\mathbf{e}_{\bar\beta}, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&\quad + \sqrt{-1}\int_0^1 ([\pi_+[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], \dot{\mathbf{e}}_\gamma], \mathbf{e}_{\bar\delta})dt - \sqrt{-1}\int_0^1 (\pi_-[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], [\mathbf{e}_\gamma, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&= \sqrt{-1}\int_0^1 (\frac{d}{dt}\pi_+[\mathbf{e}_{\bar\beta}, \mathbf{e}_\gamma], [\mathbf{e}_\alpha, \mathbf{e}_{\bar\delta}])dt + \sqrt{-1}\int_0^1 (A(\mathbf{e}_\alpha, \mathbf{e}_\gamma), [\mathbf{e}_{\bar\beta}, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&\quad + \sqrt{-1}\int_0^1 (\pi_+[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], [\dot{\mathbf{e}}_\gamma, \mathbf{e}_{\bar\delta}])dt - \sqrt{-1}\int_0^1 (\pi_-[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], [\mathbf{e}_\gamma, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&= -\sqrt{-1}\int_0^1 (\pi_+[\mathbf{e}_{\bar\beta}, \mathbf{e}_\gamma], [\dot{\mathbf{e}}_\alpha, \mathbf{e}_{\bar\delta}])dt - \sqrt{-1}\int_0^1 (\pi_+[\mathbf{e}_{\bar\beta}, \mathbf{e}_\gamma], [\mathbf{e}_\alpha, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&\quad + \sqrt{-1}\int_0^1 (A(\mathbf{e}_\alpha, \mathbf{e}_\gamma), [\mathbf{e}_{\bar\beta}, \dot{\mathbf{e}}_{\bar\delta}])dt \\
&\quad + \sqrt{-1}\int_0^1 (\pi_+[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], [\dot{\mathbf{e}}_\gamma, \mathbf{e}_{\bar\delta}])dt - \sqrt{-1}\int_0^1 (\pi_-[\mathbf{e}_\alpha, \mathbf{e}_{\bar\beta}], [\mathbf{e}_\gamma, \dot{\mathbf{e}}_{\bar\delta}])dt
\end{aligned}
$$

which completes the proof. □

4. Closability

We will show the closability of operators that were introduced in the previous sections. Operators were considered in the framework of L^2 theory. To define an L^2 space, we need a measure. So we begin with introducing a measure on the path space PG. To do this, let us consider the following stochastic differential equation. We consider the following stochastic differential equation on G:

$$
\begin{cases} d\gamma_t &= \sum_{i=1}^d \xi_i(\gamma_t) \circ db_t^i, \\ \gamma_0 &= e \end{cases} \tag{4.1}
$$

where $(b_t^1, \dots, b_t^d)_{t \in [0,T]}$ is a d-dimensional Brownian motion, and $\circ$ stands for the Stratonovich symmetric stochastic integral. Setting $b_t = \sum_i b_t^i \xi_i$, (b_t) is a Brownian motion on $\mathfrak{g}$. (b_t) induces a measure on the space $P\mathfrak{g}$ where $P\mathfrak{g}$ is the $\mathfrak{g}$-valued path space. The measure is called the Wiener measure and is denoted by P^W. There exists the unique strong solution to (4.1), i.e., there exists a measurable function $I \colon P\mathfrak{g} \to PG$ such that $\gamma = I(b)$ is the unique solution to (4.1). We call this map I the Itô map. We denote the image measure of P^W under I by μ. Then $(P\mathfrak{g}, P^W) \cong (PG, \mu)$ as measure spaces. We sometimes regard a function on (PG, μ) as a function on $(P\mathfrak{g}, P^W)$. We can restrict the measure to ΩG by taking a conditional probability. We set $m = E[\,\cdot\,|\gamma(1) = e]$. m is called a pinned measure or a bridge measure.

For function $u \in \mathcal{F}C_b^\infty(\Omega G)$, du is characterized by

$$\langle du, X\rangle = Xu, \quad \text{for } X \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G)).$$

In the section 2, X was an H_0-valued function. But we have replaced H_0 by H_1. Since $H_0 \subseteq H_1$, we have $H_1^* \subseteq H_0^*$ and du must be in H_1^*. Let us give an example of such a function. Set

$$u(\gamma) = \int_0^1 \varphi(t) f(\gamma(t))dt$$

where $\varphi \in C^\infty([0,1])$ and $f \in C^\infty(M)$. Then clearly

$$\langle du(\gamma), \mathbf{h}\rangle = \int_0^1 \varphi(t)\langle df(\gamma(t)), \mathbf{h}(t)\rangle dt.$$

It is now easy to see that $du(\gamma) \in H_1^*$. Since those functions are dense in $L^2(\Omega G)$, we can define the following pre-Dirichlet form:

$$\mathcal{E}(u, v) := \int_{\Omega G} (du, dv)_{H_1^*} m(d\gamma). \tag{4.2}$$

The closability of the pre-Dirichlet form is a fundamental problem. It is equivalent to the closability of the operator d. To show the closability, we are enough to show the existence of the dual operator.

