The logarithmic Sobolev inequality and the convergence of a semigroup in the Zygmund space

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1. Entropy and the Zygmund space

Zygmund space

- $(M, \mathcal{B}, m)$: a measure space
- $m(M) = 1$
- $\langle f \rangle = E[f] = \int_M f dm$
Let $\phi: [0, \infty) \rightarrow \mathbb{R}$ be defined by

$$\phi(x) = \log(1 + x)$$

graph of $\phi(x) = \log(1 + x)$
Set

(2) \[ \Phi(x) = \int_0^x \phi(y) \, dy = (1 + x) \log(1 + x) - x. \]
The **Zygmund space** \( Z = L \log L \) is defined by

\[
Z = \{ f; E[\Phi(|f|)] < \infty \}.
\]

The norm \( N_\Phi \) in \( Z \) is defined by

\[
N_\Phi(f) = \inf \{ \lambda; E[\Phi(|f|/\lambda)] \leq 1 \}.
\]

The dual space of \( Z \) can be defined as follows. Let \( \psi \) be the inverse function of \( \phi \):

\[
\psi(x) = e^x - 1.
\]

Set

\[
\Psi(x) = \int_0^x \psi(y) \, dy = \int_0^x (e^y - 1) \, dy = e^x - x - 1.
\]
The dual space of $Z$ is the Orlicz space associated with $\Psi$. The following inequality is fundamental:

\begin{equation}
xy \leq \Phi(x) + \Psi(y).
\end{equation}

By using this inequality we can show

\begin{equation}
\| f \|_1 \leq (e - 1) N_\Phi(f).
\end{equation}

So $Z$ is smaller than $L^1$. Moreover we have

\[ N_\Phi(f - \langle f \rangle) = 2N_\Phi(f). \]
**Entropy**

Define an entropy of $f \geq 0$ by

$$\text{Ent}(f) = E[f \log(f/\langle f \rangle)].$$

We discuss the relation between the Zygmunt space and the entropy.

**Proposition 1.** For any non-negative function $f$, we have

$$\langle f \rangle E[\Phi(||(f - \langle f \rangle)/\langle f \rangle||)] \leq \text{Ent}(f)$$

If $\langle f \rangle \geq 1$, we have another inequality.
Proposition 2. For any nonnegative function $f$ with $\langle f \rangle \geq 1$, we have

$$E[\Phi(|f - \langle f \rangle|)] \leq \langle f \rangle \text{Ent}(f).$$

Now we have

Proposition 3. For any non-negative function $f$, we have

$$N_{\Phi}(f - \langle f \rangle) \leq \max\{\sqrt{\langle f \rangle}, \sqrt{\text{Ent}(f)}\} \sqrt{\text{Ent}(f)}.$$
Now we will prove the reversed inequality. Recall

\begin{equation}
\text{Ent}(f) = E[f \log(f/\langle f \rangle)]
\end{equation}

**Proposition 4.** For any non-negative function $f$, we have

\begin{equation}
\text{Ent}(f) \leq \frac{\langle f \rangle}{\log(4/e)} E[\Phi(||(f - \langle f \rangle)/\langle f \rangle||)].
\end{equation}

If $f$ satisfy $\langle f \rangle \leq 1$, we have the following.
Proposition 5. For any non-negative function $f$ with $\langle f \rangle \leq 1$, we have

$$
\text{Ent}(f) \leq E[\Phi(|f - \langle f \rangle|)] + 2.
$$

Proposition 6. For any non-negative function $f$, we have

$$
\text{Ent}(f) \leq 3N_\Phi(f - \langle f \rangle).
$$
The logarithmic Sobolev inequality

Let us recall the logarithmic Sobolev inequality.

