Witten Laplacian for an unbounded spin system

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We consider Witten Laplacians acting on differential forms on an unbounded lattice spin system. The configuration space is \( \mathbb{R}^d \) and we are given a Gibbs measure on it. The Gibbs measure in this note is expressed formally as

\[
\nu = Z^{-1} \exp \left\{ -2 \mathcal{J} \sum_{i,j \in \mathbb{Z}^d} (x^i - x^j)^2 - 2 \sum_{i \in \mathbb{Z}^d} U(x^i) \right\} \prod_{i \in \mathbb{Z}^d} dx^i.
\]

(1)

Here \( \mathcal{J} \) is a positive constant and \( U \) is an \( \mathbb{R} \)-valued function on \( \mathbb{R} \) and \( i \sim j \) means that \(|i - j|^2 = (i_1 - j_1)^2 + \cdots + (i_1 - j_1)^2 = 1\). Precise characterization is formulated through Dobrushin-Lanford-Ruelle equation. We show that there is no harmonic \( p \)-forms (\( p \geq 1 \)) on this space. To do this, we take a finite set \( \Lambda \subset \mathbb{Z}^d \) and a boundary condition \( \eta \) and define a finite volume Gibbs measure on \( \mathbb{R}^\Lambda \) and then take limit.

For this model, the logarithmic Sobolev inequality [4, 3], the spectral gap and the vanishing theorem for 1-forms [1] under suitable condition, were already established. So we intend to generalize the vanishing theorem for general \( p \)-forms.

Finite volume Gibbs measure and Witten Laplacian

For a finite set \( \Lambda \subset \mathbb{Z}^d \) and a boundary condition \( \eta \), we define a Hamiltonian by

\[
\Phi_{\Lambda,\eta}(x) = \sum_{i,j \in \Lambda} \mathcal{J} (x^i - x^j)^2 + \sum_{i \in \mathbb{Z}^d} U(x^i) + 2 \sum_{i \in \Lambda, j \in \Lambda^c} \mathcal{J} (x^i - \eta^j)^2
\]

(2)

and define a measure on \( \mathbb{R}^\Lambda \) by

\[
\nu_{\Lambda,\eta} = Z^{-1} e^{-2\Phi_{\Lambda,\eta}(x)} dx_\Lambda.
\]

(3)

By a standard argument we can define the exterior differentiation \( d \) and its adjoint operator \( d^\ast \) with respect to the measure \( \nu_{\Lambda,\eta} \).

We call the operator \( dd^\ast + d^\ast d \) as the Witten Laplacian acting on differential forms.

Vanishing theorem and Hodge-Kodaira decomposition

Suppose that the potential \( U \) is decomposed as \( U = V + W \) so that \( V \) is convex and \( W \) is bounded. We denote the supremum and the infimum of \( W \) by \( W_{\text{sup}} \) and \( W_{\text{inf}} \), respectively. Then we have

**Theorem 1.** Suppose \( V'' \geq c > 0 \), and \( 2(c + 8d \mathcal{J}) e^{-2(W_{\text{sup}} - W_{\text{inf}})} > 16d \mathcal{J} \). Then the lowest eigenvalue of \( dd^\ast + d^\ast d \) on \( p \)-forms is not less than \( \{2(c + 8d \mathcal{J}) e^{-2(W_{\text{sup}} - W_{\text{inf}})} - 16d \mathcal{J}\} p \). Therefore there is no harmonic \( p \)-forms (\( p \geq 1 \)).

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If $U$ has a special form $U(t) = at^4 - bt^2$, we can give different kind of criterion:

**Theorem 2.** If $\sqrt{3a} - b - 4d > 0$, then the lowest eigenvalue of $dd^* + d^*d$ on $p$-forms is not less than $2(\sqrt{3a} - b - 4d)p$. Therefore there is no harmonic $p$-forms ($p \geq 1$).

Using these theorems, we can show the Hodge-Kodaira decomposition theorem.

**Theorem 3.** We have the following: for $p = 0$,

$$L^2(\nu_{A,\eta}) = \{ \text{constant functions} \} \oplus \text{Ran}(d^*),$$

and for $p \geq 1$,

$$L^2(\nu_{A,\eta}; \Lambda^p(\mathbb{R}^A)^*) = \text{Ran}(d) \oplus \text{Ran}(d^*)$$

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**References**