Since $|\mathbf{h}|_{H_1} = |\sqrt{T}\mathbf{h}|_{H_0}$, we easily show that $\boldsymbol{\theta} \in H_1^*$ if and only if $\boldsymbol{\theta} \in \mathrm{Dom}(\sqrt{T^{*-1}})$ and

$$|\boldsymbol{\theta}|_{H_1^*} = |\sqrt{T^{*-1}}\boldsymbol{\theta}|_{H_0^*},$$

where $T^*\colon H_0^* \to H_0^*$ is the dual operator of T. We also notice that

$$(\mathbf{h}, \mathbf{k})_{H_1} = (T\mathbf{h}, \mathbf{k})_{H_0} \quad \text{for } \mathbf{h}, \mathbf{k} \in H_1$$

and similarly

$$(\boldsymbol{\theta}, \boldsymbol{\eta})_{H_1^*} = (T^{*-1}\boldsymbol{\theta}, \boldsymbol{\eta})_{H_0^*} \quad \text{for } \boldsymbol{\theta} \in \mathrm{Ran}(T^*),\ \boldsymbol{\eta} \in H_1^*.$$

Now we have

$$\int_{\Omega G} (du, \boldsymbol{\theta})_{H_1^*} dm = \int_{\Omega G} (du, T^{*-1}\boldsymbol{\theta})_{H_0^*} dm = \int_{\Omega G} u d'(T^{*-1}\boldsymbol{\theta}) dm$$

where d' is the dual operator with respect to the inner product $(\ ,\ )_{H_0}$. This implies that $d^* = d'T^{*-1}$.

We shall give a rather explicit expression of d^*. We recall the notion of divergence. For $X \in \mathcal{F}\Gamma_b^\infty(T_0^1(\Omega G))$, $\operatorname{div} X$ is characterized by the following identity:

$$\int_{\Omega G} Xudm = -\int_{\Omega G} (\operatorname{div} X)udm$$

If X is left invariant, we can give an explicit formula of $\operatorname{div} X$:

$$\operatorname{div} X^{\mathbf{h}} = -\int_0^1 (\dot{\mathbf{h}}(s), db(s))_{\mathfrak{g}}.$$

Notice that this is valid only for $\mathbf{h} \in H_0$. Because we are given a Riemannian metric, a 1-form can be identified with a vector field, i.e., for any 1-form ω, there exists a unique vector field, which we denote by $\omega^\sharp$, such that

$$\langle \omega, X \rangle = B(\omega^\sharp, X).$$

We take $\boldsymbol{\theta} \in H_1^*$ (a constant 1-form). Then,

$$\int_{\Omega G} (du, \boldsymbol{\theta})_{H_1^*} dm = \int_{\Omega G} \langle du, \boldsymbol{\theta}^\sharp \rangle dm = \int_{\Omega G} \boldsymbol{\theta}^\sharp udm = -\int_{\Omega G} (\operatorname{div} \boldsymbol{\theta}^\sharp)udm$$

which implies $d^*\boldsymbol{\theta} = -\operatorname{div} \boldsymbol{\theta}^\sharp$. Of course, we have to assume that $\boldsymbol{\theta}^\sharp \in H_0$. Notice that $\boldsymbol{\theta}^\sharp \in H_0$ if and only if $\boldsymbol{\theta} \in T^*(H_0^*)$ and in this case $|\boldsymbol{\theta}^\sharp|_{H_0} = |T^{*-1}\boldsymbol{\theta}|_{H_0^*}$.

It is now easy to see that $\varphi = u\boldsymbol{\theta} \in \operatorname{Dom}(d^*)$ if $u \in \mathcal{F}C_b^\infty(\Omega G)$ and $\boldsymbol{\theta} \in \operatorname{Dom}(T^{*-1})$. Here and after, we identify an element of H_1^* with a constant 1-form (i.e., a left invariant 1-form) for notational simplicity. Hence $\operatorname{Dom}(d^*)$ is dense and thereby $\mathcal{E}$ is closable. It is easy to see that the closure of $\mathcal{E}$ is a Dirichlet form.

The closability of the exterior differentiation is valid for general p-forms. Of course, we have replaced H_0 by H_1 in the argument in the section 2. Hence p-form is a function on ΩG taking values in $\bigwedge^p = H_1^* \wedge \cdots \wedge H_1^*$. Now we notice that T^* can be naturally extended to the tensor product of H_1^* as follows:

$$\Gamma_k(T^*)(\boldsymbol{\theta}_1 \otimes \cdots \otimes \boldsymbol{\theta}_k) = (T^*\boldsymbol{\theta}_1) \otimes \cdots \otimes (T^*\boldsymbol{\theta}_k).$$

We also have

$$\begin{aligned} &(\boldsymbol{\theta}_1 \otimes \cdots \otimes \boldsymbol{\theta}_k, \boldsymbol{\eta}_1 \otimes \cdots \otimes \boldsymbol{\eta}_k)_{H_1^{*\otimes k}} \\ &\quad = (\Gamma_k(T^{*-1})\boldsymbol{\theta}_1 \otimes \cdots \otimes \boldsymbol{\theta}_k, \boldsymbol{\eta}_1 \otimes \cdots \otimes \boldsymbol{\eta}_k)_{H_0^{*\otimes k}} \end{aligned}$$

for $\boldsymbol{\theta}_i \in \operatorname{Ran}(T^*)$, $\boldsymbol{\eta}_i \in H_1^*$.

