One dimensional diffusions conditioned to be non-explosive

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1. Introduction

- \{ (X_t), P_x \} : a diffusion on a state space \( D \).
- \( \zeta \) : the explosion time.

The diffusion conditioned to be non-explosive is defined as follows:

1. If \( P_x[\zeta = \infty] > 0 \),
   \[
P_x[\cdot | \zeta = \infty] = \frac{P_x[\cdot \cap \zeta = \infty]}{P_x[\zeta = \infty]}.
   \]

2. If \( P_x[\zeta = \infty] = 0 \),
   \[
   \lim_{T \to \infty} P_x[\cdot | \zeta > T].
   \]

The limit (1.1) is called a surviving diffusion.
We discuss the following issues:

1. **When does the surviving diffusion exist?**
2. **Characterization of the surviving diffusion.**

**Strategy:**

Since

\[
E_x[\cdot \mid \zeta > T] = E_x\left[\cdot \frac{1_{\{\zeta > t\}} P_{X_t}[\zeta > T - t]}{P_x[\zeta > T]}\right],
\]

our problem is reduced to show the existence of the limit

\[
M_t = \lim_{T \to \infty} \frac{1_{\{\zeta > t\}} E_{X_t}[\zeta > T - t]}{P_x[\zeta > T]}
\]

and to show that \((M_t)\) is a martingale.
To do this, we show that there exist a \( \varphi \) with \(- \frac{d}{dm} \frac{d}{ds} \varphi = \lambda \varphi\) so that

\[
\lim_{T \to \infty} \frac{P_y[\zeta > T - t]}{P_x[\zeta > T]} = \frac{\varphi(y) e^{\lambda t}}{\varphi(x)}
\]

and

\[
M_t = 1_{\{\zeta > t\}} \varphi(X_t) e^{\lambda t} / \varphi(x).
\]

The surviving diffusion is given by

\[
\hat{E}_x[\cdot] = E_x \left[ \cdot 1_{\{\zeta > t\}} \frac{\varphi(X_t) e^{\lambda t}}{\varphi(x)} \right].
\]
2. One dimensional diffusion processes

\[ D = (l_-, l_+). \]

\( \{(X_t), P_x\} \): a (minimal) diffusion on \( D \) (Dirichlet boundary condition)

\( s(x) \): the scale function

\( dm(x) \): the speed measure (standard measure)

\( \zeta \): the explosion time

\[ \frac{d}{dm} \frac{d}{ds} \]: the generator

Dirichlet form \[ \mathcal{E}(f, g) = \int_D \frac{df}{ds} \frac{dg}{ds} ds \]
From $dm$, we define a right continuous non-decreasing function $m$ as

$$m(y) - m(x) = \int_{(x,y]} dm.$$ 

Take any $a \in (l_-, l_+)$ and define

$$S(x) = \int_{(a,x]} \{m(y) - m(a)\} ds(y) = \int_{(a,x]} \{s(x) - s(u)\} dm(u),$$

$$M(x) = \int_{(a,x]} \{s(y) - s(a)\} dm(y) = \int_{(a,x]} \{m(x) - m(u)\} ds(u).$$

- $S(l_+) < \infty \Rightarrow l_+ \text{ is called } \text{exit}.$
- $S(l_+) = \infty \Rightarrow l_+ \text{ is called } \text{non-exit}.$
- $M(l_+) < \infty \Rightarrow l_+ \text{ is called } \text{entrance}.$
- $M(l_+) = \infty \Rightarrow l_+ \text{ is called } \text{non-entrance}.$
Feller’s criterion:

\((X_t)\) is conservative \(\iff S(l_+) = \infty\) and \(S(l_-) = \infty\)

**h-transformation**

Let \(v\) be a \(\lambda\)-harmonic function, i.e.,

\[ \frac{d}{dm} \frac{d}{ds} v = \lambda v. \]