- $\mathcal{E}$ : a Dirichlet form
- $\{T_t\}$ : a Markovian semigroup in $L^2(m)$
- $\mathcal{A}$ : the generator of $\{T_t\}$

The following inequality is called a logarithmic Sobolev inequality:

\begin{equation}
\int_M f^2(x) \log\left(\frac{f(x)^2}{\|f\|_2^2}\right) \, dm \leq \frac{2}{\gamma_{LS}} \mathcal{E}(f, f).
\end{equation}
If we assume the logarithmic Sobolev inequality (15), it is known that for any non-negative function $f$, we have

\begin{equation}
\text{Ent}(T_t f) \leq e^{-2\gamma L s t} \text{Ent}(f).
\end{equation}

We set

\begin{equation}
\gamma_{z \rightarrow z} = - \lim_{t \to 0} \frac{1}{t} \log \| T_t - m \|_{z \rightarrow z}
\end{equation}
Combining the previous results, we have

**Theorem 7.** We have the following inequality:

\[(18) \quad \gamma_{LS} \leq \gamma_{Z \rightarrow Z}\]

Under the assumption of the logarithmic Sobole inequality, we can show that the independence of the spectrum.

(Kusuoka - S [2015])

Assume $\mathcal{A}$ is normal. Then $\sigma(\mathcal{A}_p)$ is independent of $p$ ($1 < p < \infty$).

Here $\mathcal{A}_p$ is the generator in $L^p$.

Question: What happens in the Zygmund space?
2. Operators in Zygmund space

We define an Orlitz norm \( \| \| \Phi \) as follows:

\[
(19) \quad \| f \| \Phi = \sup \{ E[g|f|]; E[\Psi(g)] \leq 1 \}.
\]

Here non-negative functions \( g \) run over all functions with \( E[\Psi(g)] \leq 1 \). We also have

\[
(20) \quad \| f \| \Phi = \sup \{ E[g|f|]; E[e^g - g] \leq 2 \}.
\]

Two norms \( N_\Phi \) and \( \| \| \Phi \) are equivalent:

\[
(21) \quad N_\Phi(f) \leq \| f \| \Phi \leq 2N_\Phi(f).
\]
Proposition 8. A linear operator $T$ in $Z$ is bounded if and only if there exist positive constants $A, B$ such that

$$\|Tf\|_\Phi \leq AE[\Phi(|f|)] + B. \tag{22}$$

Corollary 9. A linear operator $T$ in $Z$ is bounded if and only if there exist positive constants $A, B$ such that for all non-negative function $g$ with $E[e^g] \leq 4$, we have

$$E[g|Tf|] \leq AE[|f| \log |f|] + B. \tag{23}$$
3. Spectrum of the Kummer operator

In this section, we consider the Kummer operator

- \( M = [0, \infty) \)
- \( m(dx) = \frac{1}{\Gamma(\alpha+1)} x^\alpha e^{-x} dx \)
- \( H = L^2([0, \infty), m) \)
- \( \mathcal{A} = x \frac{d^2}{dx^2} + (1 + \alpha - x) \frac{d}{dx} \)

We assume that \( \alpha > 0 \).

We give a representation of the resolvent by using the confluent hypergeometric functions.
A confluent hypergeometric functions is defined by

\[
_{1}F_{1}(a; c; x) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n n!} x^n.
\]

Here \((a)_n\) stands for the Pochhammer symbol:

\[
(a)_n = \frac{\Gamma(a + n)}{\Gamma(a)} = \begin{cases} a(a+1) \cdots (a+n-1) & n \geq 1 \\ 1 & n = 0 \end{cases}
\]

\(1_{1}F_{1}(a; c; x)\) satisfies the following Kummer differential equation:

\[
xu'' + (c - x)u' = au.
\]
This means that \( _1F_1(a; c; x) \) is an “eigen-function” of the Kummer operator in the case of \( 1 + \alpha = c \).