Proposition 4.1. *The dual operator of d in $\bigwedge^k$ is given by*

$$d^* = \Gamma_{k-1}(T^*)d'\Gamma_k(T^{*-1}).$$

Proof. It is easy to see that

$$\begin{aligned}\int_{\Omega G}\frac{1}{k!}(\omega,d\varphi)_{H_1^{*\otimes k}}dm &= \int_{\Omega G}\frac{1}{k!}(\Gamma_k(T^{*-1})\omega,d\varphi)_{H_0^{*\otimes k}}dm\\ &= \int_{\Omega G}\frac{1}{(k-1)!}(d'\Gamma_k(T^{*-1})\omega,\varphi)_{H_0^{*\otimes k}}dm\\ &= \int_{\Omega G}\frac{1}{(k-1)!}(\Gamma_{k-1}(T^*)d'\Gamma_k(T^{*-1})\omega,\varphi)_{H_1^{*\otimes k}}dm.\end{aligned}$$

This implies the assertion. □

The closability of ∂, $\bar{\partial}$ can be shown in a similar manner.

Lastly we consider the covariant differentiation. We show the closability of covariant differentiation for 1-forms. Let ω be a 1-form, i.e., H_1^*-valued function on ΩG. Recall that the cotangent bundle $T^*(\Omega G)$ is identified with $\Omega G\times H_1^*$. Hence a section of $T^*(\Omega G)$ is nothing but a H_1^*-valued function. We define the covariant derivative of 1-form ω by

$$\langle\nabla_X\omega,Y\rangle=X\langle\omega,Y\rangle-\langle\omega,\nabla_XY\rangle.$$

Setting $\nabla\omega(X,Y)=\langle\nabla_X\omega,Y\rangle$, ∇ is a differential operator from $\Gamma(T_1^0(\Omega G))$ to $\Gamma(T_2^0(\Omega G))$. Let $(\mathbf{e}_\alpha,\mathbf{e}_{\bar{\alpha}})$ be a basis in H_0 defined by (2.6). If $\boldsymbol{\theta}\in H_1^{*(1,0)}$, then, by Theorem 3.1, we have $\nabla\boldsymbol{\theta}(\cdot,\mathbf{e}_{\bar{\beta}})=0$. Further, we have

$$\begin{aligned}\nabla\boldsymbol{\theta}(\mathbf{e}_\alpha,\mathbf{e}_\beta) &= -\boldsymbol{\theta}(A(\mathbf{e}_\alpha,\mathbf{e}_\beta))\\ \nabla\boldsymbol{\theta}(\mathbf{e}_{\bar{\alpha}},\mathbf{e}_\beta) &= -\boldsymbol{\theta}([\mathbf{e}_{\bar{\alpha}},\mathbf{e}_\beta]).\end{aligned}$$

Similar formula holds for $\boldsymbol{\theta}\in H_1^{*(0,1)}$ (just take a complex conjugate).

We will obtain the dual operator of ∇ acting on 1-forms. To do this, we recall the definition of the divergence operator, which is essentially same as the dual operator of d. Now, taking 1-forms $\boldsymbol{\theta}$, $\boldsymbol{\eta}$ and $\boldsymbol{\xi}\in H_1^*$ (constant 1-forms), we have

$$\begin{aligned}\int_{\Omega G}(\boldsymbol{\theta}\otimes\boldsymbol{\eta},\nabla\boldsymbol{\xi})_{H_1^*\otimes H_1^*}dm &= \int_{\Omega G}(\boldsymbol{\eta},\nabla_{\boldsymbol{\theta}^\sharp}\boldsymbol{\xi})_{H_1^*}dm\\ &= \int_{\Omega G}\{\boldsymbol{\theta}^\sharp(\boldsymbol{\eta},\boldsymbol{\xi})_{H_1^*}-(\nabla_{\boldsymbol{\theta}^\sharp}\boldsymbol{\eta},\boldsymbol{\xi})_{H_1^*}\}dm\\ &= \int_{\Omega G}(-(\operatorname{div}\boldsymbol{\theta}^\sharp)\boldsymbol{\eta}-\nabla_{\boldsymbol{\theta}^\sharp}\boldsymbol{\eta},\boldsymbol{\xi})_{H_1^*}dm.\end{aligned}$$

Thus we have

$$\nabla^*(\boldsymbol{\theta}\otimes\boldsymbol{\eta})=-(\operatorname{div}\boldsymbol{\theta}^\sharp)\boldsymbol{\eta}-\nabla_{\boldsymbol{\theta}^\sharp}\boldsymbol{\eta}.$$

Now we can see that the domain of ∇^* is dense.

5. The Ricci curvature

We have obtained the Riemannian curvature. In this section we will get the Ricci curvature. We adopt the following definition of Ricci curvature.

$$dd^* + d^*d = \nabla^*\nabla + \mathrm{Ric} \tag{5.1}$$

To compute the Ricci curvature, we recall the following fact. For $\mathbf{h} \in H_0$, set

$$u = \int_0^1 (\dot{\mathbf{h}}(t), db(t))_{\mathfrak{g}}.$$

Then we have (see, e.g., Gross [5, Lemma 3.5])

$$\begin{aligned} \langle du, \mathbf{k} \rangle &= \int_0^1 (\dot{\mathbf{h}}(t), \dot{\mathbf{k}}(t))_{\mathfrak{g}} dt + \int_0^1 ([\dot{\mathbf{h}}(t), \mathbf{k}(t)], db(t))_{\mathfrak{g}} \\ &= (T^{-1}\mathbf{h}, \mathbf{k})_{H_1} + \int_0^1 ([\dot{\mathbf{h}}(t), \mathbf{k}(t)], db(t))_{\mathfrak{g}}. \end{aligned} \tag{5.2}$$

We prepare a proposition for later use.