Define \(d\hat{m} = v^2 dm\), \(d\hat{s} = \frac{ds}{v^2}\). Then

\[ \frac{1}{v} \left( \frac{d}{dm} \frac{d}{ds} - \lambda \right) (vf) = \frac{d}{d\hat{m}} \frac{d}{d\hat{s}} f. \]  

\(\frac{d}{d\hat{m}} \frac{d}{d\hat{s}}\) is the **h-transform** of \(\frac{d}{dm} \frac{d}{ds} - \lambda\).
3. The case $P_x[\zeta = \infty] > 0$

Theorem 3.1. Let $(X_t)$ be a diffusion process on $(0, l)$ with a natural scale $s(x) = x$ and a speed measure $dm$. Assume that 0 is exit and $l$ is non-exit. Then $P_x[\zeta = \infty] > 0$ and the associated surviving diffusion has the scale $-1/x$ and the speed measure $x^2 dm$. 
4. Exit - exit boundaries

\( D = (0, l) \), the natural scale \( s(x) = x \), the speed measure \( dm \).

\[
\int_0^{l/2} x dm(x) < \infty.
\]

\[
\int_{l/2}^l (l - x) dm(x) < \infty.
\]

We assume that there exists \( \gamma > 0 \) and \( M \) so that

\[
\int_0^y x dm(x) \leq M y^\gamma.
\]

\[
\int_{l-y}^l (l - x) dm(x) \leq M y^\gamma.
\]
In this case, the Green operator is of trace class. We define $\lambda_0 > 0$ to be a lowest eigenvalue of $-\frac{d}{dm} \frac{d}{ds}$ and $\varphi_0$ be its eigenfunction. $\varphi_0$ has the following asymptotics:

$$\varphi_0(x) \sim c_1 x \quad \text{as} \quad x \to 0$$

$$\varphi_0(x) \sim c_2 (l - x) \quad \text{as} \quad x \to l.$$
Under these conditions,

**Theorem 4.1.**

\[
\lim_{T \to \infty} e^{\lambda_0 T} P_x[\zeta > T] = \varphi_0(x) \int_D \varphi_0(y) dm(y).
\]

In particular,

\[
\lim_{T \to \infty} \frac{P_y[\zeta > T - t]}{P_x[\zeta > T]} = e^{\lambda_0 t} \frac{\varphi_0(y)}{\varphi_0(x)}.
\]

The surviving diffusion exists and it has a scale \(d\hat{s} = ds/\varphi_0^2\) and a speed measure \(d\hat{m} = \varphi_0^2 dm\).
5. (exit & entrance) - (non-exit & non-entrance) boundaries

\[ D = (0, \infty), \text{ the natural scale } s(x) = x, \text{ the speed measure } dm. \]

We assume

\[ m(x) \sim x^{1/\mu - 1}K(x) \quad \text{as } x \to \infty \tag{5.1} \]

where \(0 < \mu < 1\) and \(K\) is a slowly varying function. Define a slowly varying function \(L\) so that the function \(y \mapsto y^{\mu}L(y)\) is an inverse of the function \(y \mapsto y^{1/\mu}K(y)\).
Under these conditions,

**Theorem 5.1.**

\[ P_x[\zeta > t] \sim x\{\mu(1 - \mu)\}^\mu \Gamma(1 + \mu)^{-1} t^{-\mu} L(t)^{-1} \quad \text{as} \; t \to \infty. \]

In particular,

\[ \lim_{T \to \infty} \frac{P_y[\zeta > T - t]}{P_x[\zeta > T]} = \frac{y}{x}. \]

The surviving diffusion exists and it has a scale \( \hat{s}(x) = -1/x \) and a speed measure \( d\hat{m} = x^2 dm \).
6. exit - (non-exit & entrance) boundaries

\( D = (0, \infty) \), the natural scale \( s(x) = x \), the speed measure \( dm \). From the boundary condition,

\[
\int_0^\infty x dm(x) < \infty.
\]

We assume that there exists \( \gamma > 0 \) and \( M \) so that

\[
\int_0^y x dm(x) \leq My^\gamma, \quad y > 0.
\]

