Witten Laplacian for an unbounded spin system*

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We consider Witten Laplacians acting on differential forms on an unbounded lattice spin system. The configulation space is $\mathbb{R}^{\mathbb{Z}^d}$ and we are given a Gibbs measure on it. The Gibbs measure in this note is expressed formally as

$$\nu = Z^{-1} \exp\Big\{-2\mathscr{J} \sum_{\substack{i,j\in\mathbb{Z}^d \\ i\sim j}} (x^i - x^j)^2 - 2\sum_{i\in\mathbb{Z}^d} U(x^i)\Big\} \prod_{i\in\mathbb{Z}^d} dx^i. \tag{1}$$

Here $\mathscr{J}$ is a positive constnat and U is an $\mathbb{R}$-valued function on $\mathbb{R}$ and $i \sim j$ means that $|i-j|^2 = (i_1 - j_1)^2 + \cdots + (i_1 - j_1)^2 = 1$. Precise characterization is fomulated through Dobrushin-Lanford-Ruelle equation. We show that there is no harmonic p-forms ($p \geq 1$) on this space. To do this, we take a finite set $\Lambda \subset \mathbb{Z}^d$ and a boundary condition η and define a finite volume Gibbs measure on $\mathbb{R}^\Lambda$ and then take limit.

For this model, the logarithmic Sobolev inequality [4, 3], the spectral gap and the vanishing theorem for 1-fomrs [1] under suitable condition, were already established. So we intend to generalize the vanishing theorem for general p-fomrs.

Finite volume Gibbs measure and Witten Laplacian

For a finite set $\Lambda \subset \mathbb{Z}^d$ and a boundary condition η, we define a Hamiltonian by

$$\Phi_{\Lambda,\eta}(x) = \sum_{\substack{i,j\in\Lambda \\ i\sim j}} \mathscr{J}(x^i - x^j)^2 + \sum_{i\in Z^d} U(x^i) + 2 \sum_{\substack{i\in\Lambda, j\in\Lambda^c \\ i\sim j}} \mathscr{J}(x^i - \eta^j)^2 \tag{2}$$

and define a measure on $\mathbb{R}^\Lambda$ by

$$\nu_{\Lambda,\eta} = Z^{-1} e^{-2\Phi_{\Lambda,\eta}(x)} dx_\Lambda. \tag{3}$$

By a standard argument we can define the exterior differentiation d and its adjoint operator d^* with respect to the measure $\nu_{\Lambda,\eta}$.

We call the operator $dd^* + d^*d$ as the Witten Laplacian acting on differential forms.

Vanishing theorem and Hodge-Kodaira decomposition

Suppose that the potential U is decomposed as $U = V + W$ so that V is convex and W is bounded. We denote the supremum and the infimum of W by $W_{\sup}$ and $W_{\inf}$, respectively. Then we have

Theorem 1. Suppose $V'' \geq c > 0$, and $2(c+8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})} > 16d\mathscr{J}$. Then the lowest eigenvalue of $dd^* + d^*d$ on p-forms is not less than $\{2(c+8d\mathscr{J})e^{-2(W_{\sup}-W_{\inf})} - 16d\mathscr{J}\}p$. Therefore there is no harmonic p-forms ($p \geq 1$).

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If U has a special form $U(t) = at^4 - bt^2$, we can give different kind of criterion:

Theorem 2. If $\sqrt{3a} - b - 4d\mathscr{J} > 0$, then the lowest eigenvalue of $dd^* + d^*d$ on p-forms is not less than $2(\sqrt{3a} - b - 4d\mathscr{J})p$. Therefore there is no harmonic p-forms $(p \geq 1)$.

Using these theorems, we can show the Hodge-Kodaira decomposition theorem.

Theorem 3. We have the following : for $p = 0$,

$$L^2(\nu_{\Lambda,\eta}) = \{ \text{ constant functions } \} \oplus \mathrm{Ran}(d^*), \tag{4}$$

and for $p \geq 1$,

$$L^2(\nu_{\Lambda,\eta}; \textstyle\bigwedge^p(\mathbb{R}^\Lambda)^*) = \mathrm{Ran}(d) \oplus \mathrm{Ran}(d^*) \tag{5}$$

References

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