Proposition 5.1. *Set Define Q and $\tilde{Q}$ as follows.*

$$Q(\mathbf{h}, \mathbf{k}) := \sqrt{-1} \sum_\beta \int_0^1 ([\mathbf{h}, \dot{\mathbf{e}}_\beta], \pi_+[\mathbf{k}, T^{-1}\mathbf{e}_{\bar{\beta}}])_{\mathfrak{g}} dt$$

for $\mathbf{h} \in H_0^{(0,1)}$ and $\mathbf{k} \in H_0^{(1,0)}$, and

$$\tilde{Q}(\mathbf{h}, \mathbf{k}) = \sqrt{-1} \sum_\beta \int_0^1 ([\mathbf{e}_\beta, \dot{\mathbf{h}}], A(T^{-1}\mathbf{e}_{\bar{\beta}}, \mathbf{k}))_{\mathfrak{g}} dt$$

for $\mathbf{h}, \mathbf{k} \in H_0^{(0,1)}$. Then it holds that

$$Q(\mathbf{h}, \mathbf{k}) = \frac{1}{2\pi} \int_0^1 K(J\mathbf{h}, \dot{\mathbf{k}}) dt, \quad \text{for } \mathbf{h} \in H_0^{(0,1)} \text{ and } \mathbf{k} \in H_0^{(1,0)}, \tag{5.3}$$

$$\tilde{Q}(\mathbf{h}, \mathbf{k}) = 0, \quad \text{for } \mathbf{h}, \mathbf{k} \in H_0^{(0,1)} \tag{5.4}$$

where K is the Killing form: $K(\xi, \eta) = \mathrm{tr}_{N(S)}(\mathrm{ad}\xi \mathrm{ad}\eta)$.

Proof. We compute Q by using a basis $(\mathbf{e}_\alpha, \mathbf{e}_{\bar{\alpha}})$.

$$
\begin{aligned}
&Q(\mathbf{e}_{-m,j},\mathbf{e}_{l,k})\\
&= \sqrt{-1}\sum_{i=1}^{d}\sum_{n}\int_0^1([\mathbf{e}_{-m,j},,\dot{\mathbf{e}}_{n,i}],\pi_+[\mathbf{e}_{l,k},T^{-1}\mathbf{e}_{-n,i}])_{\mathfrak{g}}dt\\
&= \sqrt{-1}\sum_{i=1}^{d}\sum_{n}\int_0^1(e_{-m}\dot{e}_n[\xi_j,\xi_i],(2\pi n)\pi_+\{e_l e_{-n}[\xi_k,\xi_i]\})_{\mathfrak{g}}dt\\
&= \sqrt{-1}\sum_{i=1}^{d}\sum_{n}\int_0^1(\frac{1}{-2\pi m\sqrt{-1}}(e^{-2\pi\sqrt{-1}mt}-1)e^{2\pi\sqrt{-1}nt}[\xi_j,\xi_i],\\
&\quad\times(2\pi n)\frac{1}{(2\pi\sqrt{-1}l)(-2\pi\sqrt{-1}n)}\\
&\quad\times\pi_+\{(e^{2\pi\sqrt{-1}(l-n)t}-1)-(e^{2\pi\sqrt{-1}l}-1)-(e^{-2\pi\sqrt{-1}n}-1)\}[\xi_k,\xi_i])_{\mathfrak{g}}dt\\
&= \sum_n\frac{1}{4\pi^2 ml}\int_0^1(e^{-2\pi\sqrt{-1}mt}-1)e^{2\pi\sqrt{-1}nt}\\
&\quad\times\{1_{\{l>n\}}(e^{2\pi\sqrt{-1}(l-n)t}-1)-(e^{2\pi\sqrt{-1}l}-1)\}K(\xi_j,\xi_k)dt\\
&= \sum_n\frac{1}{4\pi^2 ml}\int_0^1\big\{1_{\{l>n\}}(e^{2\pi\sqrt{-1}(l-m)t}-e^{2\pi\sqrt{-1}(n-m)t})\\
&\quad-(e^{2\pi\sqrt{-1}(n-m+l)}-e^{2\pi\sqrt{-1}(n-m)t})\big\}K(\xi_j,\xi_k)dt\\
&= \left\{\frac{1}{4\pi^2 ml}(l-1)\delta_{l,m}-\frac{1}{4\pi^2 ml}1_{\{l>m\}}\right.\\
&\quad\left.-\frac{1}{4\pi^2 ml}1_{\{m>l\}}+\frac{1}{4\pi^2 ml}\right\}K(\xi_i,\xi_j)\\
&= \left\{\frac{1}{4\pi^2 ml}(l-1)\delta_{l,m}+\frac{1}{4\pi^2 ml}\delta_{l,m}\right\}K(\xi_i,\xi_j)\\
&= \frac{1}{4\pi^2 m}\delta_{l,m}K(\xi_i,\xi_j).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
&\frac{1}{2\pi}\int_0^1 K(J\mathbf{e}_{-m,j},\dot{\mathbf{e}}_{l,k})dt\\
&\qquad= \frac{-\sqrt{-1}}{2\pi}\int_0^1\frac{1}{-2\pi\sqrt{-1}m}(e^{-2\pi\sqrt{-1}mt}-1)e^{2\pi\sqrt{-1}lt}K(\xi_j,\xi_k)dt\\
&\qquad= \frac{1}{4\pi^2 m}\delta_{l,m}K(\xi_i,\xi_j).
\end{aligned}
$$