In this case, the Green operator is of trace class. We define \( \lambda_0 > 0 \) to be a lowest eigenvalue of \(-\frac{d}{dm} \frac{d}{ds}\) and \( \varphi_0 \) be its eigenfunction.

\[
\varphi_0(x) \sim c_1 x \quad \text{as } x \to 0
\]

\[
\varphi_0(x) \sim c_2 \quad \text{as } x \to \infty.
\]
Under these conditions,

**Theorem 6.1.**

\[
\lim_{T \to \infty} e^{\lambda_0 T} P_x[\zeta > T] = \varphi_0(x) \int_D \varphi_0(y) dm(y).
\]

In particular,

\[
\lim_{T \to \infty} \frac{P_y[\zeta > T - t]}{P_x[\zeta > T]} = e^{\lambda_0 t} \frac{\varphi_0(y)}{\varphi_0(x)}.
\]

(6.1)

The surviving diffusion exists and it has a scale \( d\hat{s} = ds/\varphi_0^2 \) and a speed measure \( d\hat{m} = \varphi_0^2 dm \).
7. Examples

Exploding diffusion | Surviving diffusion

\[\nu < 0 \quad \nu > 0\]

**Bessel diffusions on \((0, \infty)\)**

\[
\frac{d}{dm} \frac{d}{ds} = \frac{1}{2} \frac{d^2}{dx^2} - \frac{d - 1}{2x} \frac{d}{dx}
\]

\[d = \text{dimension}\]

\[\nu = \frac{d - 2}{2}\]
Brownian motion on an interval $(0, l)$

ground state: $\sin \frac{\pi x}{l}$

The radial motion of the Brownian motion on a 3-dimensional sphere

radial part of $\frac{1}{2} \Delta$:

$$\frac{1}{2} \frac{d}{dx^2} + \sqrt{\kappa} \cot \sqrt{\kappa} x \frac{d}{dx}$$

$$\kappa = \frac{\pi^2}{l^2}$$
8. Proof of Theorem 4.1

Since the Green operator is compact, the transition function has the following expression

\[
p(t, x, y) = \sum_{i=0}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y)
\]

Here \( \lambda_i \) are eigenvalues of \( -\frac{d}{dm} \frac{d}{ds} \) and \( \varphi_i \) are eigenfunctions. The following estimate is crucial: there exist \( C > 0 \) and \( N \) so that

\[
\int_0^{l} |\varphi_i(y)| dm(x) \leq C \lambda_i^N \left\{ \int_0^{l} \varphi_i(y)^2 dm(x) \right\}^{1/2}
\]
9. Invariant function

\( p(t, x, dy) \): a transition probability

\( \varphi \) is called a invariant function if

\[
\varphi(x) = \int_D \varphi(y) p(t, x, dy), \quad \forall t \geq 0.
\]

It is easy to see

\( \varphi \) is invariant \( \iff \) \( h \)-transform by \( \varphi \) is conservative.

By the argument before, we can show that any one-dimensional (minimal) diffusion has a invariant function if the lowest eigenvalue is 0.
<table>
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<th>right</th>
<th>$D$</th>
<th>eigenvalue</th>
<th>$h$-transform</th>
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<tbody>
<tr>
<td>case 1</td>
<td>exit ←→ exit</td>
<td>(0, $l$)</td>
<td>$\lambda_0 &gt; 0$</td>
<td>$\varphi_0(x)$</td>
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<tr>
<td>case 2</td>
<td>exit ←→ non-exit ←← non-entrance</td>
<td>(\begin{cases} (0, \infty) \ (0, l) \end{cases})</td>
<td>$\lambda_0 \geq 0$</td>
<td>$s(x) = x$</td>
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<tr>
<td>case 3</td>
<td>exit ←→ non-exit ←← entrance</td>
<td>(0, $\infty$)</td>
<td>$\lambda_0 &gt; 0$</td>
<td>$\varphi_0(x)$</td>
<td></td>
</tr>
</tbody>
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Thank you.