If \( _1F_1(a; c; x) \in L^2 \), then \( _1F_1(a; c; x) \) is really an eigen-value. We set

\[
M(a, 1 + \alpha; x) = _1F_1(a; 1 + \alpha; x).
\]

This function is called the Kummer function. Another independent solution is

\[
U(a, 1+\alpha; x) = \frac{\Gamma(-\alpha)}{\Gamma(a - \alpha)} M(a, 1+\alpha; x) + \frac{\Gamma(\alpha)}{\Gamma(a)} x^{-\alpha} M(a-\alpha, 1-\alpha; x)
\]

which is called the Kummer function of the second kind. Their
Wronskian is

\[ W(M(a, 1 + \alpha; \cdot), U(a, 1 + \alpha; \cdot))(x) = -\frac{\Gamma(1 + \alpha)}{\Gamma(a)} x^{-\alpha - 1} e^x. \]

It is known that Laguerre polynomials are eigen-functions. In fact, we have

\[ L_n^\alpha(x) = \frac{(\alpha + 1)^n}{n!} M(-n, \alpha + 1; x). \]

Thus the spectrum of \( \mathcal{A} \) is \( \{0, -1, -2, \cdots\} \).

The asymptotic behavior is crucial in the computation of the resolvent.
When \( x \to 0 \), we have

\[
M(a, 1 + \alpha; x) \to 1,
\]

(30)

\[
U(a, 1 + \alpha; x) \sim \frac{\Gamma(\alpha)}{\Gamma(a)} x^{-\alpha}.
\]

(31)

When \( \alpha = 0 \), \( x^{-\alpha} \) should be \( \log x \).

When \( x \to \infty \), we have

\[
M(a, 1 + \alpha; x) \sim \frac{\Gamma(1 + \alpha)}{\Gamma(a)} e^{x} x^{a-1-\alpha},
\]

(32)

\[
U(a, 1 + \alpha; x) \sim x^{-\alpha}
\]

(33)

Here \( a, 1 + \alpha \neq 0, -1, -2, \ldots \).

Recall that \( \alpha > 0 \). We also assume that \( a \neq 0, -1, -2, \ldots \). Then
the resolvent $G_a = (a - A)^{-1}$ has the following kernel expression.

\[
G_a f(x) = \int_0^\infty G_a(x, y) f(y) \, dy
\]

where

\[
G_a(x, y) = \begin{cases} 
-M(a, 1 + \alpha; y)U(a, 1 + \alpha, x) \frac{1}{yW(y)} & y < x, \\
-M(a, 1 + \alpha; x)U(a, 1 + \alpha, y) \frac{1}{yW(y)} & y > x.
\end{cases}
\]

$W$ is the Wronskian. Hence

\[
G_a(x, y) = \begin{cases} 
\frac{\Gamma(a)}{\Gamma(1 + \alpha)} M(a, 1 + \alpha; y)U(a, 1 + \alpha, x)e^{-y}y^\alpha & y < x, \\
\frac{\Gamma(a)}{\Gamma(1 + \alpha)} M(a, 1 + \alpha; x)U(a, 1 + \alpha, y)e^{-y}y^\alpha & y > x.
\end{cases}
\]
$G_\alpha$ is a bounded operator in $L^2$. What happens in the case of Zygmund space?
4. The spectrum of the Kummer operator in \( \mathbb{Z} \)

Now we can compute the spectrum of \( \mathcal{A} \) in \( \mathbb{Z} \). Since we have the kernel expression of the resolvent, we can compute the spectrum.

**Theorem 10.** The set of point spectrums of \( \mathcal{A} \) is \( \{ z; \Re z < -1 \} \cup \{-1\} \cup \{0\} \).

**Theorem 11.** When \( \Re a > -1 \), \( a \) belongs to the resolvent set.
The spectrum in \( \mathbb{Z} \).
In Theorem 7, we have shown $\gamma_{LS} \leq \gamma_{Z \rightarrow Z}$. In this example $\gamma_{LS} = \frac{1}{2}$ and $\gamma_{Z \rightarrow Z} = 1$, which means that $\gamma_{LS} \neq \gamma_{Z \rightarrow Z}$. 
Thanks !