Littlewood-Paley inequality for a diffusion satisfying the logarithmic Sobolev inequality:

Let \((X_t)\) be a symmetric diffusion process on a space \((M, \mathcal{B}(M), \mu)\), where \(\mathcal{B}(M)\) is the Borel \(\sigma\)-algebra and \(\mu\) is a probability measure. We denote the associated Dirichlet form by \(\mathcal{E}\). We assume that \(\mathcal{E}\) is given as

\[
\mathcal{E}(u, v) = \int_M (\nabla u, \nabla v) d\mu(x)
\]

for some closed operator \(\nabla: L^2(\mu) \to L^2(\mu; K)\) which satisfies the derivation property. Here \(K\) is a Hilbert space and \(L^2(\mu; K)\) may be possibly a \(L^2\) section of a vector bundle over \(M\). We assume that \(\mathcal{E}(1, 1) = 0\) and the following logarithmic Sobolev inequality holds: there exist \(\alpha > 0\) and \(\beta \geq 0\) such that

\[
\int_M u^2 \log \frac{u}{\|u\|_2} \mu(dx) \leq \alpha \mathcal{E}(u, u) + \beta (u, u).
\]

We need another semigroup \(\{\hat{T}_t\}\) in \(L^2(\mu; K)\) with the generator \(\hat{L}\). We assume that \(|\hat{T}_t\theta|_K \leq T_t \|\theta\|_K\) and \(\nabla \hat{L} = (\hat{L} - R)\nabla\). We also impose the exponential integrability of the negative part of \(R\). We formulate this condition as follows. Let \(V\) be a scalar function satisfying \((R(x)k, k)_K \geq V(x)|k|_K^2\). We denote the negative part of \(V\) by \(V^-\) and assume that \(e^{V^-} \in L^{\infty} = \bigcap_{p\geq 1} L^p\).

Under these conditions, we have the following theorem:

Theorem 1 For any \(1 < p < q < \infty\), there exist constants \(C_1, C_2\), so that

\[
\|\nabla u\|_p \leq C_1 \|\sqrt{1-L}u\|_q, \quad (1)
\]

\[
\|\sqrt{1-L}u\|_p \leq C_2 (\|\nabla u\|_q + \|u\|_q). \quad (2)
\]

Reference