Now we have (5.3). Next we show (5.4)

$$
\begin{aligned}
&\tilde{Q}(\mathbf{e}_{-m,j},\mathbf{e}_{-l,k})\\
&= \sqrt{-1}\sum_{n,i}\int_0^1\big([e_n(t)\xi_i,\dot{e}_{-m}(t)\xi_j],\int_0^t 2\pi n[e_{-n}(s)\xi_i,\dot{e}_{-l}(s)\xi_k]ds\big)_{\mathfrak{g}}dt
\end{aligned}
$$

$$\begin{aligned}
&= \sqrt{-1}\sum_{n,i}\int_0^1 \Big(\frac{1}{2\pi\sqrt{-1}n}(e^{2\pi\sqrt{-1}nt}-1)e^{-2\pi\sqrt{-1}mt}[\xi_i,\xi_j], \\
&\quad \Big\{\int_0^t (2\pi n)\frac{1}{-2\pi\sqrt{-1}n}(e^{-2\pi\sqrt{-1}ns}-1)e^{-2\pi\sqrt{-1}ls}[\xi_i,\xi_k]ds\Big\}\Big)_{\mathfrak{g}}dt \\
&= -\sqrt{-1}\sum_n K(\xi_j,\xi_k)\int_0^1 \frac{1}{2\pi n}(e^{2\pi\sqrt{-1}nt}-1)e^{-2\pi\sqrt{-1}mt} \\
&\quad \times\Big\{\frac{-1}{2\pi\sqrt{-1}(n+l)}(e^{-2\pi\sqrt{-1}(n+l)t}-1)+\frac{1}{2\pi\sqrt{-1}l}(e^{-2\pi\sqrt{-1}lt}-1)\Big\}dt \\
&= -\sqrt{-1}\sum_n K(\xi_j,\xi_k) \\
&\quad \times\int_0^1\Big\{\frac{-1}{(2\pi n)(2\pi\sqrt{-1}(n+l)}(e^{2\pi\sqrt{-1}(n-m-n-l)t}-e^{2\pi\sqrt{-1}(n-m)t}) \\
&\quad +\frac{1}{(2\pi n)(2\pi\sqrt{-1}l)}(e^{2\pi\sqrt{-1}(n-m-l)t}-e^{2\pi\sqrt{-1}(n-m)t})\Big\}dt \\
&= \sum_n K(\xi_j,\xi_k)\Big\{\frac{-\delta_{n,m}}{4\pi^2 n(n+l)}-\frac{\delta_{n,m+l}}{4\pi^2 nl}+\frac{\delta_{n,m}}{4\pi^2 nl}\Big\} \\
&= K(\xi_j,\xi_k)\Big\{\frac{-1}{4\pi^2 m(m+l)}-\frac{1}{4\pi^2(m+l)l}+\frac{1}{4\pi^2 ml}\Big\} \\
&= 0.
\end{aligned}$$

This completes the proof. □

The Ricci curvature is given by the following theorem.

Theorem 5.1. *Set*

$$\begin{aligned}
\mathrm{Ric}(\mathbf{h},\mathbf{k}) &= (T^{-1}\mathbf{h},\mathbf{k})_{H_1}+\frac{1}{2\pi}\int_0^1 K(J\mathbf{h},\dot{\mathbf{k}})dt+\Big(\int_0^1[\dot{\mathbf{h}}(u),\mathbf{k}(u)]du,b(1)\Big)_{\mathfrak{g}} \\
&\quad +\int_0^1(\rho_+[\dot{\mathbf{h}},\mathbf{k}],db(t))_{\mathfrak{g}}-\int_0^1(\rho_-[\mathbf{h},\dot{\mathbf{k}}],db(t))_{\mathfrak{g}} \qquad (5.5)
\end{aligned}$$

for $\mathbf{h}\in H_0^{(0,1)}$ *and* $\mathbf{k}\in H_0^{(1,0)}$, *and*

$$\mathrm{Ric}(\mathbf{h},\mathbf{k})=0 \qquad (5.6)$$

for $\mathbf{h},\mathbf{k}\in H_0^{(0,1)}$. *If* $\mathbf{h}\in H_0^{(1,0)}$, *we define* Ric *by complex conjugation. Then identity (5.1) holds. Here* $\rho_\pm$ *in (5.5) is defined by*

$$\begin{aligned}
\rho_+\mathbf{l} &= \sum_\beta\Big\{\int_0^1\mathbf{l}(t)\dot{\mathbf{e}}_{\bar\beta}(s)ds\Big\}\dot{\mathbf{e}}_\beta \qquad (5.7) \\
\rho_-\mathbf{l} &= \sum_\beta\Big\{\int_0^1\mathbf{l}(t)\dot{\mathbf{e}}_\beta(s)ds\Big\}\dot{\mathbf{e}}_{\bar\beta}. \qquad (5.8)
\end{aligned}$$

Proof. We first note that $d\boldsymbol{\eta}(X,Y) = \nabla\boldsymbol{\eta}(X,Y) - \nabla\boldsymbol{\eta}(Y,X)$. So we set $\check{\nabla}\boldsymbol{\eta}(X,Y) = \nabla\boldsymbol{\eta}(Y,X)$. Since $d^* = \nabla^*$, we have

$$\begin{aligned} dd^*\boldsymbol{\eta} + d^*d\boldsymbol{\eta} - \nabla^*\nabla\boldsymbol{\eta} &= dd^*\boldsymbol{\eta} + \nabla^*(\nabla\boldsymbol{\eta} - \check{\nabla}\eta) - \nabla^*\nabla\boldsymbol{\eta} \\ &= dd^*\boldsymbol{\eta} - \nabla^*\check{\nabla}\eta. \end{aligned}$$

