Uniqueness of Gibbs measures on $C(\mathbb{R} \to \mathbb{R})$*

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We consider the uniqueness problem of Gibbs measures on $C(\mathbb{R} \to \mathbb{R})$. Suppose we are given a potential function $V : \mathbb{R} \mapsto \mathbb{R}$. We assume that V is continuous and non-negative. In this talk, a Gibbs measure associated with V is formally expressed as

$$\nu(dx) = Z^{-1} \exp\left\{-\frac{1}{2}\int_{-\infty}^{\infty} |\dot{x}(t)|^2\, dt - \int_{-\infty}^{\infty} V(x(t))\, dt\right\} \prod_{t\in\mathbb{R}} dx(t). \tag{1}$$

Precise characterization is fomulated through Dobrushin-Lanford-Ruelle equation as follows. For $I \subseteq \mathbb{R}$, we set $\mathcal{F}_I = \sigma\{x(t);\, t \in I\}$. Let $P_{s,x}^{t,y}$ be the pinned Brownian motion with $x(s) = x$ and $x(t) = y$. Then a probability measure μ is called a Gibbs measure if it satisfies

$$\mu(\,\cdot\,|\mathcal{F}_{[s,t]^c})(x(\cdot)) = Z^{-1} \exp\left\{-\int_s^t V(x(u))du\right\} P_{s,x}^{t,y} \otimes \delta_{x_{[s,t]^c}}. \tag{2}$$

Here Z is a normlizing constant. In this talk, we only deal with Gibbs measures satisfying the tightness condition: we set

$$\mathcal{G} = \{\mu \text{ satisfies DLR equation (1) and the family } \{\mu \circ x(t)^{-1}\} \text{ is tight}\}. \tag{3}$$

This type of measures, or more general classes, were discussed by many orthors, e.g., [1, 2]. We are interested in the uniqueness of $\mathcal{G}$. This measure is closely related to an operator $H = \frac{1}{2}\triangle - V$. H is a self-adjoint operator in $L^2(\mathbb{R})$ and we deonote the spectrum of $-H$ by $\sigma(-H)$. Now define

$$\lambda_0 = \inf \sigma(-H). \tag{4}$$

λ_0 is called a principal eigen-value in general. It is not always an eigenvalue but we can always find a positive solution ϕ such that $-H\phi = \lambda_0\phi$. If $\phi \in L^2(\mathbb{R})$, then λ_0 is an eigenvalue. Our main theorem is the following:

Theorem 1. If λ_0 is an eigenvalue then $\sharp(\mathcal{G}) = 1$, i.e., the uniqueness holds.

References

[1] K. Iwata, An infinite-dimensional stochastic differential equation with state space $C(R)$. *Probab. Theory Related Fields*, **5** (1987), 141–159.

[2] H. Osada and H. Spohn, Gibbs measures relative to Brownian motion, *Ann. Probab.*, **27** (1999), 1183–1207.

*October 25–27, 2006, "Stochastic analysis and related fields" at Kyoto University

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