Set $\boldsymbol{\eta}^\sharp = \mathbf{h}$, $u = \int_0^1 (\dot{\mathbf{h}}(s), db(s))_{\mathfrak{g}}$. Note that $\mathbf{h} \in H_0^{(0,1)}$ since $\boldsymbol{\eta} \in H_1^{*(1,0)}$ (we have assumed $\mathbf{h} \in H_0^{\mathbb{C}}$). As we saw in the previous section, $d^*\boldsymbol{\eta}$ is given by

$$d^*\boldsymbol{\eta} = -\operatorname{div}\boldsymbol{\eta}^\sharp = u$$

and hence

$$dd^*\boldsymbol{\eta} = du.$$

Let $(\boldsymbol{\theta}^\alpha, \boldsymbol{\theta}^{\bar{\alpha}})$ be a dual basis of $(\mathbf{e}_\alpha, \mathbf{e}_{\bar{\alpha}})$, i.e., $\langle\boldsymbol{\theta}^\alpha, \mathbf{e}_\beta\rangle = \delta^\alpha_\beta$, $\langle\boldsymbol{\theta}^\alpha, \mathbf{e}_{\bar{\beta}}\rangle = 0$, etc. By noting $\nabla\boldsymbol{\eta}(\cdot, \mathbf{e}_{\bar{\beta}}) = 0$, we can expand $\nabla\boldsymbol{\eta}$ as

$$\nabla\boldsymbol{\eta} = \sum_{\alpha,\beta}\{\nabla\boldsymbol{\eta}(\mathbf{e}_\alpha, \mathbf{e}_\beta)\boldsymbol{\theta}^\alpha \otimes \boldsymbol{\theta}^\beta + \nabla\boldsymbol{\eta}(\mathbf{e}_{\bar{\alpha}}, \mathbf{e}_\beta)\boldsymbol{\theta}^{\bar{\alpha}} \otimes \boldsymbol{\theta}^\beta\}.$$

Reversing the order, we have

$$\check{\nabla}\boldsymbol{\eta} = \sum_{\alpha,\beta}\{-\langle\boldsymbol{\eta}, A(\mathbf{e}_\alpha, \mathbf{e}_\beta)\rangle\boldsymbol{\theta}^\beta \otimes \boldsymbol{\theta}^\alpha - \langle\boldsymbol{\eta}, \pi_+[\mathbf{e}_{\bar{\alpha}}, \mathbf{e}_\beta]\rangle\boldsymbol{\theta}^\beta \otimes \boldsymbol{\theta}^{\bar{\alpha}}\}.$$

Since $(\boldsymbol{\theta}^\alpha)^\sharp = T^{-1}\mathbf{e}_{\bar{\alpha}}$, we have

$$\nabla^*(\boldsymbol{\theta}^\beta \otimes \boldsymbol{\theta}^\alpha) = -(\operatorname{div} T^{-1}\mathbf{e}_{\bar{\beta}})\boldsymbol{\theta}^\alpha - \nabla_{T^{-1}\mathbf{e}_{\bar{\beta}}}\boldsymbol{\theta}^\alpha.$$

Therefore

$$\begin{aligned} \nabla^*\check{\nabla}\boldsymbol{\eta} &= \sum_{\alpha,\beta}\{(\operatorname{div} T^{-1}\mathbf{e}_{\bar{\beta}})\boldsymbol{\theta}^\alpha + \nabla_{T^{-1}\mathbf{e}_{\bar{\beta}}}\boldsymbol{\theta}^\alpha\}\langle\boldsymbol{\eta}, A(\mathbf{e}_\alpha, \mathbf{e}_\beta)\rangle \\ &\quad + \sum_{\alpha,\beta}\{(\operatorname{div} T^{-1}\mathbf{e}_{\bar{\beta}})\boldsymbol{\theta}^{\bar{\alpha}} + \nabla_{T^{-1}\mathbf{e}_{\bar{\beta}}}\boldsymbol{\theta}^{\bar{\alpha}}\}\langle\boldsymbol{\eta}, \pi_+[\mathbf{e}_{\bar{\alpha}}, \mathbf{e}_\beta]\rangle. \end{aligned}$$

Hence, if $\mathbf{k} \in H_0^{(1,0)}$,

$$\begin{aligned} &\langle dd^*\boldsymbol{\eta} - \nabla^*\check{\nabla}\boldsymbol{\eta}, \mathbf{k}\rangle \\ &\quad = \langle du - \nabla^*\check{\nabla}\boldsymbol{\eta}, \mathbf{k}\rangle \\ &\quad = (\mathbf{h}, \mathbf{k})_{H_0} + \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], db(t))_{\mathfrak{g}} - \sum_{\alpha,\beta}\{(\operatorname{div} T^{-1}\mathbf{e}_{\bar{\beta}})\langle\boldsymbol{\theta}^\alpha, \mathbf{k}\rangle\langle\boldsymbol{\eta}, A(\mathbf{e}_\alpha, \mathbf{e}_\beta)\rangle \\ &\qquad - \sum_{\alpha,\beta}\langle\nabla_{T^{-1}\mathbf{e}_{\bar{\beta}}}\boldsymbol{\theta}^\alpha, \mathbf{k}\rangle\langle\boldsymbol{\eta}, A(\mathbf{e}_\alpha, \mathbf{e}_\beta)\rangle\} \\ &\quad = (T^{-1}(\boldsymbol{\eta}^\sharp), \mathbf{k})_{H_1} + \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], db(t))_{\mathfrak{g}} - \sum_{\beta}(\operatorname{div} T^{-1}\mathbf{e}_{\bar{\beta}})\langle\boldsymbol{\eta}, A(\mathbf{k}, \mathbf{e}_\beta)\rangle \end{aligned}$$

$$
\begin{aligned}
&\quad + \sum_{\alpha,\beta} \langle \boldsymbol{\theta}^\alpha, \pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}]\rangle \langle \boldsymbol{\eta}, A(\mathbf{e}_\alpha, \mathbf{e}_\beta)\rangle \\
&= \langle T^{*-1}\boldsymbol{\eta}, \mathbf{k}\rangle + \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], db(t))_{\mathfrak{g}} \\
&\quad - \sum_\beta (\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta}) \langle \boldsymbol{\eta}, A(\mathbf{k}, \mathbf{e}_\beta)\rangle + \sum_\beta \langle \boldsymbol{\eta}, A(\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \mathbf{e}_\beta)\rangle \}.
\end{aligned}
$$

We first compute the last term.

$$
\begin{aligned}
\langle \boldsymbol{\eta}, A(\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \mathbf{e}_\beta)\rangle &= (\mathbf{h}, A(\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \mathbf{e}_\beta))_{H_1} \\
&= \sqrt{-1}(J\mathbf{h}, A(\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \mathbf{e}_\beta))_{H_1} \\
&= \sqrt{-1} S(\mathbf{h}, A(\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \mathbf{e}_\beta)) \\
&= -\sqrt{-1} \int_0^1 (\mathbf{h}, \frac{d}{dt} A(\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \mathbf{e}_\beta))_{\mathfrak{g}} dt \\
&= -\sqrt{-1} \int_0^1 (\mathbf{h}, [\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \dot{\mathbf{e}}_\beta])_{\mathfrak{g}} dt \\
&= \sqrt{-1} \int_0^1 ([\mathbf{h}, \dot{\mathbf{e}}_\beta], \pi_+[\mathbf{k}, T^{-1}\mathbf{e}_{\bar\beta}])_{\mathfrak{g}} dt.
\end{aligned}
$$

By virtue of Proposition 5.1, we can get

$$
\sum_\beta \langle \boldsymbol{\eta}, A(\pi_+[T^{-1}\mathbf{e}_{\bar\beta}, \mathbf{k}], \mathbf{e}_\beta)\rangle = \frac{1}{2\pi} \int_0^1 K(J\mathbf{h}, \dot{\mathbf{k}}) dt.
$$

Now we turn to the stochastic integral.

$$
\begin{aligned}
(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})\langle \boldsymbol{\eta}, A(\mathbf{k}, \mathbf{e}_\beta)\rangle &= (\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})(\mathbf{h}, A(\mathbf{k}, \mathbf{e}_\beta))_{H_1} \\
&= \sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})(J\mathbf{h}, A(\mathbf{k}, \mathbf{e}_\beta))_{H_1} \\
&= \sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta}) S(\mathbf{h}, A(\mathbf{k}, \mathbf{e}_\beta)) \\
&= -\sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta}) \int_0^1 (\mathbf{h}, [\mathbf{k}, \dot{\mathbf{e}}_\beta])_{\mathfrak{g}} dt \\
&= -\sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta}) \int_0^1 ([\mathbf{h}, \mathbf{k}], \dot{\mathbf{e}}_\beta)_{\mathfrak{g}} dt \\
&= -\sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta}) S(\mathbf{e}_\beta, [\mathbf{h}, \mathbf{k}]) \\
&= -\sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})(J\mathbf{e}_\beta, [\mathbf{h}, \mathbf{k}])_{H_1} \\
&= (\operatorname{div} \mathbf{e}_{\bar\beta})(T^{-1}\mathbf{e}_\beta, [\mathbf{h}, \mathbf{k}])_{H_1} \\
&= (\operatorname{div} \mathbf{e}_{\bar\beta})(\mathbf{e}_\beta, [\mathbf{h}, \mathbf{k}])_{H_0} \\
&= (\operatorname{div} \mathbf{e}_{\bar\beta}) \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}] + [\mathbf{h}, \dot{\mathbf{k}}], \dot{\mathbf{e}}_\beta)_{\mathfrak{g}} dt
\end{aligned}
$$

Further

$$\begin{aligned}
&\int_0^1 ([\dot{\mathbf{h}},\mathbf{k}],db(t))_{\mathfrak{g}} - \sum_\beta (\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})\langle \boldsymbol{\eta}, A(\mathbf{k},\mathbf{e}_\beta)\rangle \\
&= \int_0^1 (\rho_+[\dot{\mathbf{h}},\mathbf{k}],db(t))_{\mathfrak{g}} + \int_0^1 \big(\rho_-[\dot{\mathbf{h}},\mathbf{k}],db(t)\big)_{\mathfrak{g}} \\
&\quad + \int_0^1 (\int_0^1 [\dot{\mathbf{h}}(u),\mathbf{k}(u)]du, db(t))_{\mathfrak{g}} - \int_0^1 (\rho_-[\dot{\mathbf{h}},\mathbf{k}] + \rho_-[\mathbf{h},\dot{\mathbf{k}}], db(t))_{\mathfrak{g}} \\
&= \int_0^1 (\rho_+[\dot{\mathbf{h}},\mathbf{k}],db(t))_{\mathfrak{g}} + (\int_0^1 [\dot{\mathbf{h}}(u),\mathbf{k}(u)]du, b(1))_{\mathfrak{g}} - \int_0^1 (\rho_-[\mathbf{h},\dot{\mathbf{k}}],db(t))_{\mathfrak{g}}.
\end{aligned}$$

Next we consider the case $\mathbf{k} \in H_1^{(0,1)}$.

$$\begin{aligned}
&\langle dd^*\boldsymbol{\eta} - \nabla^*\check{\nabla}\boldsymbol{\eta}, \mathbf{k}\rangle \\
&= \langle du - \nabla^*\check{\nabla}\boldsymbol{\eta}, \mathbf{k}\rangle \\
&= \langle T^{*-1}\boldsymbol{\eta}, \mathbf{k}\rangle + \int_0^1 ([\dot{\mathbf{h}},\mathbf{k}],db(t))_{\mathfrak{g}} \\
&\quad - \sum_{\alpha,\beta}\{(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})\langle \theta^{\bar\alpha},\mathbf{k}\rangle\langle \boldsymbol{\eta}, \pi_+[\mathbf{e}_{\bar\alpha},\mathbf{e}_\beta]\rangle + \langle \nabla_{T^{-1}\mathbf{e}_{\bar\beta}}\boldsymbol{\theta}^\alpha,\mathbf{k}\rangle \boldsymbol{\eta}(\pi_+[\mathbf{e}_{\bar\alpha},\mathbf{e}_\beta])\} \\
&= \int_0^1 ([\dot{\mathbf{h}},\mathbf{k}],db(t))_{\mathfrak{g}} \\
&\quad - \sum_\beta (\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})\langle \boldsymbol{\eta}, [\mathbf{k},\mathbf{e}_\beta]\rangle + \sum_\beta \langle \boldsymbol{\eta}, [A(T^{-1}\mathbf{e}_{\bar\beta},\mathbf{k}),\mathbf{e}_\beta]\rangle.
\end{aligned}$$

Further,

$$\begin{aligned}
\langle \boldsymbol{\eta}, [A(T^{-1}\mathbf{e}_{\bar\beta},\mathbf{k}),\mathbf{e}_\beta]\rangle &= (\mathbf{h}, [A(T^{-1}\mathbf{e}_{\bar\beta},\mathbf{k}),\mathbf{e}_\beta])_{H_1} \\
&= \sqrt{-1}(J\mathbf{h}, [A(T^{-1}\mathbf{e}_{\bar\beta},\mathbf{k}),\mathbf{e}_\beta])_{H_1} \\
&= \sqrt{-1}S(\mathbf{h}, [A(T^{-1}\mathbf{e}_{\bar\beta},\mathbf{k}),\mathbf{e}_\beta]) \\
&= \sqrt{-1}\int_0^1 (\dot{\mathbf{h}}, [A(T^{-1}\mathbf{e}_{\bar\beta},\mathbf{k}),\mathbf{e}_\beta])_{\mathfrak{g}} dt \\
&= \sqrt{-1}\int_0^1 ([\mathbf{e}_\beta,\dot{\mathbf{h}}], A(T^{-1}\mathbf{e}_{\bar\beta},\mathbf{k}))_{\mathfrak{g}} dt.
\end{aligned}$$

The stochastic integral vanishes because, for $\beta = (n,i)$,

$$\begin{aligned}
&(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})\langle \boldsymbol{\eta}, [\mathbf{k},\mathbf{e}_\beta]\rangle \\
&= (\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})(\mathbf{h}, [\mathbf{k},\mathbf{e}_\beta])_{H_1} \\
&= \sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})(J\mathbf{h}, [\mathbf{k},\mathbf{e}_\beta])_{H_1} \\
&= \sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})S(\mathbf{h}, [\mathbf{k},\mathbf{e}_\beta]) \\
&= \sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta})\int_0^1 (\dot{\mathbf{h}}, [\mathbf{k},\mathbf{e}_\beta])_{\mathfrak{g}} dt
\end{aligned}$$

$$\begin{aligned} &= \sqrt{-1}(\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta}) \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], \mathbf{e}_\beta])_{\mathfrak{g}} dt \\ &= \sqrt{-1}(\operatorname{div} \mathbf{e}_{\bar\beta})(2\pi n) \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], \frac{1}{2\pi\sqrt{-1}n}(e^{2\pi\sqrt{-1}nt} - 1)\xi_j)_{\mathfrak{g}} dt \\ &= (\operatorname{div} \mathbf{e}_{\bar\beta}) \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], e^{2\pi\sqrt{-1}nt}\xi_j)_{\mathfrak{g}} dt \\ &= (\operatorname{div} \mathbf{e}_{\bar\beta}) \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], \dot{\mathbf{e}}_\beta)_{\mathfrak{g}} dt. \end{aligned}$$

Combining this with $\int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], \dot{\mathbf{e}}_{\bar\beta})_{\mathfrak{g}} dt = 0$, we have

$$\sum_\beta (\operatorname{div} T^{-1}\mathbf{e}_{\bar\beta}) \langle \boldsymbol{\eta}, [\mathbf{k}, \mathbf{e}_\beta] \rangle = \int_0^1 ([\dot{\mathbf{h}}, \mathbf{k}], db(t))_{\mathfrak{g}}.$$

This completes the proof. □

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